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An algorithm to Solve the Linear Programming Problem Constrained with the Harmonic–Fuzzy Relational Equalities

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ABSTRACT

In this paper, a linear programming problem is investigated in which the feasible region is formed as the intersection of fuzzy relational equalities and the harmonic mean operator is considered as fuzzy composition. Theoretical properties of the feasible region are derived. It is proved that the feasible solution set is comprised of one maximum solution and a finite number of minimal solutions. Furthermore, some necessary and sufficient conditions are additionally presented to determine the feasibility of the problem. Moreover, an algorithm is presented to find the optimal solutions of the problem and finally, an example is described to illustrate the algorithm.

Keyword: fuzzy relational equalities, mean operators, harmonic mean, fuzzy compositions, linear programming.

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1 Introduction

We will study the following linear optimization model whose constraints are formed as a fuzzy system defined by the harmonic mean operator:

$$\begin{aligned} \min\ & cx \\ & A\varphi x = b \\ & x \in [0,1]^n \end{aligned} \tag{1}$$

where $I = \{1, 2, ..., m\}$, $J = \{1, 2, ..., n\}$, $A = (a_{ij})_{m\times n}$, $0 \le a_{ij} \le 1$ ($\forall i \in I$ and $\forall j \in J$), is a fuzzy matrix, $b = (b_i)_{m\times 1}$,$0 \le b_i \le 1$ ($\forall i \in I$), is an m–dimensional fuzzy vector, and "φ" is the max-Harmonic composition, that is:

$$\varphi(x, y) = \begin{cases} \frac{2}{\frac{1}{x}+\frac{1}{y}} & x \neq 0, y \neq 0 \\ 0 & otherwise \end{cases}$$

Furthermore, let denote $S(A, b)$ the feasible solutions set of the problem (1), that is, $S(A, b) = \{x \in [0, 1]^n : A\varphi x = b\}$. Additionally, if a_i denotes the i^{th} row of the matrix A, then problem (1) can be also expressed as follows:

$$\begin{aligned} \min\ & cx \\ & \varphi\left(a_i, x\right) = b_i\,, \quad i \in I \\ & x \in [0,1]^n \end{aligned} \tag{2}$$

where the constraints mean $\varphi\left(a_i, x\right) = \max_{j\in J}\{\varphi\left(a_{ij}, x_j\right)\} = b_i$, ($\forall i \in I$) and,

$$\varphi(a_{ij}, x_i) = \begin{cases} \frac{2}{\frac{1}{a_{ij}}+\frac{1}{x_j}} & a_{ij} \neq 0, x_j \neq 0 \\ 0 & otherwise \end{cases}$$

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [39]. Nowadays, it is well known that many issues associated with body knowledge can be treated as FRE problems [35].
In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [7, 25, 33].

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subjects concerning FRE problems. Actually, the solutions set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [2]. This non-convexity property is one of the two bottlenecks making a major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [32]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2] and [3, 16, 12, 13, 15, 29, 28, 32].

Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [16, 15, 34, 36, 37, 40, 43, 42, 45, 48, 22]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI) [12, 13, 14, 17, 18, 26, 21]. Li and Yang [26] studied an FRI with addition-min composition and presented an algorithm to search for minimal solutions. Ghodousian et al. [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where φ is an operator with (closed) convex solutions.

The problem of optimization subject to FRE and FRI is one of the most interesting and ongoing research topics among the problems related to FRE and FRI theory [2] and [8, 11, 16, 12, 13, 15, 14, 23, 27, 30, 38, 41, 46, 21]. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using the jump-tracking technique. In [23] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider. Wu et al. [44] improved the method used by Fang and Li, by decreasing the search domain. The topic of the linear optimization problem was also investigated with max-product operation [20, 31]. Loetamonphong and Fang defined two subproblems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two subproblems [31]. Also, in [20] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [46] and max-t-norm composition [16, 15, 19, 27, 41].

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems, and other ideas [6, 10, 16, 15, 18, 24, 30, 47]. For example, Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients.
The optimization problem subjected to various versions of FRI could be found in the literature as well [12, 13, 14, 17, 18, 49, 21]. Xiao et al. [21] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et

al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.
In this paper, an algorithm is proposed to find all the optimal solutions to the problem (1). Firstly, we describe some structural details of harmonic-FREs such as the theoretical properties of harmonic-fuzzy equalities and necessary and sufficient conditions for the feasibility of the problem. Then, the feasible region is completely determined by a finite number of convex cells. Finally, an algorithm is presented to solve the main problem.
The remainder of the paper is organized as follows. Section 2 gives some basic results on the harmonic-fuzzy equalities. Also, some feasibility conditions are derived. In section 3, the feasible region is characterized in terms of a finite number of closed convex cells. The optimal solution of the problem is described in Section 4, and finally, in section 5 an example is presented to illustrate the algorithm.

2 Basic properties of harmonic – FREs

The fuzzy system $A\varphi x = b$ consists of m fuzzy relational equalities $\varphi(a_i, x) = \max_{j\in J}\{\varphi(a_{ij}, x_j)\} = b_i$. In this section, the structural properties of each fuzzy equation $\varphi(a_i, x) = b_i$ is investigated and its solutions are found. As will be shown later, the feasible solutions of the main problem can be derived based on the solutions of these fuzzy equations. Let $S(a_i, b_i)$ denote the feasible solutions set of i^{th} equation, that is, $S(a_i, b_i) = \{x \in [0,1]^n : \varphi(a_i, x) = b_i\}$. So, it is clear that $S(A, b) = \cap_{i\in I} S(a_i, b_i)$. From the definition of the harmonic mean operator φ, the following three basic properties are obtained.

Lemma 1. *Let $i \in I$, and $j \in J$. If $a_{ij} < \frac{b_i}{2-b_i}$, then $\varphi(a_{ij}, x_j) < b_i$, $\forall x_j \in [0,1]$.*

Proof. The result follows from the equality $\varphi\left(\frac{b_i}{2-b_i}, 1\right) = b_i$ and the fact that φ is an increasing function on $[0,1]^2$ in both variables. □

Lemma 2. *Let $i \in I$, and $j \in J$. If $a_{ij} \geq \frac{b_i}{2-b_i}$, and $a_{ij} > 0$, then $x_j = \frac{a_{ij} b_i}{2a_{ij} - b_i}$ is the unique solution to the equality $\varphi(a_{ij}, x_j) = b_i$.*

Proof. Since $a_{ij} \geq \frac{b_i}{2-b_i}$, and $\frac{b_i}{2-b_i} \geq \frac{b_i}{2}$, then $a_{ij} \geq \frac{b_i}{2}$ that means $\frac{a_{ij} b_i}{2a_{ij} - b_i} \geq 0$. Also, $a_{ij} \geq \frac{b_i}{2-b_i}$ implies that $\frac{a_{ij} b_i}{2a_{ij} - b_i} \leq 1$. Hence, $\frac{a_{ij} b_i}{2a_{ij} - b_i} \in [0,1]$. Moreover, it is easy to verify that $\varphi\left(a_{ij}, \frac{a_{ij} b_i}{2a_{ij} - b_i}\right) = b_i$. Now, since|*varphi* is an increasing function on $[0,1]^2$, we have $\varphi(a_{ij}, x_j) < b_i$ if $x_j < \frac{a_{ij} b_i}{2a_{ij} - b_i}$, and $\varphi(a_{ij}, x_j) > b_i$ if $x_j > \frac{a_{ij} b_i}{2a_{ij} - b_i}$. This completes the proof. □

Lemma 3. *Let $i \in I$, and $j \in J$. If $a_{ij} = b_i = 0$, then $\varphi(a_{ij}, x_j) = b_i$, $\forall x_j \in [0,1]$.*

Proof. The proof is directly resulted from the definition of harmonic operator φ. □

The following lemma gives a necessary and sufficient condition for the feasibility of the set $S(a_i, b_i)$.

Lemma 4. *For a fixed $i \in I$, $S(a_i, b_i) \neq \emptyset$ if and only if there exists at least some $j \in J$ such that $a_{ij} \geq \frac{b_i}{2-b_i}$.*

Proof. By contradiction, suppose that $x' \in S(a_i, b_i)$, and $a_{ij} \leq \frac{b_i}{2-b_i}$, $\forall j \in J$. So, from Lemma 1, we have $\varphi\left(a_{ij}, x'_j\right) < b_i$, $\forall j \in J$. Therefore, $\varphi(a_i, x') = \max_{j\in J}\{\varphi\left(a_{ij}, x'_j\right)\} < b_i$ that contradicts $x' \in S(a_i, b_i)$. □

Definition 1. *Let $J_1(i) = \{j \in J : a_{ij} > 0, a_{ij} \geq \frac{b_i}{2-b_i}\}$, $J_2(i) = \{j \in J : a_{ij} = b_i = 0\}$, and $J_3(i) = \{j \in J : a_{ij} < \frac{b_i}{2-b_i}\}$.*

Corollary 1. *Let $x' \in S(a_i, b_i)$. Then, $x_j \leq \frac{a_{ij} b_i}{2a_{ij}-b_i}$, $\forall j \in J_1(i)$. Also, either $J_2(i) \neq \emptyset$ or there exists some $j_0 \in J_1(i)$, such that, $x_{j_0} = \frac{a_{aj_0} b_i}{2a_{ij_0}-b_i}$.*

Lemma 5. *(a) Let $j_0 \in J_1(i)$, for some $i \in I$, and $j_0 \in J$. Also, suppose that $x' \in [0,1]^n$, such that,*

$$x'_j = \begin{cases} \frac{a_{aj_0} b_i}{2a_{ij_0}-b_i} & , if \;\; j = j_0 \\ r & , if \;\; j \in J_1(i) - \{j_0\} \\ s & , if \;\; j \in J_2(i) \\ t & , if \;\; j \in J_3(i) \end{cases}$$

where $0 \leq r \leq \frac{a_{aj} b_i}{2a_{ij}-b_i}$, and $s, t \in [0,1]$. Then, $x' \in S(a_i, b_i)$.

(b) Let $j_0 \in J_2(i)$, for some $i \in I$, and $j_0 \in J$. Also, suppose that $x' \in [0,1]^n$, such that,

$$x'_j = \begin{cases} r & , if \;\; j \in J_1(i) \\ s & , if \;\; j \in J_2(i) \\ t & , if \;\; j \in J_3(i) \end{cases}$$

where $0 \leq r \leq \frac{a_{aj} b_i}{2a_{ij}-b_i}$, and $s, t \in [0,1]$. Then, $x' \in S(a_i, b_i)$.

Proof. (a) The result follows from Lemmas 1, 2, and 3, and the following equations:

$$\varphi(a_i, x') = \max_{j\in J}\{\varphi\left(a_{ij}, x'_j\right)\} = \max\Big\{\max_{j\in J_1(i)}\{\varphi\left(a_{ij}, x'_j\right)\},$$
$$\max_{j\in J_2(i)}\{\varphi\left(a_{ij}, x'_j\right)\}, \max_{j\in J_3(i)}\{\varphi\left(a_{ij}, x'_j\right)\}\Big\}$$
$$= \max\Big\{\max_{j\in J_1(i)}\{\varphi\left(a_{ij}, x'_j\right)\}, \max_{j\in J_2(i)}\{\varphi\left(a_{ij}, x'_j\right)\}\Big\}$$
$$= \max\Big\{\varphi\left(a_{ij_0}, x'_{j_0}\right), \max_{j\in J_2(i)}\{\varphi\left(a_{ij}, x'_j\right)\}\Big\} = b_i$$

(b) The proof is similar to that of part (a). □

Definition 2. *Suppose that $S(a_i, b_i) \neq \emptyset$. We define $\overline{X}(i) \in [0,1]^n$, such that,*

$$\overline{X}(i)_j = \begin{cases} \frac{a_{aj} b_i}{2a_{ij} - b_i} & , if \;\; j \in J_1(i) \\ 1 & , otherwise \end{cases}$$

Theorem 1. *Suppose that $S(a_i, b_i) \neq \emptyset$. Then, $\overline{X}(i)$ is the maximum solution of $S(a_i, b_i)$.*

Proof. Based on Lemma 5, $\overline{X}(i) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$. So, from Lemmas 1, 2, and 3, $x'_j \leq \frac{a_{aj} b_i}{2a_{ij} - b_i}$, $\forall j \in J_1(i)$, and $x'_j \leq 1$, $\forall j \in J_2(i) \cup J_3(i)$. Therefore, $x'_j \leq \overline{X}(i)_j$, $\forall j \in J$. □

Definition 3. *Let $i \in I$, and $S(a_i, b_i) \neq \emptyset$. For each $j \in J_1(i) \cup J_2(i)$, define $\underline{X}_{(i,j)} \in [0,1]^n$, such that,*

$$\underline{X}(i,j)_k = \begin{cases} \frac{a_{aj} b_i}{2a_{ij} - b_i} & , k = j \;\; , j \in J_1(i) \\ 0 & , otherwise \end{cases}$$

Remark 1. *Suppose that $S(a_i, b_i) \neq \emptyset$, and $j \in J_1(i)$. Then, from Definitions 2, and 3, we have $\overline{X}(i)_j = \underline{X}(i,j)_j$.*

Theorem 2. *Suppose that $S(a_i, b_i) \neq \emptyset$, and $j_0 \in J_1(i) \cup J_2(i)$. Then, $\underline{X}(i, j_0)$ is a minimal solution of $S(a_i, b_i)$.*

Proof. From Lemma 5, $\underline{X}(i, j_0) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$, $x' \leq \underline{X}(i, j_0)$, and $x' \neq \underline{X}(i, j_0)$. So, $x'_j \leq \underline{X}(i, j_0)_j$, $\forall j \in J$, and $x' \neq \underline{X}(i, j_0)$. Therefore, $x'_j = 0$, $\forall j \in J - \{j_0\}$, and $x'_{j_0} < \frac{a_{aj} b_i}{2a_{ij} - b_i}$ (if $j_0 \in J_2(i)$, we must have x'_{j_0} that is a contradiction). However, in this case we will have

$$\varphi(a_i, x') = \max_{j \in J}\{\varphi(a_{ij}, x'_j)\} = \max\left\{\max_{j \in J - \{j_0\}}\{\varphi(a_{ij}, x'_j)\}, \varphi(a_{ij_0}, x'_{j_0})\right\} = b_i$$

$$= \varphi(a_{ij_0}, x'_{j_0}) < b_i$$

that contradicts $x' \in S(a_i, b_i)$. □

By Theorem 3 below, the solutions set $S(a_i, b_i) \neq \emptyset$ is completely determined. The theorem shows that $S(a_i, b_i)$ can be stated in terms of a finite number of closed convex cells.

Theorem 3. $S(a_i, b_i) = \bigcup_{j \in J_1(i) \cup J_2(i)} \left[\underline{X}(i,j), \overline{X}(i)\right]$.

Proof. Let $x' \in S(a_i, b_i)$. From Theorem 1, $x' \leq \overline{X}(i)$. On the other hand, from Corollary 1, either $J_2(i) \neq \emptyset$ (e.g. there exists $j_0 \in J_2(i)$), or there exists some $j_0 \in J_1(i)$, such that, $x_{j_0} = \frac{a_{aj} b_i}{2a_{ij} - b_i}$. In both cases, $\underline{X}(i, j_0) \leq x'$. Hence, $x' \in [\underline{X}(i, j_0), \overline{X}(i)]$. Conversely, let $x' \in \bigcup_{j \in J_1(i) \cup J_2(i)} \left[\underline{X}(i,j), \overline{X}(i)\right]$. Therefore, $\varphi(a_{ij}, x'_j) \leq \varphi(a_{ij}, \overline{X}(i)_j) \leq b_i$, $\forall j \in J$.

Moreover, there exists some $j_0 \in J_1(i)$ or $j_0 \in J_2(i)$, such that, $x' \in [\underline{X}(i,j_0), \overline{X}(i)]$. In the former case, Remark 1 implies $x'_{j_0} = \overline{X}(i,j_0)_{j_0}$, and therefore, $\varphi(a_{ij_0}, x'_{j_0}) = b_i$. In the latter case, Lemma 3 implies $\varphi(a_{ij_0}, x'_{j_0}) = \varphi(0, x'_{j_0}) = 0 = b_i$. Thus, we have

$$\varphi(a_i, x') = \max_{j\in J}\{\varphi(a_{ij}, x'_j)\} = \max\left\{\max_{j\in J-\{j_0\}}\{\varphi(a_{ij}, x'_j)\}, \varphi(a_{ij_0}, x'_{j_0})\right\} = \varphi(a_{ij_0}, x'_{j_0}) = b_i$$

which implies that $x' \in S(a_i, b_i)$. □

3 Feasible region of Problem (1)

In this section, a necessary and sufficient condition is derived to determine the feasibility of the main problem.

Definition 4. *Let $\overline{X}(i)$ be as in Definition 1, $\forall i \in I$. We define $\overline{X} = \min_{i\in I} \overline{X}(i)$.*

Definition 5. *Let $e : I \to \bigcup_{i\in I} J_i$, so that $e(i) \in J_i$, $\forall i \in I$ and let E be the set of all vectors e. For the sake of convenience, we represent each $e \in E$ as an m–dimensional vector $e = [j_1, j_2, ..., j_m]$ in which $j_k = e(k)$, $k = 1, 2, ..., m$.*

Definition 6. *Let $e = [j_1, j_2, ..., j_m] \in E$. We define $\underline{X}(e) \in [0,1]^n$, such that, $\underline{X}(e)_j = \max_{i\in I}\{\underline{X}(i, e(i))_j\} = \max_{i\in I}\{\underline{X}(i, j_i)_j\}$, $\forall j \in J$.*

The following theorem indicates that the feasible region of problem 1 is completely found by a finite number of closed convex cells.

Theorem 4. *$S(A,b) = \bigcup_{e\in E}[\underline{X}(e), \overline{X}]$.*

Proof. Since $S(A,b) = \bigcap_{i\in I} S(a_i, b_i)$, from Theorem 3, we have

$$S(A,b) = \bigcap_{i\in I}\bigcup_{j\in J_i}[\underline{X}(i,j), \overline{X}(i)]$$

Now, from Definitions 5 and 6, $S(A,b) = \bigcup_{e\in E}\bigcap_{i\in I}[\underline{X}(i,e(i)), \overline{X}(i)]$, i.e., $S(A,b) = \bigcup_{e\in E}\left[\max_{i\in I}\{\underline{X}(i,e(i))\}, \min_{i\in I}\{\overline{X}(i)\}\right]$. Now, the result follows from Definitions 4 and 6. □

The following Corollary gives a simple necessary and sufficient condition for the feasibility of $S(A,b)$.

Corollary 2. *$S(A,b) \neq \emptyset$ iff $\overline{X} \in S(A,b)$.*

4 Resolution of Problem (1)

It can be easily verified that $\overline{X}$ is the optimal solution for $\min\{Z_1 = \sum_{j=1}^{n} c_j^- x_j : A\varphi x = b, x \in [0,1]^n\}$, and the optimal solution for $\min\{Z_2 = \sum_{j=1}^{n} c_j^+ x_j : A\varphi x = b, x \in [0,1]^n\}$ is $\underline{X}(e^*)$ for some $e^* \in E$, where $c_j^+ = \max\{c_j, 0\}$, and $c_j^- = \min\{c_j, 0\}$ for $j = 1, 2, ..., n$ [9, 13, 19, 28]. According to the foregoing results, the following theorem shows that the optimal solution to the problem (1) can be obtained by the combination of $\overline{X}$, and $\underline{X}(e^*)$.

Theorem 5. *Suppose that $S(A,b) \neq \emptyset$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of subproblems Z_1 and Z_2, respectively. Then, $c^T x^*$ is the lower bound of the optimal objective function in 1, where $x^* = [x_1^*, x_2^*, ..., x_n^*]$ is defined as follows:*

$$x_j^* = \begin{cases} \overline{X}_j & , c_j < 0 \\ \underline{X}(e^*)_j & , c_j \geq 0 \end{cases}$$

for $j = 1, 2, ..., n$.

Proof. For a general case, see the proof of Theorem 4.1 in [13]. □

Corollary 3. *Suppose that $S(A,b) \neq \emptyset$. Then, x^* as defined in Theorem 5, is the optimal solution to the problem 1.*

Proof. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \leq x_j^* \leq \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E} [\underline{X}(e), \overline{X}] = S(A,b)$. □

5 Numerical example

Consider the following linear programming problem constrained with a fuzzy system defined by the harmonic operator:

$$\min Z = -4.1634x_1 + 2.4461x_2 + 4.3181x_3 - 4.3854x_4 - 1.7545x_5 - 2.7559x_6 + 5.6278x_7$$

$$\begin{bmatrix} 0.0752 & 0.0240 & 0.0609 & 0.7027 & 0.0026 & 0.0006 & 0.0688 \\ 0.4476 & 0.9959 & 0.4585 & 0.6420 & 0.6218 & 0.2835 & 0.3448 \\ 0.2198 & 0.0237 & 0.0422 & 0.0323 & 0.0693 & 0.1433 & 0.2735 \\ 0.8347 & 0.1614 & 0.3268 & 0.1427 & 0.2021 & 0.2888 & 0.7064 \\ 0.0334 & 0.4273 & 0.9385 & 0.0093 & 0.4441 & 0.1234 & 0.5963 \end{bmatrix} \varphi x = \begin{bmatrix} 0.1355 \\ 0.9022 \\ 0.2896 \\ 0.4995 \\ 0.7836 \end{bmatrix}$$

$x \in [0,1]^7$.

From Definition 1, we have $J_1(1) = \{1,4\}$, $J_1(2) = \{2\}$, $J_1(3) = \{1,7\}$, $J_1(4) = \{1,7\}$, and $J_1(5) = \{3\}$. According to Definition 2, the maximum solution of $S(a_i, b_i)$ $(i = 1, ..., 5)$

are as follows:

$\overline{X}(1) = [0.6839, 1, 1, 0.0750, 1, 1, 1]$,

$\overline{X}(2) = [1, 0.8246, 1, 1, 1, 1, 1]$,

$\overline{X}(3) = [0.4244, 1, 1, 1, 1, 1, 0.3077]$,

$\overline{X}(4) = [0.3564, 1, 1, 1, 1, 1, 0.3863]$,

$\overline{X}(5) = [1, 1, 0.6726, 1, 1, 1, 1]$.

So, by Definition 4, we obtain $\overline{X} = [0.3564, 0.8246, 0.6726, 0.0750, 1, 1, 0.3077]$. It is easy to verify that $\overline{X} \in S(A, b)$. Therefore, Corollary 2 implies that $S(A, b)$ is feasible.

In this example, $|E| = 8$, that is, there are 8 solutions $\underline{X}(e)$ that may be minimal solutions of the feasible region (see Definitions 5 and 6). By a pairwise comparison between the solutions $\underline{X}(e)$, it turns out that the feasible region has only one minimal solution. This unique minimal solution is generated by $e = [4, 2, 7, 1, 3]$ as follows:

$\underline{X}(e) = [0.3564, 0.8246, 0.6726, 0.0750, 0, 0, 0.3077]$

Since the minimal solutions set is a singleton set, the unique minimal is converted to the minimum solution. Also, it is obvious that $\underline{X}(e^*) = \underline{X}(e)$ is the optimal solution of the subproblem Z_2 that is obtained by $e^* = e$. Finally, based on Theorem 5, the optimal solution to the Problem 1 is resulted as:

$x^* = [0.3564, 0.8246, 0.6726, 0.0750, 1, 1, 0.3077]$

with optimal objective value $Z^* = 0.33017$.

6 Conclusion

In this paper, we proposed an algorithm to solve the linear optimization model constrained with harmonic fuzzy relational equalities. The feasible solutions set of each harmonic-FRE was obtained and their feasibility conditions were described. Based on the foregoing results, the feasible region of the problem is completely resolved. It was shown that the feasible solutions set can be write in terms of a finite number of closed convex cells. As future works, we aim at testing our algorithm in other type of fuzzy systems and linear optimization problems whose constraints are defined as FRI with other averaging operators.

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