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Multipole Structure of Kerr Spacetime: A Proposal for Applications in Geodesy

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ABSTRACT

The multipole structure of the Kerr spacetime is investigated. It is believed that there is no stationary exact solution of the Einstein field equations which successfully describes the spacetime of the Earth. The problem is related to the pole structure of the well-known exact solutions which can be shown that none of them meet the principal criteria for a suitable physical solution. We try to identify a probable multipole structure that is compatible with the Newtonian limit, i.e., having all mass multipole moments in the metric. This is crucially important in the pursuit of finding an exact metric to simulate the gravitational field of the Earth. The presence of all arbitrary mass multipole moments, along with at least one angular moment in the corresponding Ernst potential of the spacetime metric, will eventually be ended up with finding an exact solution to Einstein's field equations that matches the current precise observational satellite data.

KEYWORDS

The Earth's Gravitational Potential,
The Earth's Metric,
Relativistic Geodesy

1. Introduction

The recent trend in the application of the general theory of relativity in geodesy goes around the precise definition of some basic notions such as geoid, gravimetry, and gradiometry (Buchin et al., 2011; Denker et al., 2017). The ability to carry out precise measurements has been the main motivation for the community to formulate classical geodesy in the framework of a more reliable theory of gravitation, i.e. Einstein's general relativity (Buchin et al., 2011). To identify the best spacetime for modeling the Earth's gravity, people have investigated the non-black hole solutions of Einstein's field equations (EFEs) (Denker et al., 2018; Ehlers, 1993). One of the main problems with the modeling of the Earth's gravitational field underlies with its multipole structure. As we know, the Newtonian potential possesses all the even and odd mass multipoles while there is no angular moment in its spectrum. However, in the relativistic side, almost all of the known vacuum solutions of the Einstein's field equations which may be a probable candidate for modeling of the Earth's gravitational field, do not have a full spectrum of multipoles in their Newtonian limit. In other words, the multipole structure

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of the relativistic gravitational theory does not compatible with that of the Newtonian one (Ehlers, 1993).

Thus, it seems there is no exact relativistic spacetime that is capable of describing all aspects of the Earth's gravitational field. Alternatively, some studies have been devoted to the approximation methods which employ the linearized theory of gravity, say for instance the weak field limit of well-known spacetimes (Foster & Nightingale, 2006). Other studies use a post-Newtonian approach to analyze the multipole structure of some exact spacetimes including the Schwarzchild, the Erez-Rozen, and the Kerr spacetimes. None of the just mentioned spacetimes have conquered the difficulties with the multipole structure of the gravitational potential (Ehlers, 1993; Kopejkin et al., 2018). Following recent research which enables us to find exact stationary axisymmetric solutions of EFEs with arbitrary multipole structure, we try to address the above-mentioned problem here and propose a probable solution.

In the next two sections, we present the main results in the passage of the exact solutions of EFEs in prolate spheroidal coordinates. The main concern is on Quevedo's solutions both in static and stationary axisymmetric spacetimes. In section IV, we summarize the derivations of Newtonian and relativistic moments of mass and spin and propose our model of relativistic potential. The last section is devoted to some discussions and conclusions.

2. The General Static Axisymmetric Solution to EFEs in the Prolate Spheroidal Coordinates

The general static axisymmetric spacetime in the prolate spheroidal coordinates which have a full spectrum of mass multipole moments was first discovered by Quevedo (Quevedo, 1989; Kopejkin et al., 2016). The metric is given by

$$ds^2 = e^{2\psi}dt^2 - \sigma^2 e^{-2\psi} \times \left\{e^{2\gamma}(X^2-Y^2)\left[\frac{dX^2}{X^2-1}+\frac{dY^2}{1-Y^2}\right]+(X^2-1)(1-Y^2)d\phi^2\right\}z \tag{1}$$

in which ψ solves the Laplace equation

$$[(X^2-1)\psi_{,X}]_{,X}+[(1-Y^2)\psi_{,Y}]_{,Y}=0, \tag{2}$$

and γ is obtained from

$$\begin{aligned}\gamma_{,X} &= \frac{1-Y^2}{X^2-Y^2}\left[X(X^2-1)\psi_{,X}^2 - X(1-Y^2)\psi_{,Y}^2 - 2Y(X^2-1)\psi_{,X}\psi_{,Y}\right],\\ \gamma_{,Y} &= \frac{X^2-1}{X^2-y^2}\left[Y(X^2-1)\psi_{,X}^2 - Y(1-Y^2)\psi_{,Y}^2 + 2X(1-Y^2)\psi_{,X}\psi_{,Y}\right].\end{aligned} \tag{3}$$

This spacetime is asymptotically flat and γ vanishes at the symmetry axis. The general solution to the Laplace equation in (2) is

$$\psi = \sum (-1)^{n+1} q_n Q_n(X) P_n(Y), \tag{4}$$

in which P_n and Q_n are the Legendre functions of the first and second kind respectively. The solution to equation (2) is found to be

$$\gamma = \sum_{m,n=0}^{\infty} (-1)^{m+n} q_n q_n \Gamma^{mn}, \tag{5}$$

where the Γ^{mn} functions are

$$\begin{aligned}\Gamma^{(mn)} &= \frac{1}{2}\ln\left(\frac{X^2-1}{X^2-Y^2}\right) + (\epsilon_n+\epsilon_m-2\epsilon_n\epsilon_m)\ln\left(\frac{X+Y}{X-1}\right)\\ &+(X^2-1)\left[X\left(A_{n,m}Q'_nQ_m + A_{m,n}Q'_mQ_n\right) + \left[(n+1)A_{n,m} - B_{n+1,m}\right]Q_nQ_m\right] +\\ &+(X^2-1)\left[(1-\epsilon_n)S_m + \epsilon_n S_{m+1} - \frac{\epsilon_n}{m+1[P_m-(-1)^m]Q'_m}\right]\\ &+(X^2-1)^2\left[Q_m\mathbb{B}_{m,n} - Q'_m\mathbb{A}_{m,n} + \frac{1}{n+1}A_{m,n}Q'_mQ'_n\right]\end{aligned} \tag{6}$$

In the above equation $\epsilon_n = \frac{1+(-1)^n}{2}$ and

$$A_n = YP_nQ'_n - XP'_nQ_n, \tag{7}$$

$$B_n = YP'_nQ_n - XP_nQ'_n, \tag{8}$$

$$S_n = \sum_{k=0}^{\left[\frac{n}{2}\right]-1}\left(\frac{1}{n-2k-1}+\frac{1}{n-2k}\right)(P_{m,n-2k-1} + (-1)^{n+1})Q'_{n-2k-1}. \tag{9}$$

$$\mathbb{A}_{m,n} = \sum_{k=0}^{\left[n-\frac{1}{2}\right]}\left[\frac{1}{n-2k+1}\frac{1}{n-2k}\right]A_{m,n-2k}Q'_{n-2k}, \tag{10}$$

$$\mathbb{B}_{m,n} = \sum_{k=0}^{\left[\frac{n}{2}\right]-1}\left[\frac{1}{n-2k-1}+\frac{1}{n-2k}\right]B_{m,n-2k-1}Q'_{n-2k-1}, \tag{11}$$

Using the coordinate-invariant definition of multipole moments (Geroch,1970), the Newtonian limit of the potential function ψ has been found as

$$\psi = G\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n!}{(2n+1)!!}q_n m^{n+1}\frac{P_n(cos\theta)}{r^{n+1}}, \tag{12}$$

in which we have introduced the polar coordinates $X = \frac{r-m}{m}$, $Y = cos\theta$ quation (12) shows that the Newtonian mass moments N_n is related to q_n moments by

$$N_n = (-1)^{n+1}\frac{n!}{(2n+1)!!}q_n m^{n+1}. \tag{13}$$

The relativistic moments M_n are related to the Newtonian one by $M_n = N_n + R_n$ where R_n is the corresponding relativistic correction (Quevedo,1989)

$$\begin{aligned} R_0 &= R_1 = R_2 = 0,\\ R_3 &= -\frac{2}{5}N_1,\\ R_4 &= -\frac{2}{7}N_2 - \frac{6}{7}N_1^2,\\ R_5 &= -\frac{2}{9}m^2 N_3 - \frac{48}{20}mN_2N_1 - \frac{2}{7}N_1^3 - \frac{4}{105}m^4 N_1, \end{aligned} \tag{14}$$

Since N_1 can be made to vanish by choosing the center of mass coordinates, the first relativistic correction appears at $n = 4$. Note that Schwarzchild spacetime corresponds to $q_0 = 1,\ q_{k>0} = 0$ or equivalently to

$$\begin{aligned} \psi &= -Q_0 = -\frac{1}{2}\ln\frac{X-1}{X+1},\\ \gamma &= \frac{1}{2}\ln(\frac{X^2-1}{X^2-Y^2}) \end{aligned} \tag{15}$$

3. The General Stationary Axisymmetric Spacetime in the Prolate Spheroidal Coordinates

Astrophysical objects may properly be described by stationary spacetime. Extraction of such a solution can be achieved by applying Hoenslaers-Kinnersley-Xantopoulos transformation (HKZ) to the static spacetime (1) as a seed metric. Such transformation was first done for the metric in equation (1) by Quevedo (Quevedo, 1989) and the result is as follows:

$$\begin{aligned} ds^2 = -\sigma^2 f^{-1}\Big\{ & e^{2\eta}(X^2-Y^2)\left[\frac{dX^2}{X^2-1}+\frac{dY^2}{1-Y^2}\right]\\ & + (X^2-1)(1-Y^2)d\phi^2\Big\}\\ & + f(dt-\omega d\phi)^2 \end{aligned} \tag{16}$$

in which σ is an arbitrary constant and

$$\begin{aligned} f = 2R[(1+cos\tau)L_+ & \left[\frac{X-1}{X+1}\right]^{1-\delta} e^{-2\sigma\psi}\\ & + (1\\ & - cos\tau)L_- \left[\frac{X-1}{X+1}\right]^{\delta-1} e^{2\sigma\psi}\\ & + 4sin\tau(XN_- + YN_+)]^{-1} \end{aligned} \tag{17}$$

$$\begin{aligned} \boldsymbol{\omega} = K_1 + \sigma sin\tau[\sigma\hat{p} + 2Y(1-\delta)] - \frac{\sigma}{R}[(1 & \\ & + cos\tau)M_+[\frac{X-1}{X+1}]^{1-\delta}e^{-2\sigma\psi} + (1\\ & - cos\tau)M_-[\frac{X-1}{X+1}]^{\delta-1}e^{2\sigma\psi}\\ & + 2sin\tau[X(\lambda^2-\mu^2)(1-Y^2) + Y(1\\ & - \lambda^2\mu^2)(X^2-1)]] \end{aligned} \tag{18}$$

$$\boldsymbol{e^{2\eta}} = K_2 e^{2\sigma^2\gamma}(1-\lambda\mu)^2 - \frac{1-Y^2}{X^2-1}(\lambda+\mu)^2. \tag{19}$$

Other parameters in the above equation are given by

$$\begin{aligned} \lambda &= \alpha_1(X^2-1)^{1-\delta}(X+Y)^{2\delta-2}\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^n q_n\beta_n^-\right]\\ \mu &= \alpha_1(X^2-1)^{1-\delta}(X+Y)^{2\delta-2}\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^n q_n\beta_n^+\right]\\ R &= (X^2-1)(1-\lambda\mu)^2 - (1-Y^2)(\lambda+\mu)^2 \end{aligned} \tag{20}$$

where $\beta_n^\pm$ and the parameters $L_\pm, M_\pm, N_\pm$ are given by

$$\begin{aligned} \beta_n^\pm = (\pm 1)^n\frac{1}{2}\ln\frac{(X\mp Y)^2}{X^2-1} - (\pm 1)^n Q_1(X) &\\ + P_n(X)Q_{n-1}(X) & \end{aligned} \tag{21}$$

$$-\sum_{k=1}^{n-1}(\pm 1)^k P_{n-k}(Y)[Q_{n-k+1}(X) - Q_{n-k-1}(X)], \quad n > 1, \tag{22}$$

$$\begin{aligned} L_\pm = (1-\mu\lambda)[(X\pm 1)^2 - \lambda\mu(X\mp 1)^2] + (\lambda &\\ + \mu)[\mu(1\pm Y)^2 + \lambda(1\mp Y)^2], & \end{aligned} \tag{23}$$

$$\begin{aligned} M_\pm = (X^2-1)(1-\lambda\mu)[\lambda+\mu\mp Y(\lambda-\mu)] &\\ + (1-Y^2)(\lambda+\mu)[1-\lambda\mu &\\ \mp X(1+\lambda\mu)], & \end{aligned} \tag{24}$$

$$N_\pm = (\lambda+\mu)(1\pm\lambda\mu). \tag{25}$$

The parameters K_1, K_2 and α_1, α_2 are arbitrary and mostly chosen in a way that the metric is asymptotically flat. The last parameter $\hat{p}$ is defined by

$$\hat{p} = -2(1-Y^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}q_n[P_n(X)]_{,Y}[Q_{n+1}(X) - Q_{n-1}(X)] \quad (26)$$

Note that the transformation

$$X = \frac{r-\sigma}{\sigma}, \qquad Y = cos\theta \quad (27)$$

will bring us to the traditional spherical coordinates. As a result, the Kerr spacetime with all mass multipole moments will be defined through the following parameter setting (Quevedo,1990):

$$q_0 = 1, q_1 = 0 \;\; \delta = 1, \alpha_1 = -\frac{a}{\sigma}, \alpha_2 = 0$$
$$K_1 = -2a, K_2 = 1 \;\; sin\tau = -\frac{a}{m}, cos\tau = \frac{\sigma}{m}, \sigma^2 = m^2 - a^2 \quad (28)$$

Thus, for the Schwarchild spacetime the parameters are given by

$$q_0 \neq 0, \quad q_{n>0} = 0, \quad \delta = 1, \quad \alpha_1 = \alpha_2 = 0$$
$$K_1 = 0, \quad K_2 = 1, \quad sin\tau = 0, \quad cos\tau = 1, \quad \sigma = m \quad (29)$$

4. The multipole moments

The relativistic definition of moments was first given in (Geroch,1970). The Enrst potential was a real number for static spacetimes while now it is a complex number and the moments have two physically distinguished parts. The first part consists of the mass moments. For the Kerr parameters given by (28) the first four mass moments are

$$M_0 = m,$$
$$M_1 = m,$$
$$M_2 = \frac{2}{15}m^3(1-\frac{a^2}{m^2})^2 q_2 - ma^2, \quad (30)$$
$$M_3 = \frac{2}{15}m^2(1-\frac{a^2}{m^2})^{\frac{3}{2}}[q_2a^3 - \frac{3}{7}m^2q_3(1-\frac{a^2}{m^2})^2]$$

The second part identifies the angular moments as follows (again for Kerr spacetime)

$$J_0 = 0,$$
$$J_1 = ma,$$
$$J_2 = -\frac{2}{15}a(1-\frac{a^2}{m^2})^{\frac{3}{2}}m^3q_2, \quad (31)$$
$$J_3 = -ma^3 + \frac{2}{15}m^3a(1-\frac{a^2}{m^2})[2q_2 + \frac{3}{7}q_3]$$

As is evident, the arbitrary moments q_2, q_3, … are free and can be set adhoc. The idea is that we can set this free parameter using GPS data to find an exact axisymmetric stationary solution to EFEs to model the Earth's normal gravity. Recent studies (Backdahl & Herberthson,2005; Backdahl & Herberthson,2005; Backdahl,2007) are supporting this proposal according to which one can set adhoc any sequence of multipole moments q_n and find a corresponding metric. The following theorem shows such an interesting issue (see theorem 8 in (Backdahl,2007)).

Theorem: For every set of multipole moments q_n, such that the power series

$$\sum_{n=0}^{\infty}\frac{q_n}{n!}x^n. \quad (32)$$

has positive radius of convergence, there is a unique solution f to the EFEs .

5. Discussion

In this paper, we investigated the multipole structure of exact solutions of the EFEs. There is a common thought in the literature that the multipole structure of the famous solutions is not sufficiently reached to model the Earth's gravitational field. In the well-understood case of the Kerr spacetime, there is only a mass monopole and an angular momentum in the spectrum of the Ernst potential. Since there exist stationary axisymmetric solutions to the EFEs with arbitrary multipoles, we proposed an adhoc justification of the free multipole moments with help of GPS data provided by satellites. To do so one must already found at least hundreds of mutipole moments analytically. As such computation will be enormously complicated, the first four moments were demonstrated in equations (21). Other numerical simulations were postponed to separate work.

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