Journal of Computational Applied Mechanics 2023, 54(4): 501-514

DOI: 10.22059/JCAMECH.2023.366509.890

RESEARCH PAPER

Symmetrical Mechanical System Properties-Based Forced Vibration Analysis

Maria Luminita Scutaru [a], Sorin Vlase [a,b], Marin Marin [c,d,*]

[a] *Department of Mechanical Engineering, Transilvania University of Brasov, B-dul Eroilor, 29, 500036, Romania.*
[b] *Technical Sciences Academy of Romania, Bucharest.*
[c] *Department of Mathematics and Computer Science, Transilvania University of Brasov, B-dul Eroilor 29, 500036 Brasov, Romania.*
[d] *Academy of Romanian Scientists, Ilfov Street, No. 3, 050045 Bucharest, Romania.*

Abstract

Mechanical systems with structural symmetries present vibration properties that allow the calculation to be easier and the analysis time to decrease. The paper aims to use the properties involved by the symmetries that exist in mechanical systems for the analysis of the forced response to vibrations. Thus, the study of the properties of systems with symmetries or with identical parts is expanded. Based on a classic model, the characteristic properties that appear in this case are obtained and the advantages of using these properties are revealed. On an example consisting of a truck equipped with two identical engines, the way of applying these properties in the calculation and the resulting advantages is presented.

Keywords: symmetrical system; forced vibration; finite element method; eigenmode

1. Introduction

In many applications in the field of engineering, we encounter machines, devices, components that present different types of constructive symmetries. There are strong reasons why symmetries or repetitive parts are used in different mechanical systems or constructions, reasons related to ease of design, manufacturing or low logistics and maintenance costs. Last but not least, especially in the field of civil engineering, we also encounter aesthetic reasons. Symmetries have been observed for a long time by researchers and the properties they have been used especially in the static case. In the works [1, 2], different types of symmetries are presented and analyzed. A systematic study of the use of symmetries in mechanics was made by mathematicians [3-5], since they can be used beneficially in writing the equations of motion of some mechanical systems, but the results obtained are still of little interest to engineers. However, some engineering applications are studied in [6-9]. Symmetry magazine dedicated a special issue to this chapter in 2018 [10] and a project with this theme was financed within the FP5 European project competition [11]. The Center for Solid Mechanics in Udine organized special courses on this topic [12] (similarity, symmetry and group theoretical methods in mechanics). We mention that recently some interesting works have been published in the field [13] which present different ways of using the senses in order to facilitate the design and dynamic analysis of such systems. For the field of vibration study, there is a poorer literature, but we can still mention some works that present the way in which these properties can be used [14-21]. Observations in the sense of facilitating finite element method (FEM) modeling in case the studied structures present certain symmetries are made in [22]. Group theory is a favorite method for studying systems with symmetries to identify the highly structured modal properties of cyclic systems [23]. The determination of eigenvalues and eigenmodes is done by calculating them for the subsystems that define the symmetries. Obviously this is computationally efficient.

* Corresponding author. *E-mail address:* m.marin@unitbv.ro

Symmetry groups are frequently used in such analyses [24, 25]. In [26] it is shown that the vibration modes of structures with symmetries are either symmetric or skew-symmetric, useful in the vibration analysis. A numerical example using FEM demonstrates the mentioned properties. Usually, simulations are expensive and different scenarios are analyzed in the design stage to determine the optimal solution from an engineering point of view. In [27] the advantages that symmetries can bring in the process of numerical simulation of some engineering projects are analyzed. A physical example to illustrate these advantages is presented in detail in the mentioned paper.

The identical parts that constitutes the structure are identical if all the parameters that define it are the same. The geometric shape, physical properties, boundary conditions and liaisons must be identical for identical subassemblies. The most used method for studying the vibrations of such systems is the FEM, which however requires a considerable effort for modeling and discretization. The symmetry properties of a structure make the information necessary for the description of the structure less. Similarly, the stages of data pre-processing will thus be reduced and the costs will be lower. A number of properties of systems with repetitive elements have been studied with FEM in [28-33]. For continuous systems, the analysis of wave propagation in linear periodic systems was done in [34-38]. In [39, 40], some properties of discrete mechanical systems with repetitive elements are presented. Meirovitch [41]signaled, allusively, the existence of symmetries in vibrating mechanical systems and the possibility of using these symmetries to simplify the calculation. A first systematic approach to the problem can be found in [36-38] and further development is presented in [42]. New research in the field is presented in [43-54]. For problems with complex symmetries, a systematic study is not yet done. More results in domain were obtained by the researcher in the last decade [55-59].

The symmetries can lead to properties of the systems that allow the ease of dynamic analysis and the study of vibrations in these cases. The systems are characterized by the fact that they have identical geometry, properties, boundary conditions and liaisons with the main structure. They will present properties that allow the calculation and the necessary time to be eased and, as a consequence, the reduction of design time and costs.

A field in which the existence of symmetries can lead to a significant decrease in the cost price is civil engineering. In this field, the existence of symmetries is also dictated by aesthetic considerations. In all engineering fields, such as the automotive industry, aerospace, equipment and machine manufacturing, there are products, elements or components with identical, repetitive parts or with different types of symmetries.

In the paper we show that the existence of symmetries can lead to the disappearance of some terms of the forced vibration response of a mechanical system. This could cause the forced vibration amplitudes to decrease in these systems.

2. Materials and Methods

Let's consider a mechanical system with elastic elements that can be considered composed of two identical systems (S_1) linked symmetrically to a third one called (S_2).

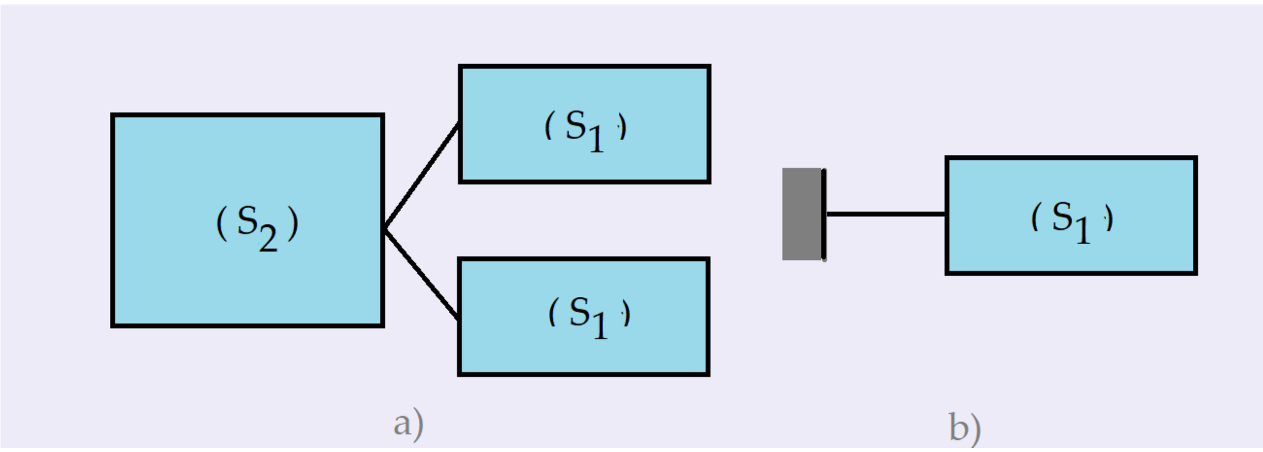


Figure 1. A mechanical system containing two identical subsystems

If we denote by m_1 the inertial matrix for the system (S_1), m_2 the inertial matrix for the system (S_2), k_1 the stiffness matrix of (S_1), k_2 the stiffness matrix of (S_2), with f_1 and f_2 the excitations that act on the two systems, with k_{12} the stiffness matrix of the elastic liaison between the system (S_1) and (S_2), in the absence of damping, the equations of motion of the system are [60, 61]:

$$\begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}_1\} \\ \{\ddot{x}'_1\} \\ \{\ddot{x}_2\} \end{Bmatrix} + \begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} \begin{Bmatrix} \{x_1\} \\ \{x'_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} \{f_1\} \\ \{f_1\} \\ \{f_2\} \end{Bmatrix} \quad (1)$$

In the following it will be used $[X]$ for square matrices and $\{X\}$ for vectors. If we consider only the free vibration of the system Eq. (1) becomes:

$$\begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}_1\} \\ \{\ddot{x}'_1\} \\ \{\ddot{x}_2\} \end{Bmatrix} + \begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} \begin{Bmatrix} \{x_1\} \\ \{x'_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (2)$$

If a harmonic solution of the form:

$$\begin{Bmatrix} \{x_1\} \\ \{x'_1\} \\ \{x_2\} \end{Bmatrix} = C \begin{Bmatrix} \{A_1\} \\ \{A'_1\} \\ \{A_2\} \end{Bmatrix} \cos(\omega t - \varphi). \quad (3)$$

it is chosen for system (2) and the condition that Eq. (3) verifies the system of differential equations (2), the following condition is obtained:

$$\begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} \begin{Bmatrix} \{A_1\} \\ \{A'_1\} \\ \{A_2\} \end{Bmatrix} = \omega^2 \begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \begin{Bmatrix} \{A_1\} \\ \{A'_1\} \\ \{A_2\} \end{Bmatrix} \quad (4)$$

which represents a linear homogeneous system. To have non-trivial solutions it is necessary that:

$$\det\left(\begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} - \omega^2 \begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \right) = 0 \quad (5)$$

or:

$$\left\| \begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} - \omega^2 \begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \right\| = 0 \quad (6)$$

The Eq.(4) is the characteristic equations of the system (1) and the solutions ω represent the undamped eigenfrequencies. If the number of degrees of freedom (DOF) is *n*, there exist *n* eigenfrequencies.

For a real mechanical system studied in engineering the matrix $[m]$ and $[k]$ are real, positively definited, so the eigenvalues are real and positive. If exists rigid body modes, the corresponding eigenvalues are zero.

From relation (3), it results that for every eigenfrequency ω there is a eigenvector that satify the equation:

$$\left(\begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} - \omega^2 \begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} \right) \begin{Bmatrix} \{A_1\} \\ \{A'_1\} \\ \{A_2\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (7)$$

that represents the modal vector. For a system having *n* DOF there are *n* modal vectors. Considering two different modal vectors, indexed with r and s $(r \neq s)$ it exists the orthogonality of this two vectors through $[m]$ and $[k]$ [55]:

$$\begin{bmatrix}\{A_1\}^T & \{A'_1\}^T & \{A_2\}^T\end{bmatrix}_r \begin{bmatrix}[k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2]\end{bmatrix} \begin{Bmatrix}\{A_1\} \\ \{A'_1\} \\ \{A_2\}\end{Bmatrix}_s = 0 \tag{8}$$

$$\begin{bmatrix}\{A_1\}^T & \{A'_1\}^T & \{A_2\}^T\end{bmatrix}_r \begin{bmatrix}[m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2]\end{bmatrix} \begin{Bmatrix}\{A_1\} \\ \{A'_1\} \\ \{A_2\}\end{Bmatrix}_s = 0 \tag{9}$$

If it is considered a mass normalization of the eigenmodes, so that if r=s,

$$\begin{bmatrix}\{A_1\}^T & \{A'_1\}^T & \{A_2\}^T\end{bmatrix}_r \begin{bmatrix}[m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2]\end{bmatrix} \begin{Bmatrix}\{A_1\} \\ \{A'_1\} \\ \{A_2\}\end{Bmatrix}_r = 1 \tag{10}$$

Conditions (5) and (7) can be written together as:

$$\begin{bmatrix}\{A_1\}^T & \{A'_1\}^T & \{A_2\}^T\end{bmatrix}_r \begin{bmatrix}[m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2]\end{bmatrix} \begin{Bmatrix}\{A_1\} \\ \{A'_1\} \\ \{A_2\}\end{Bmatrix}_s = \delta_{rs} \quad , \quad r,s = \overline{1,n} \tag{11}$$

where δ_{rs} is the Kronecker delta.

In the following, we will note the normalized modes with:

$$\{\Phi\}_i = \begin{Bmatrix}\{\Phi_1\} \\ \{\Phi'_1\} \\ \{\Phi_2\}\end{Bmatrix}_i = \begin{Bmatrix}\{A_1\} \\ \{A'_1\} \\ \{A_2\}\end{Bmatrix}_i \quad , i = \overline{1,n} \tag{12}$$

The modal analysis uses the linear transformation:

$$\{x\} = \begin{bmatrix}\Phi_1 & \Phi_2 & \dots & \Phi_n\end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} = \sum_{i=1}^{n} \{\Phi_i\} q_i \tag{13}$$

where:

$$[\Phi] = \begin{bmatrix}\Phi_1 & \Phi_2 & \dots & \Phi_n\end{bmatrix} \tag{14}$$

is the modal matrix. If it is denoted with:

$$[M] = [\Phi]^T [m][\Phi] \tag{15}$$

the modal mass matix, with:

$$[K] = [\Phi]^T [k][\Phi] \tag{16}$$

the modal stiffness and:

$$[F] = [\Phi]^T [f] \tag{17}$$

the modal force vector,. The system (1) becomes:

$$[M]\{\ddot{q}\}+[K]\{q\}=\{F\}. \tag{18}$$

If only the system (S_1) is considered (see Fig.1.b), the motion equations are:

$$[M]\{\ddot{q}\}+[K]\{q\}=\{F\}. \tag{19}$$

The free vibrations of the sstem are described by the system of differential equations:

$$[m_1]\{\ddot{x}_1\}+[k_1]\{x_1\}=0 \tag{20}$$

The characteristic equation becomes:

$$\det([k_1]-\omega^2[m_1])=0 \tag{21}$$

or:

$$\left|[k_1]-\omega^2[m_1]\right|=0\ . \tag{22}$$

In previous works, the authors demonstrated the following properties that mechanical systems with symmetries have and that can be used to analyze the free vibrations [14, 15, 17, 44, 62].

P1 – the eigenvalues for the subsystem (S_l) (see Eq.(20)) are between (included in) the eigenvalues for the system (S) (Eq.(2)).

P2 **-** for the common eigenvalues of the system presented in Figure 1, a) and b) the eigenvectors have of the form:

$$\{\Phi\}=\begin{Bmatrix}\Phi_1\\-\Phi_1\\0\end{Bmatrix} \tag{23}$$

(the components of the eigenmodes, corresponding to the two identical parts are skew symmetric, the other components are zero **-** we call these skewsymmetric eigenmodes).

P3 **-** For the other eigenvalues, the eigenvectors have the form:

$$\{\Phi\}=\begin{Bmatrix}\Phi_1\\\Phi_1\\\Phi_2\end{Bmatrix} \tag{23}$$

(the components of the eigenmodes corresponding to the two identical beams are the same**-** we call this symmetric eigenmodes).

In Eq.(18) the matrix M and K are diagonal, so the system can be written:

$$M_i\ddot{q}_i+K_iq_i=F_i\quad,\ i=\overline{1,n} \tag{24}$$

or:

$$\ddot{q}_i+\omega_i^2q_i=\frac{F_i}{M_i}\quad,\ i=\overline{1,n}. \tag{25}$$

For a system (S) subject to external excitation we have the following property:

P4. For the eigenfrequebncy of the system (S1) (common with the eigenfrequencies of the entire system (S), the term F_i/M_i is zero and the solution for the Eq.(33) corresponding to the common eigenfrequencies are the solution for the homogeneous system.

In the situation of the property **P2**, the eigenmode is: $\{\Phi\}=[\Phi_1\quad-\Phi_1\quad 0]$ and the Eq.(17) becomes:

$$\{F\} = [\Phi]^T \{f\} = [\Phi_1 \quad -\Phi_1 \quad 0] \begin{Bmatrix} \{f_1\} \\ \{f_1\} \\ \{f_2\} \end{Bmatrix} = 0 \quad . \tag{26}$$

So, the Eq.(25) becomes:

$$\ddot{q}_i + \omega_i^2 q_i = 0 \quad , \ i = \overline{1, n_1} . \tag{27}$$

$$\ddot{q}_i + \omega_i^2 q_i = \frac{F_i}{M_i} \quad , \ i = \overline{n_1 + 1, n} . \tag{28}$$

Here n_1 is the number of the DOF of the system (S_1) and n is the dimension of the entire system. There exists the relation $2n_1 + n_2 = n$ where n_2 represents the number of DOF of the system (S_2).

The solution of the forced vibration is:

$$\ddot{q}_i + \omega_i^2 q_i = \frac{F_i}{M_i} \quad , \ i = \overline{n_1 + 1, n} . \tag{29}$$

Discussion: It is observed that a number of n_1 solutions of the forced vibrations are missing. This suggests that the general solution, having fewer excitation terms, could have a smaller total response amplitude. Obviously, a study and an argumentation of this observation can only be done on a concrete system, due to the large number of parameters that must be taken into account. The example from the paper shows us that this observation can be useful and can encourage the use of systems with repetitive parts in the design activity. We also have the following property:

P5. The amplitudes of the vibrations of the forced particular solutions have, for each harmonic, the form:

$$\{x_p\} = \begin{Bmatrix} x_{1p} \\ x_{1p} \\ x_{2p} \end{Bmatrix} \tag{30}$$

(the excitation symmetry is kept in the response's symmetry).

Proof:

Considering a harmonic p where the excitation is $\{f_p\}$, the amplitude of the forced vibrations is given by:

$$([K] - p^2\omega^2[M])\{x_p\} = \{F_p\}$$

or:

$$\begin{bmatrix} [k_1] - p^2\omega^2[m_1] & 0 & [k_{12}] \\ 0 & [k_1] - p^2\omega^2[m_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] - p^2\omega^2[m_2] \end{bmatrix} \begin{Bmatrix} x_{1p} \\ x'_{1p} \\ x_{2p} \end{Bmatrix} = \begin{Bmatrix} f_{1p} \\ f_{1p} \\ f_{2p} \end{Bmatrix} \tag{31}$$

or, in other words:

$$[[k_1] - p^2\omega^2[m_1]]\{x_{1p}\} + [k_{12}]\{x_{2p}\} = \{f_{1p}\} \ ; \tag{32}$$

$$[[k_1] - p^2\omega^2[m_1]]\{x'_{1p}\} + [k_{12}]\{x_{2p}\} = \{f_{1p}\} \ ; \tag{33}$$

$$[k_{12}]^T\{x_{1p}\} + [k_{12}]^T\{x_{1p}\} + ([k_2] - p^2\omega^2[m_2])\{x_{2p}\} = \{f_{2p}\} \quad . \tag{34}$$

It is obvious that (32) and (33) have identical solution ($\{x_{1p}\}=\{x'_{1p}\}$.□

3. Results

The previously obtained properties are verified in the case of a truck, equipped with two identical engines. When moving the truck, one of the engines is used, then the equipment transported by the truck is used for drilling, where the power of both engines is used. This mobile drilling rig is built on the chassis of a truck that has the role of transporting the rig to the work site. The requirements imposed on these types of vehicles are complex, being subjected to high static and dynamic loads, with shocks and vibrations in operation, both in transport and in service. The propulsion system is ensured by two motors and the moment from the two motors is collected by means of a adding and distribution box. Usually, during transport, only one engine is used, and during the operation of the drilling rig, both engines are used. This ensures the necessary power for the drilling rig, as well as a stable service with high durability/reliability indices.

The truck model with two identical engines is presented in Figure 2.

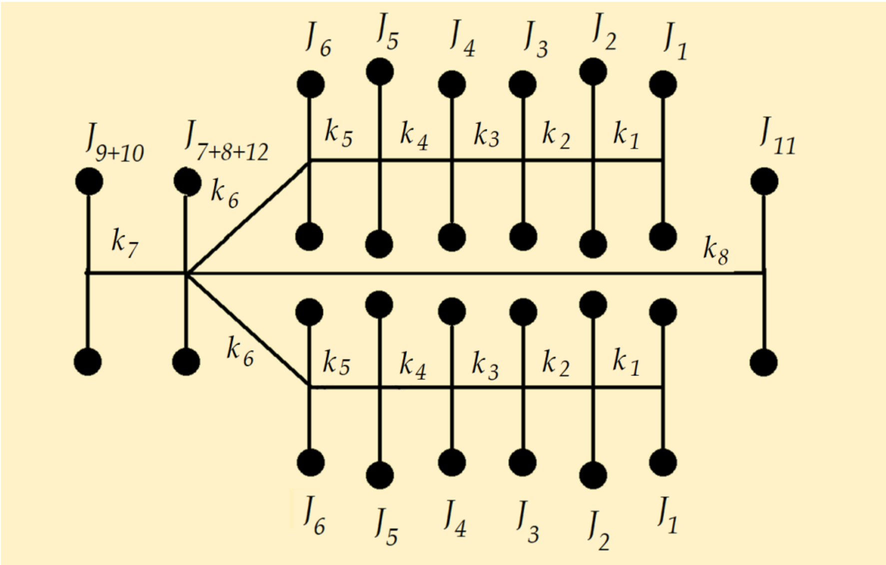


Figure 2: The truck model with two identical engines

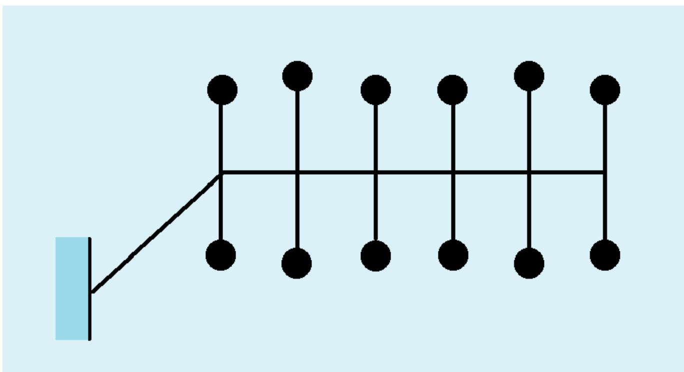


Figure 3: The model for the study of one single engine

For the entire system, the forced non-damped vibrations shall be described by the system of equations:

$$\begin{bmatrix} [J]_1 & 0 & 0 \\ 0 & [J]_1 & 0 \\ 0 & 0 & [J]_2 \end{bmatrix} \begin{Bmatrix} \{\ddot{\varphi}\}_1 \\ \{\ddot{\varphi}'\}_1 \\ \{\ddot{\varphi}\}_{23} \end{Bmatrix} + \begin{bmatrix} [K]_1 & 0 & [K]_{12} \\ 0 & [K]_1 & [K]_{12} \\ [K]_{12}^T & [K]_{12}^T & [K]_2 \end{bmatrix} \begin{Bmatrix} \{\varphi\}_1 \\ \{\varphi'\}_1 \\ \{\varphi\}_{23} \end{Bmatrix} = \begin{Bmatrix} \{f\}_1 \\ \{f\}_1 \\ \{f\}_2 \end{Bmatrix} \tag{35}$$

The free non-damped vibration equations are:

$$\begin{bmatrix} [J]_1 & 0 & 0 \\ 0 & [J]_1 & 0 \\ 0 & 0 & [J]_2 \end{bmatrix} \begin{Bmatrix} \{\ddot{\varphi}\}_1 \\ \{\ddot{\varphi}'\}_1 \\ \{\ddot{\varphi}\}_{23} \end{Bmatrix} + \begin{bmatrix} [K]_1 & 0 & [K]_{12} \\ 0 & [K]_1 & [K]_{12} \\ [K]_{12}^T & [K]_{12}^T & [K]_2 \end{bmatrix} \begin{Bmatrix} \{\varphi\}_1 \\ \{\varphi'\}_1 \\ \{\varphi\}_{23} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{36}$$

where:

$$[J]_1 = \begin{bmatrix} J_1 & & & & & \\ & J_2 & & & 0 & \\ & & J_3 & & & \\ & & & J_4 & & \\ & 0 & & & J_5 & \\ & & & & & J_6 \end{bmatrix} \; ; \; [J]_1 = \begin{bmatrix} J_9 + J_{10} & 0 & 0 \\ 0 & J_{11} & 0 \\ 0 & 0 & J_7 + J_8 + J_{12} \end{bmatrix} \tag{37}$$

$$[K]_1 = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 & 0 \\ 0 & 0 & -k_3 & k_3 + k_4 & -k_4 & 0 \\ 0 & 0 & 0 & -k_4 & k_4 + k_5 & -k_5 \\ 0 & 0 & 0 & 0 & -k_5 & k_6 \end{bmatrix} \; ; \tag{38}$$

$$[K]_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_6 \end{bmatrix} \; ; \; [K]_2 = \begin{bmatrix} k_7 & 0 & -k_7 \\ 0 & k_8 & -k_8 \\ -k_7 & -k_8 & k_7 + k_8 + 2k_6 \end{bmatrix} \; ; \tag{39}$$

$$\{\varphi\}_1 = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \end{Bmatrix} \; ; \; \{\varphi'\}_1 = \begin{Bmatrix} \varphi'_1 \\ \varphi'_2 \\ \varphi'_3 \\ \varphi'_4 \\ \varphi'_5 \\ \varphi'_6 \end{Bmatrix} \; ; \; \{f\}_1 = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \; ; \; \{f\}_2 = \begin{Bmatrix} f_{9+10} \\ f_{11} \\ f_{7+8+12} \end{Bmatrix} \; . \tag{40}$$

The inertial and stiffness parameters of the system are presented in Tables 1 and 2.

Table 1: The inertia properties of the system

No.	Moment of inertia	Detail	Values [kgm^2]
1,7	J_1	Cylinder 1	0.1048
2,8	J_2	Cylinder 2	0.0638
3,9	J_3	Cylinder 3	0.1048
4,10	J_4	Cylinder 4	0.1048

5,11	J_5	Cylinder 5	0.0638
6,12	J_6	Cylinder 6	0.1048
13,14,18	$J_{7}+ J_{8}+J_{12}$	Gears and central gear	1.4157
15,16	$J_9+ J_{10}$	Flywheel and ventilator	2.9841
17	J_{11}	Exit steering wheel	1.3382

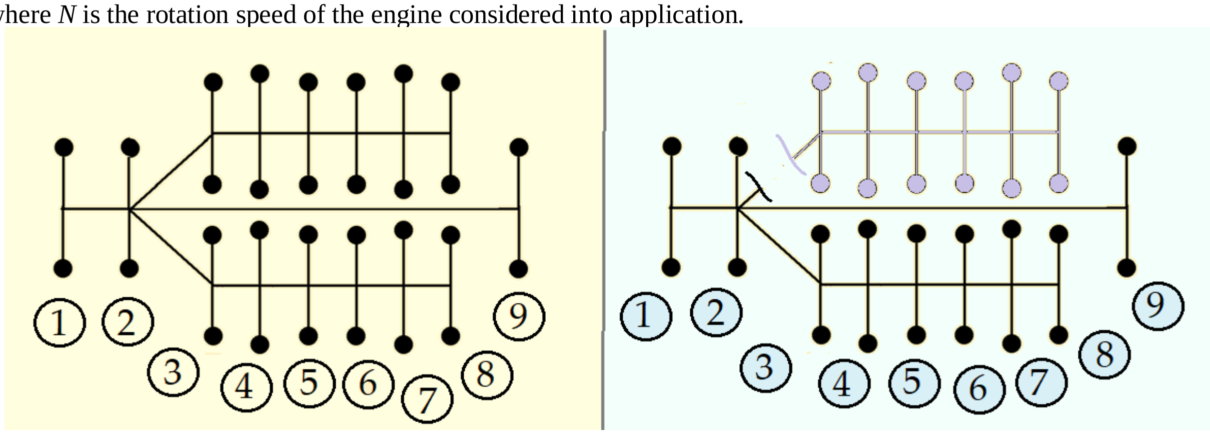
Table 2: The stiffness properties of the system

No.	Stiffness	Values [Nm/rad]
1	k_1	2.56x10^6
2	k_2	2.56x10^6
3	k_3	2.53x10^6
4	k_4	2.56x10^6
5	k_5	2.56x10^6
6	k_6	20.87 x10^6
7	k_7	12.67 x10^6
8	K_8	0.045961 x10^6

At a constant engine rotation, the movement of the crankshaft, connecting rod, piston and other moving parts of the engine is a periodic movement, with period *T:*

$$T=\frac{2\pi}{\omega} \quad , \quad \text{with} \quad \omega=\frac{\pi N}{30} \quad , \tag{41}$$

where *N* is the rotation speed of the engine considered into application.



a) b)

Figure 4: a) System with two engines; b) System with one engine

In order to be able to make the comparison, consider the system with two engines in operation and the system with one engine in operation (Fig. 4) and calculate the amplitudes of the forced vibrations for different harmonics of the exciting force.

The moment given by each of the engines due to the pressure forces appearing in the cylinder is $\{f\}_1$ and the moment given by the forces resisting the wind and air resistance is given by $\{f\}_2$:

$$\{f\}_1=\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \quad ; \quad \{f\}_2=\begin{Bmatrix} f_7 \\ f_8 \\ f_9 \end{Bmatrix} \quad ; \quad \{f\}=\begin{Bmatrix} \{f\}_1 \\ \{f\}_1 \\ \{f\}_2 \end{Bmatrix} \quad . \tag{42}$$

The excitation can be decomposed into Fourier series [63]:

$$\{f\} = \begin{Bmatrix} \{f\}_1 \\ \{f\}_1 \\ \{f\}_2 \end{Bmatrix} = \begin{Bmatrix} \{f\}_{10} \\ \{f\}_{10} \\ \{f\}_{20} \end{Bmatrix} + \begin{Bmatrix} \{f\}_{11s} \\ \{f\}_{11s} \\ \{f\}_{21s} \end{Bmatrix} \sin\omega t + \begin{Bmatrix} \{f\}_{11c} \\ \{f\}_{11c} \\ \{f\}_{21c} \end{Bmatrix} \cos\omega t + \begin{Bmatrix} \{f\}_{12s} \\ \{f\}_{12s} \\ \{f\}_{22s} \end{Bmatrix} \sin 2\omega t + \begin{Bmatrix} \{f\}_{12c} \\ \{f\}_{12c} \\ \{f\}_{22c} \end{Bmatrix} \cos 2\omega t + \ldots \quad (43)$$

In numerical calculations, a reasonable number of terms is considered [64].
If we take some term of the series, for example the k term:

$$\begin{Bmatrix} \{f\}_{1is} \\ \{f\}_{1is} \\ \{f\}_{2is} \end{Bmatrix} \sin k\omega t \quad (44)$$

and we are looking for a solution of the system of equations (1) of the form:

$$\begin{Bmatrix} \{x\}_{1is} \\ \{x\}_{1is} \\ \{x\}_{2is} \end{Bmatrix} \sin k\omega t \quad (45)$$

the linear system results:

$$\left(-k^2\omega^2 \begin{bmatrix} [m_1] & 0 & 0 \\ 0 & [m_1] & 0 \\ 0 & 0 & [m_2] \end{bmatrix} + \begin{bmatrix} [k_1] & 0 & [k_{12}] \\ 0 & [k_1] & [k_{12}] \\ [k_{12}]^T & [k_{12}]^T & [k_2] \end{bmatrix} \right) \begin{Bmatrix} \{x\}_{1is} \\ \{x\}_{1is} \\ \{x\}_{2is} \end{Bmatrix} = \begin{Bmatrix} \{f\}_{1is} \\ \{f\}_{1is} \\ \{f\}_{2is} \end{Bmatrix} \quad (46)$$

If a single engine is working, the form of the solution is:

$$\left(-k^2\omega^2 \begin{bmatrix} [m_1] & 0 \\ 0 & [m_2] \end{bmatrix} + \begin{bmatrix} [k_1] & 0 \\ 0 & [k_2] \end{bmatrix} \right) \begin{Bmatrix} \{x\}_{1is} \\ \{x\}_{2is} \end{Bmatrix} = \begin{Bmatrix} \{f\}_{1is} \\ \{f\}_{2is} \end{Bmatrix} \quad (47)$$

Obtaining the solutions can now be done numerically using the Matlab software. In Figs.4 and 5 are presented the amplitude of vibration of the wheels 1 to 9 for the two cases considered, with two engine working and with one engine working.

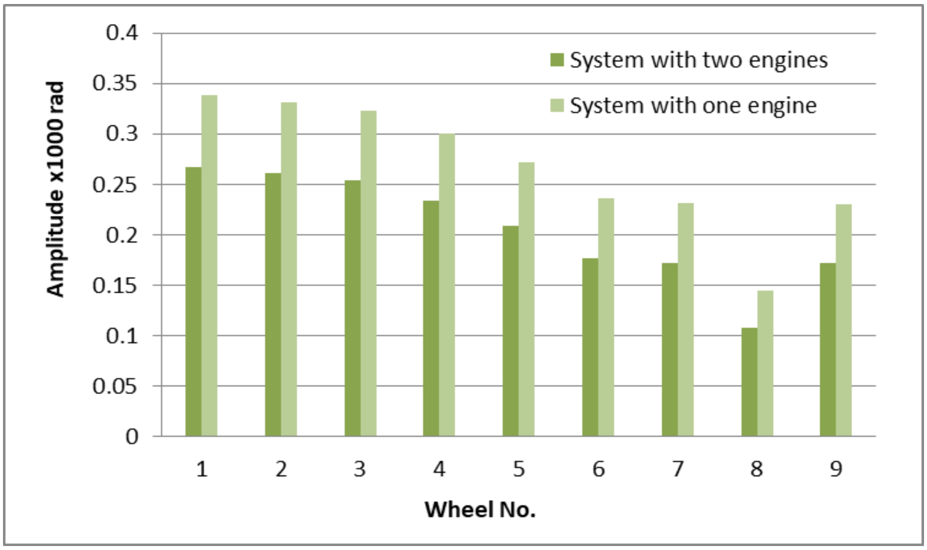


Figure 5: The amplitudes of the flywheels for the 5th harmonic of the excitations

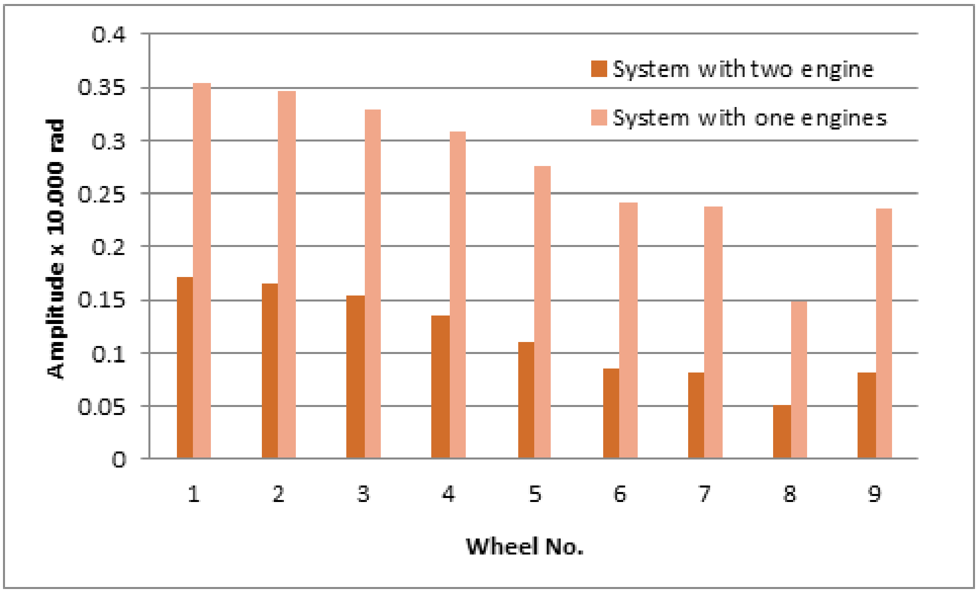


Figure 6: The amplitudes of the flywheels for the 5th harmonic of the excitations

Figures 5 and 6 illustrate the previously presented properties. Thus, if only one engine is working on the machine under study (this happens during the transport of the machine), the amplitude of the forced oscillations of the flywheels is higher than the amplitude of the forced oscillations for the same flywheels, if both engines are working (during the service of the machine). The introduction of a symmetrical part in the system leads to the improvement of the forced vibration behavior for the studied case.

4. Discussion and Conclusions

In engineering practice, many buildings, halls, stadiums, stations, airports, etc. have identical parts in their composition and have different types of symmetry, as well as the various machines and devices that we use in our everyday life. It is a situation that we have encountered since the dawn of civilization and there are many reasons for this. An easier, faster, cheaper design, cheaper manufacturing and in a shorter time, reduced maintenance and logistics costs. Apart from the listed advantages, these symmetry properties can also be useful in the field of mechanical analysis of some structures. In the static field, some advantages in the calculation are described in the materials resistance manuals. In the field of dynamic analysis, however, until now these advantages have been less used. Even if FEM programmers intuitively use these properties to make their work easier, a description and a systematic study have been done less. The present work comes precisely to present the possibility of using these properties for the vibration calculation of structures.

The paper demonstrates two properties that the existence of the symmetries of the equations of motion and finally the solutions of these equations of motion provide. Thus, for certain modes of movement, the existence of these symmetries can determine the balance of the excitation forces that act, and as a result, the load on the entire system is reduced. This suggested that the amplitude of the forced vibrations could also be reduced. An example shows us that this assumption is correct.

The work suggests the possibility that the existence of symmetries in a mechanical system can lead to the improvement of its properties from the point of view of behavior to external excitations. Of course, a much wider study of this observation, on real systems, where the number of parameters is low enough to obtain results, is required. A possibility of applicability of this observation can be in the field of civil engineering.

References

[1] C. Ambrus, *Dynamic Analysis of Stresses in the Motor-Transmission Assembly of Mobile Power Drilling Installations*, Ph. D Thesis, Transylvania University of Brasov, Brasov, Romania., 2017.

[2] C. Kaluba, A. Zingoni, Group-Theoretic Buckling Analysis of Symmetric Plane Frames, *Journal of Structural Engineering*, Vol. 147, pp. 04021153, 10/01, 2021.

[3] D. Holm, T. Schmah, C. Stoica, 2009, *Geometric Mechanics and Symmetry*,

[4] J. E. Marsden, T. S. Ratiu, 2013, *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, Springer Science & Business Media,

[5] S. F. Singer, 2004, *Symmetry in Mechanics*, Springer, Berlin/Heidelberg, Germany, 1ed.

[6] Z. Celep, On the axially symmetric vibration of thick circular plates, *Ingenieur-Archiv,* Vol. 47, No. 6, pp. 411-420, 1978/11/01, 1978.

[7] A. Zingoni, N. Enoma, Dual-purpose concrete domes: A strategy for the revival of thin concrete shell roofs, in *Proceeding of*, Elsevier, pp. 2686-2703.

[8] A. Zingoni, N. Enoma, On the strength and stability of elliptic toroidal domes, *Engineering Structures,* Vol. 207, pp. 110241, 2020.

[9] Y. Chen, J. Feng, Generalized Eigenvalue Analysis of Symmetric Prestressed Structures Using Group Theory, *Journal of Computing in Civil Engineering,* Vol. 26, pp. 488-497, 07/01, 2012.

[10] E. Zavadskas, R. Bausys, J. Antucheviciene, Civil Engineering and Symmetry, *Symmetry,* Vol. 11, pp. 501, 04/05, 2019.

[11] L. Codarcea-Munteanu, M. Marin, S. Vlase, The study of vibrations in the context of porous micropolar media thermoelasticity and the absence of energy dissipation, *Journal of Computational Applied Mechanics,* Vol. 54, No. 3, pp. 437-454, 2023.

[12] I. M. J. J. Ganghoffer, Similarity, Symmetry and Group Theoretical Methods in Mechanics, in *International Centre for Mechanical Sciences*, Udine, Italy, 2015.

[13] I. G. D. Mangeron, S. Vlase, Symmetrical Branched Systems Vibrations, *Sci. Mem. Rom. Acad,* Vol. 12, pp. 232-236, 1991.

[14] M. Luminita Scutaru, S. Vlase, M. Marin, Symmetrical Mechanical System Properties-Based Forced Vibration Analysis, *Journal of Computational Applied Mechanics*, pp. -, 2023.

[15] S. Vlase, M. Marin, M. Scutaru, I. R. Száva, Coupled transverse and torsional vibrations in a mechanical system with two identical beams, *AIP Advances,* Vol. 7, pp. 065301, 06/01, 2017.

[16] S. M. Abo-Dahab, A. E. Abouelregal, M. Marin, Generalized thermoelastic functionally graded on a thin slim strip non-Gaussian laser beam, *Symmetry,* Vol. 12, No. 7, pp. 1094, 2020.

[17] S. Vlase, C. Năstac, M. Marin, M. Mihălcică, A method for the study of the vibration of mechanical bars systems with symmetries, *ACTA TECHNICA NAPOCENSIS-Series: APPLIED MATHEMATICS, MECHANICS, and ENGINEERING,* Vol. 60, No. 4, 2017.

[18] M. Hassan, M. I. Marin, R. Ellahi, S. Z. Alamri, EXPLORATION OF CONVECTIVE HEAT TRANSFER AND FLOW CHARACTERISTICS SYNTHESIS BY Cu–Ag/WATER HYBRID-NANOFLUIDS, *Heat Transfer Research,* Vol. 49, pp. 1837-1848, 2018.

[19] A. Zingoni, Symmetry recognition in group-theoretic computational schemes for complex structural systems, *Computers & Structures,* Vol. 94-95, pp. 34-44, 2012/03/01/, 2012.

[20] A. Zingoni, Group-theoretic exploitations of symmetry in computational solid and structural mechanics, *International Journal for Numerical Methods in Engineering,* Vol. 79, pp. 253-289, 07/16, 2009.

[21] M. Marin, A. Seadawy, S. Vlase, A. Chirila, On mixed problem in thermoelasticity of type III for Cosserat media, *Journal of Taibah University for Science,* Vol. 16, No. 1, pp. 1264-1274, 2022.

[22] A. Zingoni, Group-theoretic insights on the vibration of symmetric structures in engineering, *Philosophical transactions. Series A, Mathematical, physical, and engineering sciences,* Vol. 372, pp. 20120037, 02/13, 2014.

[23] T. PERNA, ROLE OF SYMMETRY IN OPTIMIZATION OF FEM SIMULATION CALCULATIONS, *MM Science Journal*, 2022.

[24] B. Dong, R. Parker, Modal properties of cyclically symmetric systems with central components vibrating as three-dimensional rigid bodies, *Journal of Sound and Vibration,* Vol. 435, 08/01, 2018.

[25] A. Zingoni, Group-theoretic vibration analysis of double-layer cable nets of D4h symmetry, *International Journal of Solids and Structures,* Vol. 176-177, pp. 68-85, 2019/11/30/, 2019.

[26] A. Zingoni, On the best choice of symmetry group for group-theoretic computational schemes in solid and structural mechanics, *Computers & Structures,* Vol. 223, pp. 106101, 10/01, 2019.

[27] D. Bartilson, J. Jang, A. Smyth, Symmetry properties of natural frequency and mode shape sensitivities in symmetric structures, *Mechanical Systems and Signal Processing,* Vol. 143, pp. 106797, 09/01, 2020.

[28] M. G. Fernández-Godino, S. Balachandar, R. Haftka, On the Use of Symmetries in Building Surrogate Models, *Journal of Mechanical Design,* Vol. 141, 11/20, 2018.

[29] D. Wang, C. Zhou, J. Rong, Free and forced vibration of repetitive structures, *International Journal of Solids and Structures,* Vol. 40, pp. 5477-5494, 10/01, 2003.

[30] C. Cai, F. Wu, On the vibration of rotational periodic structures, *Acta Scientiarum Naturalium Universitatis Sunyatseni*, Vol. 22, No. 3, pp. 1-9, 1983.

[31] C. W. Cai, Y. K. Cheung, H. C. Chan, Uncoupling of dynamic equations for periodic structures, *Journal of Sound and Vibration*, Vol. 139, No. 2, pp. 253-263, 1990/06/08/, 1990.

[32] H. C. Chan, C. Cai, Y. K. Cheung, 1998, *Exact analysis of structures with periodicity using U-transformation*, World Scientific,

[33] D. A. Evensen, Vibration analysis of multi-symmetric structures, *AIAA Journal*, Vol. 14, No. 4, pp. 446-453, 1976.

[34] D. Wang, C.-C. Wang, Natural vibrations of repetitive structures, *Journal of Mechanics*, Vol. 16, No. 2, pp. 85-95, 2000.

[35] W. Zhong, J. Lin, The eigen-value problem of the chain of identical substructures and the expansion method solution lasted on the eigen-vectors, *Acta Mech. Sin*, Vol. 23, pp. 72-81, 1991.

[36] L. Brillouin, 1946, *Wave Propagation in Periodic Structures*, Dover Publications, New York, USA

[37] D. Mead, A general theory of harmonic wave propagation in linear periodic systems with multiple coupling, *Journal of Sound and Vibration*, Vol. 27, No. 2, pp. 235-260, 1973.

[38] M. Marin, Some estimates on vibrations in thermoelasticity of dipolar bodies, *Journal of Vibration and Control*, Vol. 16, No. 1, pp. 33-47, 2010.

[39] D. Mead, A. Bansal, Mono-coupled periodic systems with a single disorder: Free wave propagation, *Journal of Sound and Vibration*, Vol. 61, No. 4, pp. 481-496, 1978.

[40] L. Gry, C. Gontier, DYNAMIC MODELLING OF RAILWAY TRACK: A PERIODIC MODEL BASED ON A GENERALIZED BEAM FORMULATION, *Journal of Sound and Vibration*, Vol. 199, No. 4, pp. 531-558, 1997/01/30/, 1997.

[41] L. Meirovitch, 2010, *Methods of analytical dynamics*, Courier Corporation,

[42] W. Zhong, F. W. Williams, On the direct solution of wave propagation for repetitive structures, *Journal of Sound and Vibration*, Vol. 181, pp. 485-501, 1995.

[43] A. Kaveh, K. Biabani Hamedani, A. Joudaki, M. Kamalinejad, Optimal analysis for optimal design of cyclic symmetric structures subject to frequency constraints, *Structures*, Vol. 33, pp. 3122-3136, 06/17, 2021.

[44] S. Vlase, M. Marin, M. L. Scutaru, C. Pruncu, Vibration Response of a Concrete Structure with Repetitive Parts Used in Civil Engineering, *Mathematics*, Vol. 9, No. 5, pp. 490, 2021.

[45] S. Vlase, P. Teodorescu, Elasto-dynamics of a solid with a general 'rigid'motion using fem model. Part I. Theoretical approach, *Rom. J. Phys*, Vol. 58, No. 7-8, pp. 872-881, 2013.

[46] J. Zhang, E. Reynders, G. De Roeck, G. Lombaert, Model updating of periodic structures based on free wave characteristics, *Journal of Sound and Vibration*, Vol. 442, 11/01, 2018.

[47] L.-J. Wu, H.-W. Song, Band gap analysis of periodic structures based on cell experimental frequency response functions (FRFs), *Acta Mechanica Sinica*, Vol. 35, pp. 156-173, 2019.

[48] M. I. Othman, M. Fekry, M. Marin, Plane waves in generalized magneto-thermo-viscoelastic medium with voids under the effect of initial stress and laser pulse heating, *Struct. Eng. Mech*, Vol. 73, No. 6, pp. 621-629, 2020.

[49] A. Kaveh, A. Joudaki, Matrix Analysis of Repetitive Circulant Structures: New-block and Near Block Matrices, *Periodica Polytechnica Civil Engineering*, 05/28, 2019.

[50] R. Antoine, S. Hans, C. Boutin, Asymptotic analysis of high-frequency modulation in periodic systems. Analytical study of discrete and continuous structures, *Journal of the Mechanics and Physics of Solids*, Vol. 117, 04/01, 2018.

[51] S. Vlase, M. Marin, P. Bratu, O. A. O. SHRRAT, Analysis of Vibration Suppression in Multi-Degrees of Freedom Systems, *Romanian Journal of Acoustics and Vibration*, Vol. 19, No. 2, pp. 149-156, 2022.

[52] M. Marin, O. Florea, On temporal behaviour of solutions in thermoelasticity of porous micropolar bodies, *Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică,* Vol. 22, No. 1, pp. 169-188, 2014.

[53] A. Kaveh, A. Zolghadr, Optimal design of cyclically symmetric trusses with frequency constraints using cyclical parthenogenesis algorithm, *Advances in Structural Engineering*, Vol. 21, No. 5, pp. 739-755, 2018.

[54] L. Li, B. Cheng, Y. Zhang, H. Qin, Study of a smart platform based on backstepping control method, *Earthquake Engineering and Engineering Vibration*, Vol. 16, No. 3, pp. 599-608, 2017/07/01, 2017.

[55] Z. Ghemari, S. Belkhiri, Mechanical resonator sensor characteristics development for precise vibratory analysis, *Sensing and Imaging*, Vol. 22, No. 1, pp. 40, 2021.

[56] H. Çetin, G. G. Yaralioglu, Analysis of Vibratory Gyroscopes: Drive and Sense Mode Resonance Shift by Coriolis Force, *IEEE Sensors Journal*, Vol. 17, pp. 347-358, 2017.

[57] A. Z. Ter-Martirosyan, A. N. Shebunyaev, V. V. Sidorov, Mathematical Analysis of the Vibratory Pile Driving Rate, *Axioms*, Vol. 12, No. 7, pp. 629, 2023.

[58] I. Abbas, A. Hobiny, M. Marin, Photo-thermal interactions in a semi-conductor material with cylindrical cavities and variable thermal conductivity, *Journal of Taibah University for Science,* Vol. 14, No. 1, pp. 1369-1376, 2020.

[59] M. Marin, On the minimum principle for dipolar materials with stretch, *Nonlinear Analysis: Real World Applications*, Vol. 10, No. 3, pp. 1572-1578, 2009.

[60] M. Rades, Mechanical vibrations II. Structural dynamic modeling, *Rumania: University Politehnica Bucharest*, pp. 354, 2010.

[61] Shock, V. I. Center, 1986, *Shock and Vibration Monograph Series*, Shock and Vibration Information Center.,

[62] D. Chen, J. Yang, S. Kitipornchai, Free and forced vibrations of shear deformable functionally graded porous beams, *International Journal of Mechanical Sciences,* Vol. 108-109, pp. 14-22, 2016/04/01/, 2016.