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RESEARCH PAPER

Variational formulation for a generalized third order equation

Shao Yabin [a], Ji-Huan He [a, b*], Abdulrahman Ali Alsolami [c]

[a] *School of Jia Yang, Zhejiang Shuren University, Shaoxing 312028, China*
[b] *National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, 199 Ren-Ai Road, Suzhou Industrial Park, Suzhou 215123, China*
[c] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract

A universal formulation is obtained for the construction of a variational principle for a general third-order differential equation, regardless of its self-adjoint condition. Three illustrative examples are provided to demonstrate the convenience and efficacy of the proposed formulation.

Keywords: variational principle; semi-inverse method; KdV equation

1. Introduction

Here introduces the paper, and put a nomenclature if necessary, with the same font size as the rest of the paper. The paragraphs continue from here and are only separated by headings, subheadings, images and formulae. The section headings are arranged by numbers, bold and 10 pt. Here follows further instructions for authors.

The majority of engineering problems can be ultimately modelled by differential equations. Second-order differential equations, e.g., heat equation and diffusion equation, have been extensively studied, with notable methods including the homotopy perturbation method [1, 2] and the variational iteration method [3]. Additionally, a considerable number of nonlinear problems can be described by a third-order differential equation [4, 5]. The most renowned of these is the KdV equation [6], which plays a pivotal role in soliton theory.

The following general equation will be considered in this paper.

$$u_t = (g(u))_{xxx} + (f(u))_x \tag{1}$$

where g and f are continuous functions.

For the quasi-self-adjoint case, Eq. (1) admits a Lagrangian as discussed in Ref. [7]. In our study, it is not necessary for Eq. (1) to be self-adjoint or nonself-adjoint to obtain a Lagrangian or a variational principle.

2. Variational principle

The general approach to establishing a variational formulation from the governing equations is the semi-inverse method, as described in [8]. This method has been employed to establish various variational principles, including those pertaining to nano-lubrication [9], incompressible fluids [10], Rabinowitsch lubrication [11], fractal waves [12], plasma waves [13], shallow water waves [14], and optical solitons [15].

In order to obtain a generalized variational principle for Eq. (1), according to the semi-inverse method [8], a special function, Φ, is introduced

$$\Phi_x = u \tag{2}$$

* Corresponding author. Tel.: +86-139-1315-2427. *E-mail address:* hejihuan@suda.edu.cn

$$\Phi_t = g_{xx} + f \tag{3}$$

A trial-functional is established in the following form

$$J(u,\Phi) = \iint L dxdt \tag{4}$$

where L is the Lagrangian to be further determined. We begin with

$$L = au\Phi_t + b\Phi_x\Phi_t - g\Phi_{xxx} - f\Phi_x + F \tag{5}$$

The unknown function, denoted by the letter F, is a function of the variable u and/or its derivatives. The Euler-Lagrange equations can be expressed as follows:

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \Phi_t}) - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \Phi_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial L}{\partial \Phi_{xx}}) + \cdots = 0 \tag{6}$$

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial L}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial L}{\partial u_{xx}}) + \cdots = 0 \tag{7}$$

or

$$-au_t - 2b\Phi_{xt} + g_{xxx} + f_x + \frac{\delta F}{\delta \Phi} = 0 \tag{8}$$

$$a\Phi_t - g_u\Phi_{xxx} - f_u\Phi_x + \frac{\delta F}{\delta u} = 0 \tag{9}$$

The variational derivative with respect to u is defined as

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial F}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial u_{xx}}) + \cdots \tag{10}$$

If we set

$$\frac{\delta F}{\delta \Phi} = 0 \tag{11}$$

and

$$a + 2b = 1 \tag{12}$$

in Eq. (8), and use Eq. (2) as a constraint, Eq. (8) turns out to be Eq. (1). Using Eqs. (2) and (3), from Eq. (9) we obtain

$$\frac{\delta F}{\delta u} = -a\Phi_t + g_u\Phi_{xxx} + f_u\Phi_x = -ag_{xx} + g_u u_{xx} + f_u u - af \tag{13}$$

We set

$$a = -1 \tag{14}$$

to identify *F*, which reads

$$F = ug_{xx} + \xi \tag{15}$$

where ξ is defined as

$$\frac{\partial \xi}{\partial u} = f_u u + f \tag{16}$$

Consequently, we are able to derive the requisite variational principle, which is

$$J(u,\Phi) = \iint \left\{ -u\Phi_t + \Phi_x \Phi_t - g\Phi_{xxx} - f\Phi_x + ug_{xx} + \xi \right\} dxdt \tag{17}$$

This is constrained by Eq. (2).

Proof.

The Euler-Lagrange equations for Eq.(17) reads

$$u_t - 2\Phi_{xt} + g_{xxx} + f_x = 0 \tag{18}$$

$$-\Phi_t - g_u \Phi_{xxx} - f_u \Phi_x + u_{xx} g_u + g_{xx} + \xi_u = 0 \tag{19}$$

Considering the constraints, Eqs.(2) and (16), we can easily prove that Eqs.(18) and (19) lead, respectively, to Eq.(1) and Eq.(3).

3. Examples

The obtained variational principle, Eq. (17), is valid for all cases of continuous *f* and *g* in Eq. (1).

Example 1

$$u_t + u_{xxx} + de^{-au} u_x = 0 \tag{20}$$

where *d* and *a* are constants or functions of *t* and /or *x*.

We re-write Eq. (20) in the form

$$u_t + (u_{xx} - \frac{d}{a} e^{-au})_x = 0 \tag{21}$$

In this example

$$\Phi_x = u \tag{22}$$

$$\Phi_t = -(u_{xx} - \frac{d}{a} e^{-au}) \tag{23}$$

and

$$\begin{cases} g = -u \\ f = \dfrac{d}{a}e^{-au} \\ \dfrac{\partial \xi}{\partial u} = f_u u + f = -de^{-au}u + \dfrac{d}{a}e^{-au} = \dfrac{d}{a}e^{-au}(1-au) \end{cases} \tag{24}$$

The variational principle is

$$J(u,\Phi) = \iint \left\{ -u\Phi_t + \Phi_x \Phi_t + u\Phi_{xxx} - \frac{d}{a}e^{-au}\Phi_x - uu_{xx} + \xi \right\} dxdt \tag{25}$$

The variational principle is constrained by Eq. (22).

Proof.

The Euler-Lagrange equations are

$$u_t - 2\Phi_{xt} - u_{xxx} + \frac{d}{a}(e^{-au})_x = 0 \tag{26}$$

$$-\Phi_t + \Phi_{xxx} + de^{-au}\Phi_x - 2u_{xx} + \xi_u = 0 \tag{27}$$

Using the constraint, Eq. (22), we convert Eq. (26) in the form

$$-u_t - u_{xxx} - de^{-au}u_x = 0 \tag{28}$$

which is Eq. (21).

In view of Eq. (22) and Eq. (24), Eq.(27) becomes

$$-\Phi_t + u_{xx} + de^{-au}u - 2u_{xx} + \frac{d}{a}e^{-au}(1-au) = 0 \tag{29}$$

Simplifying Eq. (29) results in

$$-\Phi_t - u_{xx} + \frac{d}{a}e^{-au} = 0 \tag{30}$$

which is Eq. (23).

Example 2

$$u_t = \left(\frac{a}{u^n}\right)_{xxx} - \left(\frac{b}{u^n}\right)_x \tag{31}$$

where *a* and *b* are constants, or functions of *x* and/or *t*.

In this example

$$\begin{cases} g = \dfrac{a}{u^n} \\ f = -\dfrac{b}{u^n} \\ \dfrac{\partial \xi}{\partial u} = f_u u + f = \dfrac{b(n-1)}{u^n} \end{cases} \tag{32}$$

The variational principle reads

$$J(u,\Phi) = \iint \left\{ -u\Phi_t + \Phi_x\Phi_t - \frac{a}{u^n}\Phi_{xxx} + \frac{b}{u^n}\Phi_x - anu(\frac{u_x}{u^{n+1}})_x + \frac{b(n-1)}{u^n} \right\} dxdt \tag{33}$$

Example 3

$$u_t + u_{xxx} + \frac{u_x}{u} = 0 \tag{34}$$

We re-write Eq.(33) in the form

$$u_t + (u_{xx} + \ln u)_x = 0 \tag{35}$$

Its variational principle is

$$J(u,\Phi) = \iint \{ -u\Phi_t + \Phi_x\Phi_t + u\Phi_{xxx} + \ln u\Phi_x + uu_{xx} + 1 - \ln u \} dxdt \tag{36}$$

The variational principle is under constraint of $\Phi_x = u$.

Examples 2 and 3 pertain to singular terms. The variational principle for singular waves is a topic of considerable interest in soliton theory [16], as evidenced by the references cited in footnote 16. A MEMS system also exhibits a singular term, which has attracted attention from diverse fields, including mathematics, physics, and nanotechnology. Chun-Hui He established a variational formulation for fractal MEMS systems [17], which can be extended to other MEMS systems [18, 19].

4. Conclusion

This paper presents a general formula for constructing a variational form for a third-order differential equation as expressed in Eq. (1), which is not contingent on the equation being self-adjoint. The formula can be utilized for all continuous g and f in Eq. (1). The variational principle serves as a theoretical foundation for numerous numerical and analytical techniques. This paper presents a straightforward derivation of the requisite variational principle for practical applications.

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