Journal of Computational Applied Mechanics 2024, 55 (1): 1-7

DOI: 10.22059/JCAMECH.2024.372024.958

RESEARCH PAPER

Shock-like Waves with Finite Amplitudes

Guojuan Lv [a], Dan Tian [a] , Min Xiao [b], Chun-Hui He [c, *] , Ji-Huan He [a,d, †]

[a] *School of Science, Xi'an University of Architecture and Technology, Xi'an, P. R. China*
[b] *School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou, P. R. China*
[c] *School of Mathematics, China University of Mining and Technology, Xuzhou 221116, Jiangsu, P. R. China*
[d] *National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University,199 Ren-Ai Road, Suzhou, P. R. China*

Abstract

The tidal wave in the Qiantang River, Hangzhou City, China is quite different from that of KdV equation, it is a shock-like wave with a finite amplitude. This phenomenon has mathematicians adjusting their solitary wave models on how such waves behave. This paper applies the variational theory to insight into the energy behave of the tidal wave, which can be modelled by the Benny-Luke equation, and the exp-function method is used to figure out the solution structure. This paper provides a new window for designing energy harvesting devices from the shock-like waves.

Keywords: Semi-inverse method, variational principle, solitary wave, shock wave, singular wave;

1. Introduction

The tidal wave in the Qiantang River, Hangzhou City, China always attracts many sightseers, who might have been astonished by its fascinating phenomenon, that is the wave walks like a moving cobra ready for an attack, resulting in two obvious water surfaces with finite height, see Fig.1. The tidal bore propagation can be easily observed [1], but its properties were rarely studied.

Fig.1 The tidal bore propagation. The photos were taken with a camera near the Xiasha Bridge in Hangzhou, Zhejiang province in 2020.

The tidal wave is quite different from that arising in the well-known KdV equation [2-5], but the interaction of two waves follows the same soliton theory. The Qiantang River tidal is a shock-like wave with a finite amplitude,

* Corresponding author. *E-mail address:* mathew_he@yahoo.com (C.H. He)
† Corresponding author. *E-mail address:* hejihuan@suda.edu.cn (J.H He)

which can be modelled by the following Benny-Luke equation [6].

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0 \tag{1}$$

where u is a potential function, u_x is the height of the tidal, α and β are constants.

This equation was extensively studied to reveal the solution properties by various methods, e.g., the modified simple equation method [7], the modified exp-function method [8], and the G′/G-expansion method [9, 10]. Shock waves and other types of solitary waves were found in literature [11-13]. This paper will study its energy conservation by the semi-inverse method [14] and its shock-like wave with a finite amplitude by the exp-function method [15].

2. Energy conservation of the permanent wave of finite amplitude

A variational principle can elucidate the energy change during the tidal wave travelling, and it is also likely to suggest the solution's structure for analytical solutions [16] and boost the reliability of the variational-based numerical simulation [17]. Due to the merits of the variational theory, the variational formulation has been widely used to study various engineering problems, for examples, lubrication [18], boundary value problems [19], nanobeams [20], and singular waves [21]. It is a useful tool to numerical simulation [22-24]. The variational principle is also extremely useful to study the solution structure of various solitary waves, see for examples, Refs. [25-30].

By the semi-inverse method [14], a variational principle for Eq.(1) can be written in the form

$$J(u) = \iint \left\{ -\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + \frac{1}{2}\alpha(u_{xx})^2 - \frac{1}{2}\beta(u_{xt})^2 - \frac{1}{2}u_t(u_x)^2 \right\} dxdt \tag{2}$$

Its stationary condition is

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial L}{\partial u_{xt}}\right) = 0 \tag{3}$$

where the Lagrange function is expressed as

$$L = -\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + \frac{1}{2}\alpha(u_{xx})^2 - \frac{1}{2}\beta(u_{xt})^2 - \frac{1}{2}u_t(u_x)^2 \tag{4}$$

It is obvious that

$$\begin{aligned} \frac{\partial L}{\partial u_t} &= -u_t - \frac{1}{2}(u_x)^2 \\ \frac{\partial L}{\partial u_x} &= u_x - u_t u_x \\ \frac{\partial L}{\partial u_{xx}} &= \alpha u_{xx} \\ \frac{\partial L}{\partial u_{xt}} &= -\beta u_{xt} \end{aligned} \tag{5}$$

Eq.(4) becomes

$$-\frac{\partial}{\partial t}\left(-u_t - \frac{1}{2}(u_x)^2\right) - \frac{\partial}{\partial x}(u_x - u_t u_x) + \frac{\partial^2}{\partial x^2}(\alpha u_{xx}) + \frac{\partial^2}{\partial x \partial t}(-\beta u_{xt}) = 0 \tag{6}$$

After a simple calculation, Eq.(6) leads to Eq.(1).

3. Permanent waves of finite amplitude

In order to figure out the basic wave properties of the Benny-Luke equation, we use the following transform

$$\eta = x - ct \tag{7}$$

where c is the wave travelling velocity. The variational formulation becomes

$$\begin{aligned} J(u) &= \int \left\{ -\frac{1}{2}c^2(u_\eta)^2 + \frac{1}{2}(u_\eta)^2 + \frac{1}{2}\alpha(u_{\eta\eta})^2 - \frac{1}{2}\beta c^2(u_{\eta\eta})^2 + \frac{1}{2}(u_\eta)^3 \right\} d\eta \\ &= \int \left\{ -\frac{1}{2}(c^2-1)(u_\eta)^2 + \frac{1}{2}(\alpha-\beta c^2)(u_{\eta\eta})^2 + \frac{1}{2}(u_\eta)^3 \right\} d\eta \end{aligned} \tag{8}$$

The stationary condition of Eq.(8) is

$$(c^2-1)u'' + (\alpha - \beta c^2)u^{(4)} - 3cu'u'' = 0. \tag{9}$$

where the prime is the derivative with respect to η. Integrating Eq.(9) yields to the following equation

$$(c^2-1)u' + (\alpha - \beta c^2)u''' - \frac{3}{2}cu'^2 = 0 \tag{10}$$

In order to search for the solitary solutions, we use the exp-function method [15], and we assume the solution structure has the following forms:

$$u(\eta) = \frac{a_1 e^{\eta} + a_2 e^{-\eta} + a_0}{b_1 e^{\eta} + b_2 e^{-\eta} + b_0} \tag{11}$$

$$u(\eta) = \frac{a_1 e^{2\eta} + a_2 e^{-3\eta} + a_0}{b_1 e^{\eta} + b_2 e^{-\eta} + b_0} \tag{12}$$

$$u(\eta) = \frac{a_1 e^{\eta} + a_2 e^{-\eta} + a_0}{b_1 e^{2\eta} + b_2 e^{-3\eta} + b_0} \tag{13}$$

where $a_i (i = 0,1,2)$ and $b_i (i = 0,1,2)$ are unknown constants to be further solved. The exp-function method is useful tool in the soliton theory to find various solitary wave solutions, see for examples, Refs. [31-34].

Following the standard solving process of the exp-function method [15], we obtain the following multiple solutions

$$u_1(\eta) = \frac{a_0}{b_2 e^{-\eta} + \dfrac{a_0(\beta-1)\sqrt{\alpha-1}}{4(\alpha-\beta)\sqrt{\beta-1}}}, \eta = x - \sqrt{\frac{\alpha-1}{\beta-1}}t \quad \frac{\partial L}{\partial u_{xt}} = -\beta u_{xt} \tag{14}$$

$$u_2(\eta) = \frac{4b_0(\alpha-\beta)\sqrt{\alpha-1}}{(\alpha-1)(b_2 e^{-\eta} + b_0)\sqrt{\beta-1}}, \eta = x - \sqrt{\frac{\alpha-1}{\beta-1}}t \quad \frac{\partial L}{\partial u_{xt}} = -\beta u_{xt} \tag{15}$$

$$u_3(\eta) = \frac{a_0}{b_1 e^{\eta} - \dfrac{a_0(\beta-1)\sqrt{\alpha-1}}{4(\alpha-\beta)\sqrt{\beta-1}}}, \eta = x - \sqrt{\frac{\alpha-1}{\beta-1}}t \quad \frac{\partial L}{\partial u_{xt}} = -\beta u_{xt} \tag{16}$$

$$u_4(\eta)=\frac{12b_0(\alpha-\beta)}{(4\beta-1)(b_1e^{2\eta}+b_0)},\eta=x-\sqrt{\frac{4\alpha+1}{4\beta-1}}t\ \ \frac{\partial L}{\partial u_{xt}}=-\beta u_{xt} \tag{17}$$

$$u_5(\eta)=\frac{-18b_0(\alpha-\beta)}{(9\beta-1)(b_1e^{2\eta}+b_0)},\eta=x-\sqrt{\frac{9\alpha+1}{9\beta-1}}t\ \ \frac{\partial L}{\partial u_{xt}}=-\beta u_{xt} \tag{18}$$

The solutions are illustrated in Fig.2-6 respectively. Fig.2 and Fig.3 have the similar property except the amplitude. Fig.4 shows singularities and the surface height can be adjusted through the singularities. Fig. 5 and Fig.6 show a sharper change in the surface height than those in Fig.2 and Fig.3.

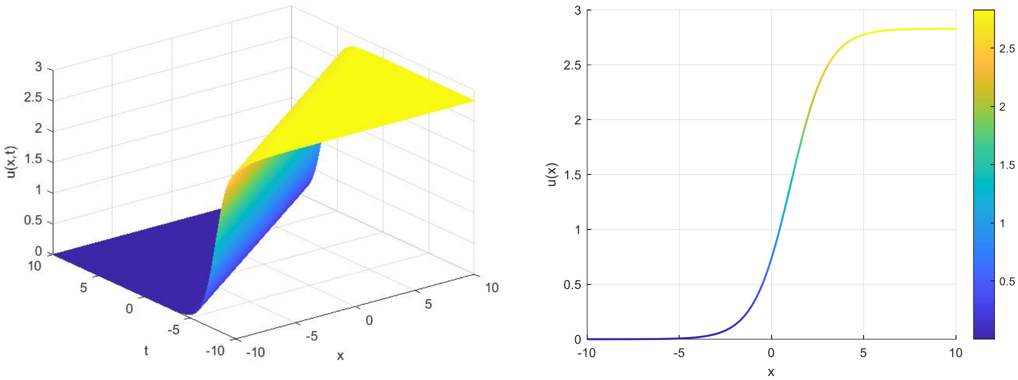

Fig.2 Solution Eq.(14) when $a_0=1,\alpha=3,\beta=2,b_2=1$.

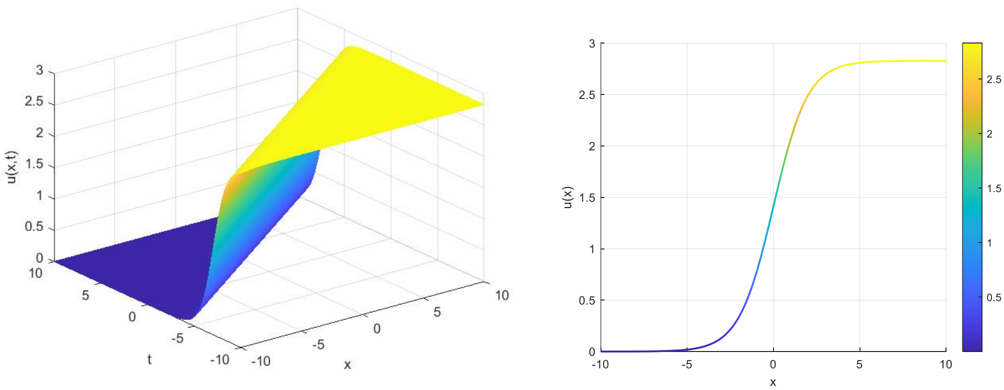

Fig.3 Solution Eq.(15) when $b_0=1,b_2=1,\alpha=3,\beta=2$.

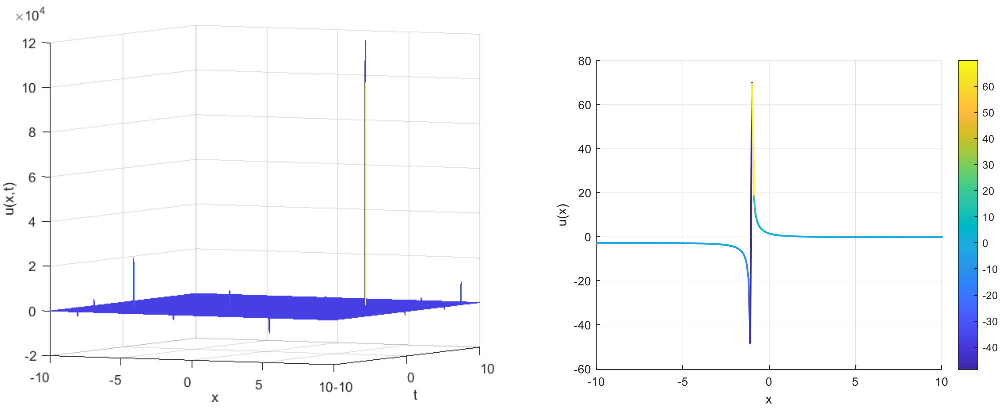

Fig.4 Solution Eq.(16) when $a_0=1,\alpha=3,\beta=2,b_1=1$.

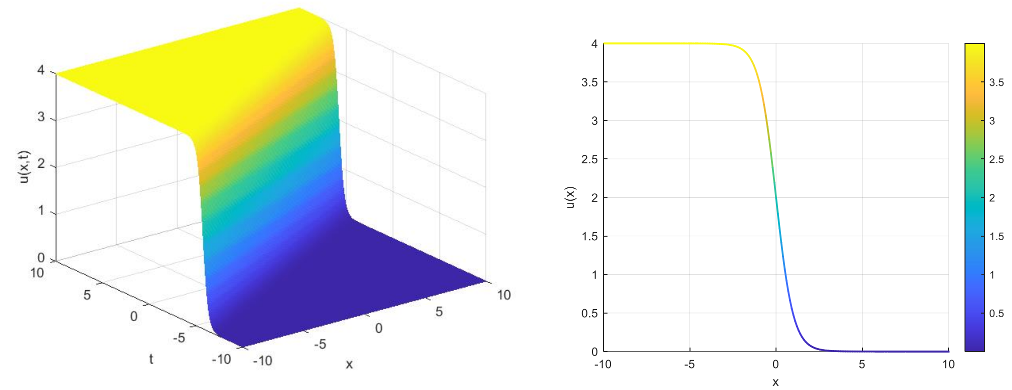

Fig.5 Solution Eq.(17) $\alpha = 2, \beta = 1, b_0 = 1, b_1 = 1$

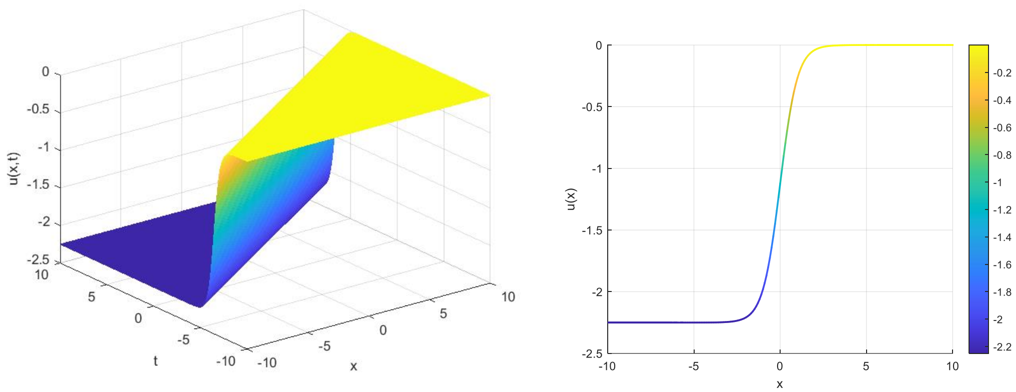

Fig. 6 Solution Eq.(18) when $\alpha = 2, \beta = 1, b_0 = 1, b_1 = 1$.

4. Discussion and conclusions

The shock-like waves with finite amplitudes can be used for energy harvesting [35, 36], the shock height and the velocity are clearly given in the solutions of Eqs.(14)-(18). The results in this paper speeds up optimization of the energy harvesting devices from shock waves and boosting the operation reliability.

This paper studies the shock-like wave by the variational principle and the exp-function method, the results shows that the wave is similar to the shock in aerodynamics, and the singularities are also found in the travelling process. The surface height through a singularity can be greatly affected, this is because that much energy is accumulated on the singularities, so that the height between two water surface becomes much less than those without singularities.

Funding

This work was supported by the Natural Science Foundation of Shaan Xi Province(No.2023-JC-QN-0016)

References

[1] Y. Li, D.-Z. Pan, H. Chanson, C.-H. Pan, Real-time characteristics of tidal bore propagation in the Qiantang River Estuary, China, recorded by marine radar, *Continental Shelf Research,* Vol. 180, pp. 48-58, 2019.

[2] H. Wu, J. Song, Q. Zhu, Consistent Riccati expansion solvability and soliton–cnoidal wave solutions of a coupled KdV system, *Applied Mathematics Letters,* Vol. 135, pp. 108439, 2023.

[3] S. Deng, Z. Deng, Approximate analytical solutions of generalized fractional Korteweg-de Vries equation, *Thermal Science,* Vol. 27, No. 3 Part A, pp. 1873-1879, 2023.

[4] J. Cui, D. Li, T.-F. Zhang, Symmetry reduction and exact solutions of the (3+1)-dimensional nKdV-nCBS equation, *Applied Mathematics Letters,* Vol. 144, pp. 108718, 05/01, 2023.

[5] R. I. Ivanov, On the modelling of short and intermediate water waves, *Applied Mathematics Letters,* Vol. 142, pp. 108653, 2023/08/01/, 2023.

[6] D. Benney, J. Luke, On the interactions of permanent waves of finite amplitude, *Journal of Mathematics and Physics,* Vol. 43, No. 1-4, pp. 309-313, 1964.

[7] J. Akter, M. Ali Akbar, Exact solutions to the Benney–Luke equation and the Phi-4 equations by using modified simple equation method, *Results in Physics,* Vol. 5, pp. 125-130, 2015/01/01/, 2015.

[8] S. M. R. Islam, Applications of the exp(-Φ(ξ))-Expansion Method to Find Exact Traveling Wave Solutions of the Benney-Luke Equation in Mathematical Physics, *American journal of Applied Mathematics,* Vol. 3, pp. 100-105, 04/29, 2015.

[9] A. K. M. K. S. Hossain, P. M. A. Akbar, Traveling wave solutions of Benny Luke equation via the enhanced (G ' / G)-expansion method, *Ain Shams Engineering Journal,* Vol. 12, 05/01, 2021.

[10] U. Khan, R. Ellahi, R. A. Khan, S. T. Mohyud-Din, Extracting new solitary wave solutions of Benny–Luke equation and Phi-4 equation of fractional order by using (G′/G)-expansion method, *Optical and Quantum Electronics,* Vol. 49, pp. 1-14, 2017.

[11] T. Houria, A. Yildirim, T. Hayat, O. Aldossary, A. Biswas, Shock wave solution of Benney-Luke equation, *Romanian Reports of Physics,* Vol. 57, 01/01, 2012.

[12] B. Ghanbari, M. Inc, A. Yusuf, D. Baleanu, New solitary wave solutions and stability analysis of the Benney-Luke and the Phi-4 equations in mathematical physics, *AIMS Mathematics,* Vol. 4, pp. 1523-1539, 09/01, 2019.

[13] M. Ablowitz, C. Curtis, On the evolution of perturbations to solutions of the Kadomtsev–Petviashvilli equation using the Benney–Luke equation, *Journal of Physics A: Mathematical and Theoretical,* Vol. 44, pp. 195202, 04/12, 2011.

[14] J.-H. He, Variational principles for some nonlinear partial differential equations with variable coefficients, *Chaos, Solitons & Fractals,* Vol. 19, pp. 847-851, 03/01, 2004.

[15] J.-H. He, X.-H. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons & Fractals,* Vol. 30, pp. 700-708, 11/01, 2006.

[16] H. Ma, SIMPLIFIED HAMILTONIAN-BASED FREQUENCY-AMPLITUDE FORMULATION FOR NONLINEAR VIBRATION SYSTEMS, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, pp. 445, 07/28, 2022.

[17] X. Li, D. Wang, T. Saeed, Multi-scale numerical approach to the polymer filling process in the weld line region, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, No. 2, pp. 363-380, 2022.

[18] H. Ma, Variational principle for a generalized Rabinowitsch lubrication, *Thermal Science,* Vol. 27, pp. 71-71, 01/01, 2022.

[19] S.-Q. Wang, A variational approach to nonlinear two-point boundary value problems, *Computers & Mathematics with Applications,* Vol. 58, No. 11-12, pp. 2452-2455, 2009.

[20] S. A. Faghidian, A. Tounsi, DYNAMIC CHARACTERISTICS OF MIXTURE UNIFIED GRADIENT ELASTIC NANOBEAMS, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 20, pp. 539, 11/30, 2022.

[21] C.-H. He, C. Liu, Variational principle for singular waves, *Chaos, Solitons & Fractals,* Vol. 172, pp. 113566, 07/01, 2023.

[22] C. Miehe, S. Mauthe, H. Ulmer, Formulation and numerical exploitation of mixed variational principles for coupled problems of Cahn–Hilliard-type and standard diffusion in elastic solids, *International Journal for Numerical Methods in Engineering,* Vol. 99, 09/07, 2014.

[23] P.-H. Kuo, T.-L. Tu, Y.-W. Chen, W.-Y. Jywe, H.-T. Yau, Thermal displacement prediction model with a structural optimized transfer learning technique, *Case Studies in Thermal Engineering,* Vol. 49, pp. 103323, 09/01, 2023.

[24] P.-H. Kuo, Y.-W. Chen, T. H. Hsieh, W. Jywe, H.-T. Yau, A Thermal Displacement Prediction System With an Automatic LRGTVAC-PSO Optimized Branch Structured Bidirectional GRU Neural Network, *IEEE Sensors Journal,* Vol. 23, pp. 12574-12586, 2023.

[25] A. Biswas, D. Milovic, D. S. Kumar, A. Yildirim, Perturbation of shallow water waves by semi-inverse variational principle, *Indian Journal of Physics,* Vol. 87, 06/01, 2013.

[26] J. Lu, L. Ma, Numerical analysis of space-time fractional Benjamin-Bona-Mahony equation, *Thermal Science,* Vol. 27, pp. 1755-1762, 01/01, 2023.

[27] Y. Wu, J.-H. He, Variational principle for the Kaup-Newell system, *Journal of Computational Applied Mechanics,* Vol. 54, No. 3, pp. 405-409, 2023.

[28] X.-Q. Cao, B.-N. Liu, M.-Z. Liu, K.-C. Peng, W.-L. Tian, Variational principles for two kinds of non-linear geophysical KdV equation with fractal derivatives, *Thermal Science,* Vol. 26, No. 3 Part B, pp. 2505-2515, 2022.

[29] Q. Ma, R. Yuan, C. Wang, Variational method to fractal long-wave model with variable coefficients, *Thermal Science,* Vol. 27, No. 3 Part A, pp. 1779-1786, 2023.

[30] J. Sun, Variational principle for fractal high-order long water-wave equation, *Thermal Science,* Vol. 27, No. 3 Part A, pp. 1899-1905, 2023.

[31] A. Bekir, Ö. Güner, A. Bhrawy, A. Biswas, Solving nonlinear fractional differential equations using exp-function and (G/G′) -expansion methods, *Romanian Journal of Physics,* Vol. 60, 01/01, 2015.

[32] M. Dehghan, J. Manafian, A. Saadatmandi, Application of the Exp-function method for solving a partial differential equation arising in biology and population genetics, *International Journal of Numerical Methods for Heat & Fluid Flow,* Vol. 21, pp. 736-753, 08/09, 2011.

[33] A. Biswas, M. Ekici, A. Sonmezoglu, M. Belić, Highly dispersive optical solitons with cubic–quintic–septic law by extended Jacobi's elliptic function expansion, *Optik,* Vol. 183, 02/01, 2019.

[34] S. T. Mohyud-Din, Y. Khan, N. Faraz, A. Yıldırım, Exp-function method for solitary and periodic solutions of Fitzhugh-Nagumo equation, *International Journal of Numerical Methods for Heat & Fluid Flow,* Vol. 22, No. 3, pp. 335-341, 2012.

[35] Z. Hadas, V. Vetiska, V. Singule, O. Andrs, J. Kovar, J. Vetiska, Energy Harvesting from Mechanical Shocks Using A Sensitive Vibration Energy Harvester Regular Paper, *International Journal of Advanced Robotic Systems,* Vol. 9, pp. 1, 05/15, 2017.

[36] M.-U. Noll, L. Lentz, U. von Wagner, On the discretization of a bistable cantilever beam with application to energy harvesting, *Facta Universitatis, Series: Mechanical Engineering,* Vol. 17, No. 2, pp. 125-139, 2019.