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Pair mean cordial labeling of graphs

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ABSTRACT

In this paper, we introduce a new graph labeling called pair mean cordial labeling of graphs. Also we investigate the pair mean cordiality of some graphs like path, cycle, complete graph, star, wheel, ladder, comb.

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1 Introduction

In this paper, we consider a finite, simple and undirected connected graphs. We follow the terminologies and different notations by the book of Harary [1]. For a detailed survey on graph labeling, we refer the book of Gallian(2021)[2]. The notion of mean labeling of graphs was introduced by S. Somasundaram and R. Ponraj [4]. The concept of pair difference cordial labeling was discussed in [5]. Motivated by these two concepts, In this paper we introduced new graph labeling called pair mean cordial labeling and also we investigate the pair mean cordial labeling behavior of several graph like path, cycle, wheel, ladder, comb, star and bistar graph.

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2 Preliminaries

Definition 1. *The ladder graph* $L_n (n \geq 2)$ *is defined by* $L_n = P_n \times K_2$ *where* P_n *is a path with* n *vertices and* $\times$ *denotes the Cartesian product and* K_2 *is a complete graph with two vertices.*

Definition 2. *The comb* $P_n \odot K_1$ *is obtained by joining a pendant edge to each vertices of the path* P_n*. It has* $2n$ *vertices and* $2n - 1$ *edges*

Definition 3. *The wheel graph* W_n *is defined to be the join of* $K_1 + C_n$ *i.e., the wheel graph consists of edges which join a vertex of* K_1 *to every vertex of* C_n*.*

Definition 4. *A helm graph* H_n *is a graph obtained from a wheel by attaching a pendant vertex at each* n*-cycle vertex. Then it has* $2n + 1$ *vertices and* $3n$ *edges.*

Definition 5. *The fan* $f_n (n \geq 2)$ *is obtained by joining all vertices of* P_n *to a further vertex called the center. It has* $n + 1$ *vertices and* $2n - 1$ *edges.*

3 Pair Mean Cordial Labeling

Definition 6. *Let a graph* $G = (V, E)$ *be a* (p, q) *graph. Define*

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ *called the set of labels. Consider a mapping* $\lambda : V \to M$ *by assigning different labels in* M *to the different elements of* V *when* p *is even and different labels in* M *to* $p - 1$ *elements of* V *and repeating a label for the remaining one vertex when* p *is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge* uv *of* G*, there exists a labeling* $\frac{\lambda(u)+\lambda(v)}{2}$ *if* $\lambda(u) + \lambda(v)$ *is even* $\frac{\lambda(u)+\lambda(v)+1}{2}$ *if* $\lambda(u) + \lambda(v)$ *is odd such that* $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ *where* $\bar{\mathbb{S}}_{\lambda_1}$ *and* $\bar{\mathbb{S}}_{\lambda_1^c}$ *respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph* G *for which there exists a pair mean cordial labeling is called a pair mean cordial graph.*

Theorem 7. *If* G *is a (p,q) pair mean cordial graph, then*

$$q \leq \begin{cases} 2p - 5 & \text{if } p \text{ is even} \\ 2p - 3 & \text{if } p \text{ is odd} \end{cases}$$

Proof. **Case 1**: p is even
The maximum number of edges with label 1 among the vertex labels $\pm 1, \pm 2, \pm 3, \ldots, \pm \frac{p}{2}$ is $p - 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1} \leq p - 3$. This implies that

$$\bar{\mathbb{S}}_{\lambda_1^c} \geq q - p + 3. \quad (1)$$

Type 1. $\bar{\mathbb{S}}_{\lambda_1^c} = \bar{\mathbb{S}}_{\lambda_1} + 1$

$$\begin{aligned} By(1), q - p + 3 &\leq \bar{\mathbb{S}}_{\lambda_1^c} \\ &\leq \bar{\mathbb{S}}_{\lambda_1} + 1 \\ &\leq p - 2 \end{aligned}$$

This implies that

$$q \leq 2p - 5 \tag{2}$$

Type 2. $\bar{\mathbb{S}}_{\lambda_1^c} = \bar{\mathbb{S}}_{\lambda_1} - 1$

$$\begin{aligned} By(1), q - p + 3 &\leq \bar{\mathbb{S}}_{\lambda_1^c} \\ &\leq \bar{\mathbb{S}}_{\lambda_1} - 1 \\ &\leq p - 4 \end{aligned}$$

This implies that

$$q \leq 2p - 7 \tag{3}$$

Type 3. $\bar{\mathbb{S}}_{\lambda_1^c} = \bar{\mathbb{S}}_{\lambda_1}$

$$\begin{aligned} By(1), q - p + 3 &\leq \bar{\mathbb{S}}_{\lambda_1^c} \\ &\leq \bar{\mathbb{S}}_{\lambda_1} \\ &\leq p - 3 \end{aligned}$$

This implies that

$$q \leq 2p - 6 \tag{4}$$

Then by (2),(3),(4), $q \leq 2p - 5$.
Case 2: p is odd
In this case, by definition of pair mean cordial labeling one vertex label is repeat. This vertex label contributes maximum two edges with label 1. Therefore, $\bar{\mathbb{S}}_{\lambda_1} \leq p - 3 + 2 = p - 1$. As in case (1), we get $q \leq 2p - 3$. □

Theorem 8. *Any path P_n is pair mean cordial for all n.*

Proof. Let P_n be the path $u_1 u_2 \dots u_n$. Then the path P_n has n vertices and $n - 1$ edges. This proof is divided into two cases:
Case 1: $n = 3$
Assign the labels $1, 1, -1$ to the vertices u_1, u_2, u_3 respectively. Then $\bar{\mathbb{S}}_{\lambda_1} = 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2$.
Case 2: $n \geq 1$
There are two subcases arises:
Subcase 1: n is even
Assign the labels $1, 2, 3, \dots, \frac{n}{2}$ to the vertices $u_1, u_3, \dots, u_{n-1}$ respectively. Next we assign the labels $-1, -2, -3, \dots, \frac{-n}{2}$ respectively to the vertices $u_2, u_4, \dots, u_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n-2}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n}{2}$.

Subcase 2: n is odd
Assign the labels $1, 2, 3, \ldots, \frac{n-1}{2}$ to the vertices $u_1, u_3, \ldots, u_{n-2}$ respectively. Next we assign the labels $-1, -2, -3, \ldots, \frac{-n+3}{2}$ respectively to the vertices $u_2, u_4, \ldots, u_{n-3}$. Finally we assign the labels $\frac{-n+3}{2}, \frac{-n+1}{2}$ to the vertices u_{n-1} and u_n respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n-1}{2}$. □

Remark 9. *P_3 is pair mean cordial but not pair difference cordial [5].*

Theorem 10. *The cycle C_n is pair mean cordial for all values of n except $n = 4$.*

Proof. Let the cycle C_n be $u_1u_2 \ldots u_nu_1$. Then the cycle C_n has n vertices and n edges. This proof is divided into two cases:
Case 1: For $n = 3$
Assign the labels $1, 1, -1$ to the vertices u_1, u_2, u_3 respectively. Then $\bar{\mathbb{S}}_{\lambda_1} = 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2$.
Case 2: $n \geq 4$
There are four subcases arises:
Subcase 1: $n \equiv 0(mod 4)$
First assign the label 1 to the vertex u_1. Assign the labels $2, 3, 4, \ldots, \frac{n+8}{4}$ to the vertices $u_2, u_4, u_6, \ldots, u_{\frac{n+4}{2}}$ respectively. Then we assign the labels $-1, -2, -3, \ldots, \frac{-n}{4}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{\frac{n+2}{2}}$. Next assign the labels $\frac{-n-8}{4}, \frac{-n-4}{4}$ to the vertices $u_{\frac{n+6}{2}}, u_{\frac{n+8}{2}}$ respectively. Now we give labels $\frac{-n-12}{4}, \frac{-n-16}{4}, \ldots \frac{-n}{2}$ respectively in the vertices $u_{\frac{n+10}{2}}, u_{\frac{n+12}{2}}, \ldots, u_{\frac{3n+8}{4}}$. Finally we give labels $\frac{n+12}{4}, \frac{n+16}{4}, \ldots \frac{n}{2}$ in the vertices $u_{\frac{3n+12}{4}}, u_{\frac{3n+16}{4}}, \ldots, u_n$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n}{2}$.
Subcase 2: $n \equiv 1(mod 4)$
First assign the label 1 to the vertex u_1. Assign the labels $2, 3, 4, \ldots, \frac{n+7}{4}$ to the vertices $u_2, u_4, u_6, \ldots, u_{\frac{n+3}{2}}$ respectively. Then we assign the labels $-1, -2, -3, \ldots, \frac{-n+1}{4}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{\frac{n+1}{2}}$. Next assign the labels $\frac{-n-7}{4}, \frac{-n-3}{4}$ to the vertices $u_{\frac{n+5}{2}}, u_{\frac{n+7}{2}}$ respectively. Now we give labels $\frac{-n-11}{4}, \frac{-n-15}{4}, \ldots \frac{-n+1}{2}$ respectively in the vertices $u_{\frac{n+9}{2}}, u_{\frac{n+11}{2}}, \ldots, u_{\frac{3n+5}{4}}$. We give labels $\frac{n+11}{4}, \frac{n+15}{4}, \ldots \frac{n-1}{2}$ in the vertices $u_{\frac{3n+9}{4}}, u_{\frac{3n+13}{4}}, \ldots, u_{n-1}$ respectively. Finally Assign the label $\frac{n-1}{4}$ to the vertex u_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+1}{2}$.
Subcase 3: $n \equiv 2(mod 4)$
First assign the label 1 to the vertex u_1. Assign the labels $2, 3, 4, \ldots, \frac{n+6}{4}$ to the vertices $u_2, u_4, u_6, \ldots, u_{\frac{n+2}{2}}$ respectively. Then we assign the labels $-1, -2, -3, \ldots, \frac{-n-2}{4}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{\frac{n+4}{2}}$. Now we give labels $\frac{-n-6}{4}, \frac{-n-10}{4}, \ldots \frac{-n}{2}$ in the vertices respectively $u_{\frac{n+6}{2}}, u_{\frac{n+8}{2}}, \ldots, u_{\frac{3n+6}{4}}$. Finally we give labels $\frac{n+10}{4}, \frac{n+14}{4}, \ldots \frac{n}{2}$ in the vertices $u_{\frac{3n+10}{4}}, u_{\frac{3n+14}{4}}, \ldots, u_n$ respectively. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n}{2}$.
Subcase 4: $n \equiv 3(mod 4)$
First assign the label 1 to the vertex u_1. Assign the labels $2, 3, 4, \ldots, \frac{n+5}{4}$ to the vertices $u_2, u_4, u_6, \ldots, u_{\frac{n+1}{2}}$ respectively. Then we assign the labels $-1, -2, -3, \ldots, \frac{-n-1}{4}$ respectively to the vertices $u_3, u_5, u_7, \ldots, u_{\frac{n+3}{2}}$. Now we give labels $\frac{-n-5}{4}, \frac{-n-9}{4}, \ldots \frac{-n+1}{2}$ in the vertices $u_{\frac{n+5}{2}}, u_{\frac{n+7}{2}}, \ldots, u_{\frac{3n+3}{4}}$ respectively. We give labels $\frac{n+9}{4}, \frac{n+13}{4}, \ldots \frac{n-1}{2}$ respectively

in the vertices $u_{\frac{3n+7}{4}}, u_{\frac{3n+11}{4}}$, $\dots, u_{n-1}$. Finally assign the label $\frac{n+1}{4}$ to the vertex u_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{n+1}{2}$. □

Remark 11. *C_4 is pair difference cordial but not pair mean cordial [5].*

Theorem 12. *The complete graph K_n is pair mean cordial if and only if $n \leq 3$.*

Proof. This proof is divided into two cases:
Case 1: $n \leq 3$
By theorem 3.5, K_1, K_2 and K_3 are pair mean cordial.
Case 2: $n > 3$
Suppose λ is a pair mean cordial labeling. If the edge uv get label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. There are two subcases arises:
Subcase 1: n is odd
In this case, the maximum number of edges with label 1 is $n-2$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq \frac{n(n-1)}{2} - (n-2) = \frac{n^2-3n+4}{2}$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq \frac{n^2-3n+4}{2} - (n-2) = \frac{n^2-5n+8}{2} > 1$, a contradiction.
Subcase 2: n is even
In this case, the maximum number of edges with label 1 is $n-3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq \frac{n(n-1)}{2} - (n-3) = \frac{n^2-3n+6}{2}$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq \frac{n^2-3n+6}{2} - (n-3) = \frac{n^2-5n+12}{2} > 1$, a contradiction. □

Theorem 13. *The star graph $K_{1,n}$ is pair mean cordial if and only if $1 \leq n \leq 6$.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Then $K_{1,n}$ has $n+1$ vertices and n edges. This proof is divided into two cases:
Case 1: $1 \leq n \leq 6$
The following table shows that $K_{1,n}$, $1 \leq n \leq 6$ is pair mean cordial.

Nature of n	u	u_1	u_2	u_3	u_4	u_5	u_6
1	1	-1					
2	1	-1	1				
3	-1	1	2	-2			
4	-1	1	2	-2	2		
5	-1	1	2	-2	3	-3	
6	-1	1	2	-2	3	-3	2

Case 2: $n > 6$
Suppose λ is a pair mean cordial labeling. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. There are two subcases arises:
Subcase 1: n is odd
In this case, the maximum number of edges with label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n-2$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n-2-2 = n-4 > 1$, a contradiction.

Subcase 2: n is even
In this case, the maximum number of edges with label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n-3$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n-3-3 = n-6 > 1$, a contradiction. □

Theorem 14. *The bistar graph $B_{1,n}$ is pair mean cordial if and only if $1 \leq n \leq 6$.*

Proof. Let $V(B_{1,n}) = \{u, v, u_1, v_i : 1 \leq i \leq n\}$ and $E(B_{1,n}) = \{uv, uu_1, vv_i : 1 \leq i \leq n\}$. Then $B_{1,n}$ has $n+3$ vertices and $n+2$ edges. This proof is divided into two cases:
Case 1: $1 \leq n \leq 6$
The following table shows that $B_{1,n}$, $1 \leq n \leq 6$ is pair mean cordial.
Define $\lambda(u) = -1$, $\lambda(u_1) = 2$

Nature of n	v	v_1	v_2	v_3	v_4	v_5	v_6
1	1	-2					
2	1	-2	1				
3	3	-2	-3	1			
4	3	-2	-3	1	1		
5	3	-2	-3	-4	4	1	
6	3	-2	-3	-4	4	1	-2

Case 2: $n > 6$
Suppose λ is a pair mean cordial labeling. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. There are two subcases arises:
Subcase 1: n is odd
In this case, the maximum number of edges with label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n+2-3 = n-1$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n-1-3 = n-4 > 1$, a contradiction.
Subcase 2: n is even
In this case, the maximum number of edges with label 1 is 4. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n+2-4 = n-2$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n-2-4 = n-6 > 1$, a contradiction. □

Theorem 15. *The bistar graph $B_{m,n}(m \geq 2, n \geq 2)$ is pair mean cordial if and only if $m+n \leq 9$.*

Proof. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(B_{1,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Then $B_{m,n}$ has $m+n+2$ vertices and $m+n+1$ edges. This proof is divided into two cases:
Case 1: $m+n \leq 9$
There are three subcases arises:
subcase 1: $m = 2, n = 2$
Define $\lambda(u) = -1, \lambda(u_1) = 1, \lambda(u_2) = 2, \lambda(v) = 3, f(\lambda_1) = -2, \lambda(v_2) = -3$. Hence $\bar{\mathbb{S}}_{\lambda_1} = 3$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 2$.
subcase 2: $m = 2, n > 2$
The following table shows that $B_{2,n}$, $3 \leq n \leq 7$ is pair mean cordial.

n	v	u_1	u_2	v_3	v_4	v_5	v_6	v_7
3	3	1	2	1				
4	3	1	2	-4	4			
5	4	2	3	-4	1	1		
6	4	2	3	-4	-5	5	1	
7	4	2	3	-4	-5	5	1	-3

Define $\lambda(u) = -1, \lambda(v_1) = -2, \lambda(v_2) = -3$
Subcase 3: $m > 2, n > 2$
If $m + n$ is even, define the function $\lambda : V(B_{m,n}) \to \{\pm 1, \pm 2, \ldots, \pm\frac{m+n+2}{2}\}$ by $\lambda(u) = -1, \lambda(v) = 4, \lambda(u_1) = 2, \lambda(u_2) = 3, \lambda(\lambda_1) = -2, \lambda(v_2) = -3, \lambda(v_3) = -2$. Next we assign the remaining labels to the remaining vertices in any order.
If $m + n$ is odd, define the function $f : V(B_{m,n}) \to \{\pm 1, \pm 2, \ldots, \pm\frac{m+n+1}{2}\}$ by $\lambda(u) = -1, \lambda(v) = 4, f(\lambda_1) = 2, f(\lambda_2) = 3, \lambda(v_1) = -2, \lambda(v_2) = -3$. Next we assign the remaining labels to the remaining vertices in any order.
Case 2: $m + n > 9$
Suppose λ is a pair mean cordial labeling. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$.
There are two subcases arises:
Subcase 1: $m + n$ is even
In this case, the maximum number of edges with label 1 is 4. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq m + n + 1 - 4 = m + n - 3$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m + n - 3 - 4 = m + n - 7 > 1$, a contradiction.
Subcase 2: $m + n$ is odd
Then the maximum number of edges with label 1 is 5. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 5$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq m + n + 1 - 5 = m + n - 4$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m + n - 4 - 5 = m + n - 9 > 1$, a contradiction. □

Theorem 16. *The comb $P_n \odot K_1$ is pair mean cordial.*

Proof. Let P_n be the path $u_1 u_2 \ldots u_n$. Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Then $P_n \odot K_1$ has $2n$ vertices and $2n - 1$ edges.
Define the function $\lambda : V(P_n \odot K_1) \to \{\pm 1, \pm 2, \ldots, \pm n\}$ by $\lambda(u_i) = i$, for $1 \leq i \leq n$, $\lambda(v_i) = -i + 1$, for $2 \leq i \leq n$ and $\lambda(v_1) = -n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = n - 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$. □

Theorem 17. *The ladder L_n is pair mean cordial for all values of n except $n \neq 2$*

Proof. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n\}$. Then the ladder graph L_n has $2n$ vertices and $3n - 2$. For $n = 2$, $L_2 \simeq C_4$ is not a pair mean cordial. This proof is divided into two cases:
Case 1: $n \equiv 0(mod 4)$
First assign the label $-1, -3, -5, \ldots, -n + 1$ to the vertices $u_1, u_3, u_5, \ldots, u_{\frac{n+1}{2}}$ respectively. Assign the labels $3, 5, 7, \ldots, \frac{n+2}{2}$ respectively to the vertices $u_2, u_4,$

$u_6, \ldots, u_{\frac{n}{2}}$. Then we assign the labels $\frac{-n-4}{2}, \frac{-n-8}{2}, \ldots, -n$ to the vertices $u_{\frac{n+4}{2}}$, $u_{\frac{n+8}{2}}, \ldots, u_n$ respectively. Next we assign the labels $2, 4, 6, \ldots, n$ respectively to the vertices $v_1, v_3, v_5, \ldots, v_{n-1}$. Now we give labels $-2, -4, -6, \ldots, \frac{-n}{2}$ in the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n}{2}}$ respectively. We give labels $\frac{-n-4}{2}, \frac{-n-8}{2}, \ldots, n$ respectively in the vertices $v_{\frac{n+6}{2}}, v_{\frac{n+10}{2}}, \ldots, v_{n-2}$. Finally assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-2}{2}n - 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-2}{2}$.

Case 2: $n \equiv 1(mod4)$

First assign the label $-1, -3, -5, \ldots, \frac{-n-1}{2}$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. Assign the labels $3, 5, 7, \ldots, n$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Then we assign the labels $\frac{n+7}{2}, \frac{n+11}{2}, \ldots, n-1$ to the vertices $u_{\frac{n+5}{2}}$, $u_{\frac{n+9}{2}}, \ldots, u_{n-2}$ respectively. Assign the label 1 to the vertex u_n. Next we assign the labels $2, 4, 6, \ldots, \frac{n+3}{2}$ respectively to the vertices $v_1, v_3, v_5, \ldots, v_{\frac{n+1}{2}}$. Now we give labels $-2, -4, -6, \ldots, -n+1$ in the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$ respectively. Finally we give labels $\frac{-n-5}{2}, \frac{-n-9}{2}, \ldots, -n$ respectively in the vertices $v_{\frac{n+5}{2}}, v_{\frac{n+9}{2}}, \ldots, v_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-4}{2}$.

Case 3: $n \equiv 2(mod4)$

First assign the label $-1, -3, -5, \ldots, \frac{-n}{2}$ to the vertices $u_1, u_3, u_5, \ldots, u_{\frac{n}{2}}$ respectively. Assign the labels $3, 5, 7, \ldots, n-1$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-2}$. Then assign the labels $\frac{n+6}{2}, \frac{n+10}{2}, \ldots, n$ to the vertices $u_{\frac{n+4}{2}}, u_{\frac{n+8}{2}}, \ldots, u_{n-1}$ respectively. Finally assign the 1 to the vertex u_n. Next we assign the labels $-2, -4, -6, \ldots, -n$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_n$. Now we give labels $2, 4, 6, \ldots, \frac{n+2}{2}$ in the vertices $v_1, v_3, v_5, \ldots, v_{\frac{n}{2}}$ respectively. Finally we give labels $\frac{-n-4}{2}, \frac{-n-8}{2}, \ldots, -n+1$ respectively in the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+8}{2}}, \ldots, v_{n-1}$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-2}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-2}{2}$.

Case 4: $n \equiv 3(mod4)$

First assign the label $-1, -3, -5, \ldots, -n$ to the vertices $u_1, u_3, u_5, \ldots, u_n$ respectively. Assign the labels $3, 5, 7, \ldots, \frac{n+3}{2}$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{\frac{n+1}{2}}$. Then we assign the labels $\frac{-n-5}{2}, \frac{-n-9}{2}, \ldots, n-1$ to the vertices $u_{\frac{n+5}{2}}, u_{\frac{n+9}{2}}, \ldots, u_{n-1}$ respectively. Next we assign the labels $-2, -4, -6, \ldots, \frac{-n-1}{2}$ respectively to the vertices $v_2, v_4, v_6, \ldots, v_{\frac{n+1}{2}}$. Now we give labels $2, 4, 6, \ldots, n-1$ in the vertices $v_1, v_3, v_5, \ldots, v_{n-2}$ respectively. We give labels $\frac{n+7}{2}, \frac{n+11}{2}, \ldots, n$ respectively in the vertices $v_{\frac{n+5}{2}}, v_{\frac{n+9}{2}}, \ldots, v_{n-1}$. Finally assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n-4}{2}$. □

Theorem 18. *$L_n \odot K_1$ is pair mean cordial for all $n \geq 2$.*

Proof. Let $V(L_n \odot K_1) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(L_n \odot K_1) = \{u_iv_i, u_ix_i, v_iy_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n\}$. Then the graph $L_n \odot K_1$ has $4n$ vertices and $5n-2$. This proof is divided into two cases:

Case 1: n is odd

First assign the labels $-1, -5, -9, \ldots, -2n+1$ to the vertices $u_1, u_3, u_5, \ldots, u_n$ respectively. Then we assign the labels $4, 8, 12, \ldots, 2n-2$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_{n-1}$. Next we assign the labels $2, 6, 10, \ldots, 2n$ to the vertices $x_1, x_3, x_5, \ldots, x_n$ respectively. Now

we give labels $-3, -7, -11, \ldots,$
$-2n+3$ respectively in the vertices $x_2, x_4, x_6, \ldots, x_{n-1}$. Assign the labels $3, 5, 7, \ldots, 2n+1$ to the vertices $v_1, v_2, v_3, \ldots, v_{n-1}$ respectively. Next we assign the label 1 to the vertex v_n. Finally we give labels $-2, -4, -6, \ldots, -2n$ respectively in the vertices $y_1, y_2, y_3, \ldots, y_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-3}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{5n-1}{2}$.
Case 2: n is even
First assign the labels $-1, -5, -9, \ldots, -2n+3$ to the vertices $u_1, u_3, u_5, \ldots, u_{n-1}$ respectively. Then we assign the labels $4, 8, 12, \ldots, 2n$ respectively to the vertices $u_2, u_4, u_6, \ldots, u_n$. Next we assign the labels $2, 6, 10, \ldots, 2n-2$ to the vertices $x_1, x_3, x_5, \ldots, x_{n-1}$ respectively. Now we give labels $-3, -7, -11, \ldots, -2n+1$ respectively in the vertices $x_2, x_4, x_6, \ldots, x_n$. Assign the labels $3, 5, 7, \ldots, 2n-1$ to the vertices $v_1, v_2, v_3, \ldots, v_{n-1}$ respectively. Next we assign the label 1 to the vertex v_n. Finally we give labels $-2, -4, -6, \ldots, -2n$ respectively in the vertices $y_1, y_2, y_3, \ldots, y_n$. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{5n-2}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{5n-2}{2}$. □

Theorem 19. *The wheel W_n is not a pair mean cordial graph for all $n \geq 3$.*

Proof. Let $W_n = C_n + K_1$ be a wheel graph where C_n is the cycle $u_1u_2\cdots u_nu_1$ and $V(K_1) = u$. Then the graph W_n has $n+1$ vertices and $2n$ edges. If possible, let there be a pair mean cordial labeling λ. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. This proof is divided into two cases:
Case 1: n is odd
In this case, the maximum number of edges with label 1 is $n-2$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n+2$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+2-n+2 = 4 > 1$, a contradiction.
Case 2: n is even
In this case, the maximum number of edges with label 1 is $n-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq n+1$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+1-n+1 = 2 > 1$, a contradiction. □

Theorem 20. *The helm H_n is a pair mean cordial graph for all $n \geq 3$.*

Proof. Let $V(H_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(H_n) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$. Then H_n consists of $2n+1$ vertices and $3n$ edges. This proof is divided into two cases:
Case 1: n is odd
Fix the the label 3 to the vertex u. Next assign the label -1 to the vertex u_1. We assign the labels $-2, -4, \cdots -n+1$ to the vertices $u_2, u_4, \cdots, u_{n-1}$ respectively. Now we give labels $4, 6, \cdots, n-1$ respectively in the vertices $u_3, u_5, \cdots, u_{n-2}$. Next we fix the label 1 to the vertex u_n. Then assign the label 2 to the vertex v_1. Then we assign the labels $3, 5, \cdots, n$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Also we assign the the labels $-3, -5, \cdots, -n$ respectively to the vertices $v_3, v_5, \cdots, v_n$. Finally assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n-1}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n+1}{2}$.
Case 2: n is even
Fix the the label 3 to the vertex u. Next we assign the label -1 to the vertex u_1. We assign the labels $-2, -4, \cdots -n$ to the vertices $u_2, u_4, \cdots, u_n$ respectively. Now we give labels $4, 6, \cdots, n$ respectively in the vertices $u_3, u_5, \cdots, u_{n-1}$. Next we fix the label 2 to the vertex v_1. Then assign the labels $3, 5, \cdots, n-1$ to the vertices $v_2, v_4, \ldots, v_{n-2}$ respectively. Also

we assign the the labels $-3, -5, \cdots, -n+1$ respectively to the vertices $v_3, v_5, \cdots, v_{n-1}$. Finally assign the label 1 to the vertex v_n. Hence $\bar{\mathbb{S}}_{\lambda_1} = \frac{3n}{2}$ and $\bar{\mathbb{S}}_{\lambda_1^c} = \frac{3n}{2}$. □

Theorem 21. *The fan graph $f_n = P_n + K_1$ is pair mean cordial if n is even and $n \neq 4$*

Proof. Let P_n be the path $u_1u_2 \dots u_n$. Let $V(f_n) = V(P_n) \cup \{v\}$, $E(f_n) = E(P_n) \cup \{u_iv : 1 \leq i \leq n\}$. Then the fan graph f_n has $n+1$ vertices and $2n-1$ edges. When $n = 4$, suppose λ is a pair mean cordial of f_4. Then the maximum number of edges label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Hence $\bar{\mathbb{S}}_{\lambda_1^c} \geq 5$. Thus $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \leq 5 - 2 = 3 > 1$ a contradiction. Let n be even and $n \neq 4$. Then we define the function $\lambda : V(f_n) \to \{\pm 1, \pm 2, \dots, \pm \frac{n}{2}\}$ by

$$\begin{aligned}
\lambda(u_1) &= 1, \\
\lambda(u_{2i}) &= i+1, \text{for } 1 \leq i \leq \frac{n-2}{2}, \\
\lambda(u_{2i+1}) &= -i, \text{for } 1 \leq i \leq \frac{n-2}{2}, \\
\lambda(u_n) &= \frac{-n}{2}, \\
\lambda(v) &= \frac{n}{2}.
\end{aligned}$$

Hence $\bar{\mathbb{S}}_{\lambda_1} = n - 1$ and $\bar{\mathbb{S}}_{\lambda_1^c} = n$. □

Theorem 22. *The fan graph $f_n = P_n + K_1$ is not pair mean cordial if n is odd and $n > 1$.*

Proof. For $n = 1$, $f_1 \simeq P_2$ is a pair mean cordial. Let n be odd and $n > 1$. Suppose λ is a pair mean cordial. Then if the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is $n + 1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq n+1$. Thus $\bar{\mathbb{S}}_{\lambda_1^c} \geq 2n-1-n-1 = n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq n+1-n+2 = 3 > 1$, a contradiction. □

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