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RESEARCH PAPER

Weak Disposability of Input and Output in a Nonparametric Production Analysis

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Abstract
In some production processes, unlike traditional data envelopment analysis (DEA), decision-making units (DMUs) consume widely used inputs since increasing them is more favorable for improving efficiency. Plastic wastes and rotten fruits are two instances of widely used inputs for recycling factories. In this paper, weak disposability of inputs will be presented with non-uniform profit factors for inputs, and then the model will be extended to cases that consume normal and widely used inputs, and produce desirable and undesirable outputs, simultaneously. Due to a nonlinear form of the final model, a linearization method is presented to provide a linear structure of the technology. A directional distance function approach, including weakly disposable inputs and outputs, is introduced to deal with widely used inputs and undesirable outputs. Finally, the approach will be tested by applying it to some domestic sewage treatment plants in China. Results show the proposed approach can discriminate between DMUs in a rational way and with less computational effort and also it can be used for the efficiency analysis when widely used inputs and undesirable outputs are presented.
Keywords: Data Envelopment Analysis, DMU, Efficiency, Production Possibility Set, Weak Disposability.
JEL Classification: C61, C67, D24.

Introduction

Economic development is the growth of the living standards of society from poor quality to rich. Based on this concept, one of the important reasons for economic development is the presence of effective manufacturing units. The high cost of materials, workforce, and consumed energies make manufacturers implement a practical effective operating plan for a production process. Since Charnes et al. (1978), data envelopment analysis (DEA) has been mostly used to calculate the relative efficiency of comparable decision-making units (DMUs) which consume multiple inputs to produce multiple outputs. In traditional DEA approaches, decreasing inputs and increasing outputs have a positive effect on operational performance. In recent years, there has been a growing interest in assessing the production process by taking undesirable outputs into account. Undesirable outputs (bad outputs) are produced together with desirable outputs (good outputs) in the production process. Greenhouse gas emission (a mixture of carbon dioxide, methane, nitrous oxide, water vapor, and ozone), noise pollution, wastewater, and hospital waste is the main instances of undesirable outputs, which their production leads to environmental pollutions and global warming. To limit the difficulties, it

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is necessary to make changes in the global economic system. Hence, firms should reduce the number of undesirable outputs to protect the environment. Shephard (1974) introduced the weak disposability axiom to consider harmful products. According to the weak disposability definition, a proportional reduction of possible products is feasible. But not much attention has been paid to the widely used inputs that need to be increased.

Therefore, in this paper, the weak disposability axiom is represented to the production possibility sets that consume widely used inputs, and then a DEA-based approach is proposed to the simultaneous presence of widely used inputs and undesirable outputs.

The remainder of this paper is organized as follows. A short literature review is represented in Section 2. Section 3 gives a brief definition of the weak disposability of outputs. Section 4 defines the weak disposability of inputs and their technology. In Section 5, new technology under the weak disposability of input-output factors is proposed, and a linearization method is described to provide a linear structure of the proposed technology. Section 6 tests the new proposed model with an application in domestic sewage treatment plants, and Section 7 concludes the paper.

Literature Review

In recent years, many studies focused on the production systems that generate undesirable outputs. Some of them are based on the non-parametric, deterministic DEA approach and the number of them is founded on the stochastic frontier approach (SFA) that is parametric, stochastic. Among cases that harmful factors can be seen are the environment and agriculture. Many studies have focused on the evaluation and improvement of these sectors. For example, Dahmardeh and Sardar Shahraki (2015) focused on the evaluation factors affecting risk production in Sistan grape growers by using SFA. Sardar Shahraki and Karim (2018) represented a DEA-based method to evaluate the efficiency of date growers in Saravan County. Karim and Sardar Shahraki (2019) evaluated the performance of Khash township pomegranate producers with a DEA-based approach. Sardar Shahraki (2019) examined the effectiveness of pomegranate gardens in the Sistan region and used the SFA method to analyze the data. Vishwakarma et al. (2012) provided a DEA-based method to evaluate the performance of municipal solid waste management services.

Some DEA pieces of literature on dealing with the undesirable outputs transform the values to their reciprocals (Lovell et al., 1995; Athanassopoulos and Thanassoulis, 1995). Scheel (2001) and Seiford and Zhu (2002) convert the detrimental outputs to the positive desirable products by a linear monotone decreasing transformation. Zhu and Chen (2012), Liang et al. (2009), and Puri and Yadav (2014) converted bad outputs to be desirable by reversing. Hailu and Veeman (2001), Khalili Damghani et al. (2015), and Reinhard et al. (2000) dealt with undesirable outputs as inputs. The treatment of detrimental outputs as inputs is incompatible with the physical laws and the axioms of production theory (Fare and Grosskopf, 2003; 2004). Hence, the authors present an alternative approach to deal with undesirable outputs as desirable outputs by providing a production technology, using the weak disposability definition by Shephard (1974). According to the weak disposability definition, a proportional reduction of possible products is feasible. Despite the mentioned studies that focus on decreasing undesirable outputs, in many real cases, some inputs should be increased to improve the firm's performance. For example, in the recycling industry, the wastes increase leads to producing more recycled materials. The problem of landfilling is an important challenge of environmental issues since leads to producing harmful gases. Hence, recycling the wastes reduces environmental pollutions. On the other hand, there is a significant cost saving in using recycled materials. For example, recycling leaves and the grass is a good procedure to produce compost. Producing this compost at home is cheaper than buying it. According to the mentioned advantages, improving the performance of recycling is one of the ways for improving the quality of life.

Huang and Li (2013) proposed an undesirable input/output with a two-phase algorithm to provide an evaluation method of environmental performance, while the traditional DEA models were incorporated with the transformation method. Mehdiloozad and Podinovski (2018) focused on the weak disposability of inputs. They presented a range of production technologies by considering the weak disposability of inputs; while they did not argue about the application of their definition of weak disposability of inputs in real cases. On the other hand, they did not design the technology of weak disposability of inputs-outputs.

In this paper, the DEA-based approach is extended to the simultaneous presence of widely used inputs and undesirable outputs, and a mathematical technology is proposed. According to the non-linear structure of the proposed technology, a linearization method is presented. Also, a directional distance function is provided for the linear technology to calculate the efficiency measure.

Weak Disposability of Outputs

The original DEA models rely on maximizing outputs and minimizing inputs to improve the efficiency measure. However, in many real applications, there are some undesirable factors (i.e. waste and polluting emissions) that should be minimized. If there are K peer DMUs $\{DMU_k : k=1,\dots,K\}$, and each DMU has N inputs, S desirable outputs, and H undesirable outputs, the input vector will be denoted by $x=(x_1,\dots,x_N)\in R_+^N$, the desirable output vector will be denoted by $v=(v_1,\dots,v_S)\in R_+^S$, and the undesirable output vector will be shown by $w=(w_1,\dots,w_H)\in R_+^H$. The production possibility set is indicated either by $P=\{(x,v,w)|x\, can\, produce(v,w)\}$ or by the output set $P(x)=\{(v,w)|(x,v,w)\in P\}$.

According to Shephard (1970), the weak disposability of outputs means that it is possible to reduce a proportion of outputs. In other words, firms can limit bad outputs by decreasing the activity level. Formally, if $(v,w)\in P(x)$ and $0\le\theta\le 1$ imply $(\theta v,\theta w)\in P(x)$, outputs will be weakly disposable (θ is the abatement factor). Fare and Grosskopf (2003) proposed the following algebraic technology using the introduced weak disposability definition.

$$
\begin{aligned}
P(x)=\{(v,w)\,|\,&\sum_{k=1}^{K} x_n^k z^k \le x_n, && n=1,\dots,N\\
&\theta\sum_{k=1}^{K} v_s^k z^k \ge v_s, && s=1,\dots,S\\
&\theta\sum_{k=1}^{K} w_h^k z^k = w_h, && h=1,\dots,H\\
&\sum_{k=1}^{K} z^k = 1, && k=1,\dots,K\\
&z^k \ge 0,\\
&0\le\theta\le 1\}
\end{aligned}
\tag{1}
$$

where the abatement factor θ allows the simultaneous contraction of desirable and undesirable outputs, and the fourth constraint follows variable returns to scale (VRS). Normal inputs and desirable outputs are strongly disposable, but outputs (bad and good) are weakly disposable. Fare and Grosskopf (2003) assumed that all DMUs use the same reducing factor. Kuosmanen (2005) claimed that Shephard's definition of weak disposability, advocated by Fare and Grosskopf (2003), was not cost-beneficial. Thus, different reducing factors for the activities are necessary. They assumed that the production technology followed the conditions of the strong disposability of inputs and good outputs, weak disposability of bad outputs, and convexity. The algebraic technology of Kuosmanen (2005) under VRS is represented as follows:

$$
\begin{aligned}
P(x)=\{(v,w)\mid & \sum_{k=1}^{K} x_n^k z^k \le x_n, && n=1,\ldots,N \\
& \sum_{k=1}^{K} \theta^k v_s^k z^k \ge v_s, && s=1,\ldots,S \\
& \sum_{k=1}^{K} \theta^k w_h^k z^k = w_h, && h=1,\ldots,H \\
& \sum_{k=1}^{K} z^k = 1, && k=1,\ldots,K \\
& z^k \ge 0, \\
& 0 \le \theta^k \le 1\}
\end{aligned} \tag{2}
$$

where θ^k and z^k are the reducing factors and intensity weight of DMU_k, respectively. Kuosmanen (2005) indicated that the proposed technology was the minimal technology that followed the above conditions, and contained all observed activities. Shephard's technology does not follow the convexity axiom and is not the smallest technology based on two remaining conditions. Yet, Kuosmanen (2005) proposed a simple method to represent a linear structure of this technology. The intensity weight of firm *k* decomposes into two parts as follows:

$$z^k = \lambda^k + \mu^k$$

where λ^k belongs to the part of DMU k's output which remains active (i.e., $\theta^k z^k = \lambda^k$), and μ^k belongs to the reduced part of DMU k's output (i.e., $(1-\theta^k)z^k = \mu^k$). Using this change of variables, the technology transforms to the following structure:

$$
\begin{aligned}
P(x)=\{(v,w)\mid & \sum_{k=1}^{K} (\lambda^k+\mu^k) x_n^k \le x_n, && n=1,\ldots,N \\
& \sum_{k=1}^{K} \lambda^k v_s^k \ge v_s, && s=1,\ldots,S \\
& \sum_{k=1}^{K} \lambda^k w_h^k = w_h, && h=1,\ldots,H \\
& \sum_{k=1}^{K} \left(\lambda^k+\mu^k\right) = 1, && k=1,\ldots,K \\
& \lambda^k, \mu^k \ge 0\}
\end{aligned} \tag{3}
$$

The dominant approach in weak disposability is the abating output into a nonparametric analysis of productivity growth. Yet, contrary to the procedure of many DEA approaches, in some manufacturing processes, some inputs should increase.

Weak Disposability of Inputs

In many manufacturing processes, the DMU consumes some inputs, which should be increased, called “widely used”. Plastic wastes and rotten fruits as the raw material of recycling factories are the instances of these inputs. The municipalities are responsible for collecting wastes from the cities and towns. The materials are transferred to the recycling factories to be converted into new materials. This process reduces energy usage and environmental pollutions. Consuming more wastes increases the performance of recycling units. On the other hand, an increase of 20% of wastes means an increase of 20% in the required electricity for conversion operations.

Today, banks give various services to attract more customers to increase their deposits. So, to evaluate the banks’ performance, several customers can be widely studied as inputs. However, the number of customers and the time spent on the services can be proportionally scaled up as inputs. The weak disposability of inputs is defined in the following.

With the assumption of notations in the previous section, we assume that the widely used vector is denoted by $c=(c_1,\ldots,c_N)$, and the production possibility set is denoted by $L=\{(x,c,v)|(x,c)\, can\ produce\, v\}$, or by the input set of $L(y)=\{(x,c)|(x,c,v)\in L\}$.

Definition 1.3: Inputs are weakly disposable if:

$$(x,c)\in L(y),\ 1\le \varphi\ \Rightarrow\ (\varphi x,\varphi c)\in L(y)$$

Using the inputs which increase the operations requires scaling the activity level. Hence, the multiplier φ can be interpreted as a profit factor. Figure 1 illustrates the mentioned definition of weak disposability of inputs. The firm has been labeled as A. Referring to the diagram, we assume that A belongs to the production technology set, which consumes one normal input and one widely used input to produce the same output.

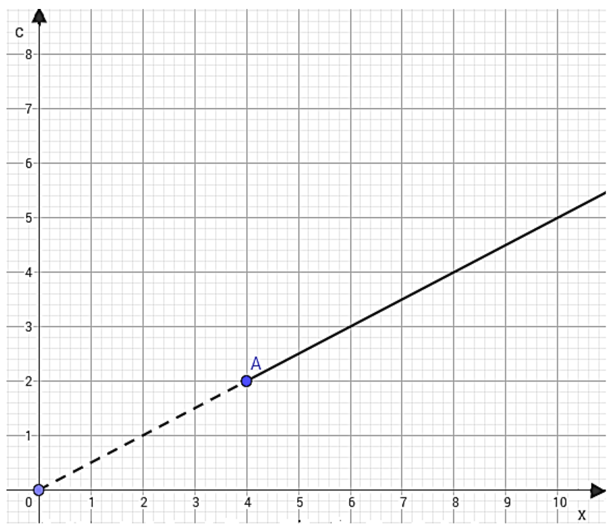

Figure 1. Weak Disposability of Inputs
Source: Research finding.

Based on definition 1.3, all points of the thick half-line of the diagram, made by the extension of line segment OA, belong to the production possibility set (PPS). Increasing the inputs is proportional according to the represented condition $1\le\varphi$ in the definition.

We assume that the production technology follows the below suppositions:

A1) Inclusion of observations: $(x^k,c^k,v^k)\in L$ for $k=1,\ldots,K$.

A2) Free disposability of outputs and normal inputs: If $(x,c,v)\in L$, $x'\geq x$, $v'\leq v$ and $c'=c$, then $(x',c',v')\in L$.

A3) Weak disposability of inputs: If $(x,c,v)\in L$ and $\varphi\geq 1$, then $(\varphi x,\varphi c,v)\in L$.

A4) Convexity: If $(x^1,c^1,v^1)\in L$ and $(x^2,c^2,v^2)\in L$, then for any scalar $\delta\in[0,1]$, $\delta(x^1,c^1,v^1)+(1-\delta)(x^2,c^2,v^2)\in L$.

A5) Minimum extrapolation: L is the smallest production possibility set which follows the suppositions A1-A4.

The above suppositions are followed by the below production possibility set under variable return to scale supposition:

$$
\begin{aligned}
L(y)=\{(x,c)\,|\,&\sum_{k=1}^{K}\varphi^k x_n^k z^k \leq x_n, && n=1,\ldots,N\\
&\sum_{k=1}^{K}\varphi^k c_m^k z^k = c_m, && m=1,\ldots,M\\
&\sum_{k=1}^{K} v_s^k z^k \geq v_s, && s=1,\ldots,S\\
&\sum_{k=1}^{K} z^k = 1, && k=1,\ldots,K\\
&z^k \geq 0,\\
&\varphi^k \geq 1\}
\end{aligned}
\tag{4}
$$

The technology is nonempty and follows suppositions A1-A5. The fourth constraint forms the convex combination of the observed DMUs. The first and the third restrictions indicate the strong disposability of normal inputs and outputs. Incorporating φ^k as the profit factor into the first and second constraints allows simultaneous increasing the normal widely used inputs. The last constraint applies to the growth of waste consumption by the recycling units. The proofs of all mentioned suppositions are provided in the appendix. To illustrate the mentioned technology set, a numerical example is provided. Table 1 indicates the hypothetical data of three DMUs (A, B, and C) that consume one normal and one widely used input to produce one output. Figure 2 indicates the input set, while it is bounded by the horizontal extension from B, the segment AB, and the extension from B to the northwest.

Table 1. Data and Efficiency Results

DMU	x	C	v	Efficiency
A	2	3	1	1
B	5	1	1	7.5
C	4	4	1	1.5

Source: Research finding.

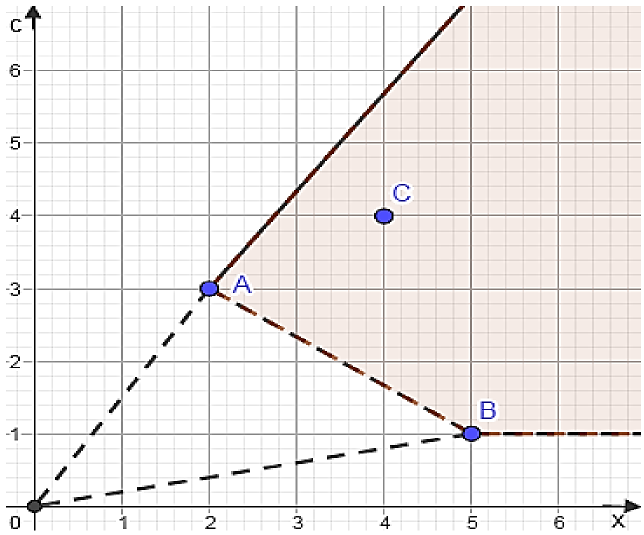

Figure 2. Input Set with Output Unit Value
Source: Research finding.

Due to a multiplication of profit factor φ^k and the intensity variable z^k in the constraints of inputs, the mentioned technology set (4) has a nonlinear structure. To provide a linear version, let $z^k = \lambda^k - \mu^k$, $\lambda^k = \varphi^k z^k$ and $\mu^k = \left(\varphi^k - 1\right) z^k$. The linear version of the technology is presented as Equation (5):

$$
\begin{aligned}
L(y) = \{(x,c) \mid & \sum_{k=1}^{K} \lambda^k x_n^k \le x_n, && n = 1,\dots,N \\
& \sum_{k=1}^{K} \lambda^k c_m^k = c_m, && m = 1,\dots,M \\
& \sum_{k=1}^{K} v_s^k \left(\lambda^k - \mu^k\right) \ge v_s, && s = 1,\dots,S \\
& \sum_{k=1}^{K} (\lambda^k - \mu^k) = 1, && k = 1,\dots,K \\
& \lambda^k, \mu^k \ge 0\}
\end{aligned} \tag{5}
$$

The intensity variable z^k has been separated as the difference of two components to decompose inputs into two parts. μ^k is related to that part of the input which has been increased due to activity level growth, and λ^k is related to that part of the input which has been remained active. To calculate the relative efficiency of DMU_o, the following measure is proposed:

$$
\begin{aligned}
& Max\,\rho \\
\text{s.t.} \quad & \sum_{k=1}^{K} \lambda^k x_n^k \le x_n^o, && n = 1,\dots,N \\
& \sum_{k=1}^{K} \lambda^k c_m^k = \rho c_m^o, && m = 1,\dots,M
\end{aligned} \tag{6}
$$

$$\sum_{k=1}^{K} v_s^k \left(\lambda^k - \mu^k\right) \geq v_s^o, \qquad s = 1,\ldots,S$$
$$\sum_{k=1}^{K} (\lambda^k - \mu^k) = 1, \qquad k = 1,\ldots,K$$
$$\lambda^k, \mu^k \geq 0$$

where under the variable returns to scale assumption, widely used inputs are maximized. DMU_o is presented as an efficient unit if the efficiency measure is equal to 1. The efficiency measure of an inefficient unit is more than 1. By applying Model (6) on the hypothetical data of Table 1, the thick half-line extended from A is identified as the efficient frontier. To gain further insight, we consider the data set in Table 2 with a normal input, a widely used input, and a non-unit output.

Table 2. Data and Efficiency Results

DMU	x	c	v	Efficiency
A	15	3	2	7.6
B	2	4	1	1
C	9	4	3	1

Source: Research finding.

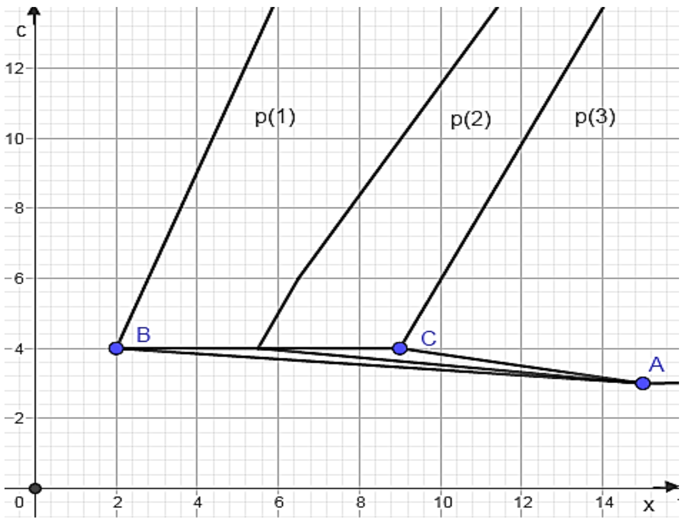

Figure 3. Input Set with Output Non-Unit Value
Source: Research finding.

B and C are efficient units. As can be seen in Figure 3, A is located in the area of the technology but it is not efficient. The technology area is limited by the horizontal line passing through A, connecting the units A, B, and the diagonal line passing through B. The lines make the input isoquants, A is the efficient unit of the technology area, which is restricted by the horizontal line passing through A, which connects the units A, C, and the diagonal line passing through C.

Weak Disposability of Inputs-Outputs

Now, the calculations are extended to the cases which consume normal and widely used inputs and produce desirable and undesirable outputs, simultaneously. In many applications, especially recycling, decision-makers deal with the processes in which raw materials are consumed, and harmful emissions are produced. The production possibility set is designed as $T=\{(x,c,v,w)|(x,c)\, can\ produce (v,w)\}$.

Definition 1.4: If $(x,c,v,w)\in T$, $0\le\theta\le 1$ and $\varphi\ge 1$ imply $(\varphi x,\varphi c,\theta v,\theta w)\in T$, the production possibility set is weakly disposable of input/output.

Implementing this definition leads to a proportional reduction in desirable and undesirable outputs by reducing the factors θ, and a proportional expansion in normal and widely used inputs by the profit factors φ.

We assume that the production technology follows the below suppositions:

P1) Inclusion of observations
P2) Free disposability of desirable outputs and normal inputs
P3) Weak disposability of inputs
P4) Weak disposability of outputs
P5) Convexity
P6) Minimum extrapolation

The production possibility set under VRS supposition is as follows:

$$
\begin{aligned}
T=\{(x,c,v,w)\,|\, & \sum_{k=1}^{K}\varphi^k x_n^k z^k \le x_n, && n=1,\ldots,N\\
& \sum_{k=1}^{K}\varphi^k c_m^k z^k = c_m, && m=1,\ldots,M\\
& \sum_{k=1}^{K}\theta^k v_s^k z^k \ge v_s, && s=1,\ldots,S\\
& \sum_{k=1}^{K}\theta^k w_h^k z^k = w_h, && h=1,\ldots,H\\
& \sum_{k=1}^{K} z^k = 1, && k=1,\ldots,K\\
& z^k\ge 0,\\
& \varphi^k \ge 1,\\
& 0\le\theta^k\le 1\}
\end{aligned}
\tag{7}
$$

The technology is nonempty and follows the suppositions P1-P6. The first and the third restrictions show the disposability of normal inputs and outputs. By incorporating φ^k and θ^k as the profit factor of inputs and reducing factor of the outputs, respectively, into the first to the fourth restrictions, simultaneously increasing of inputs (normal and widely used), and decreasing of outputs (good and bad) is applied. Strong disposability of normal inputs and desirable outputs has been provided in the first and the third constraints.

To present a linear structure of the proposed production possibility set, let $z^k=\lambda^k+\mu^k=\alpha^k+\beta^k$. Two different partitions have been considered for the intensity variable. If the two considered partitions are multiplied by θ^k and φ^k, we have:

$$\varphi^k z^k=\varphi^k\lambda^k+\varphi^k\mu^k \Rightarrow \varphi^k=\frac{\varphi^k\lambda^k+\varphi^k\mu^k}{z^k}\ge 1\Rightarrow \varphi^k\lambda^k+\varphi^k\mu^k\ge z^k$$

$$\theta^k z^k = \theta^k \alpha^k + \theta^k \beta^k \Rightarrow \theta^k = \frac{\theta^k \alpha^k + \theta^k \beta^k}{z^k} \le 1 \Rightarrow \theta^k \alpha^k + \theta^k \beta^k \le z^k$$

If $\theta^k \lambda^k = \overline{\lambda^k}$, $\theta^k \mu^k = \overline{\mu^k}$, $\varphi^k \alpha^k = \overline{\alpha^k}$ and $\varphi^k \beta^k = \overline{\beta^k}$, then $\overline{\alpha^k} + \overline{\beta^k} \le z^k \le \overline{\lambda^k} + \overline{\mu^k}$. Hence, we have:

$$\sum_{k=1}^{K}(\overline{\alpha^k} + \overline{\beta^k}) \le \sum_{k=1}^{K} z^k \le \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) \Rightarrow \sum_{k=1}^{K}(\overline{\alpha^k} + \overline{\beta^k}) \le 1 \le \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right)$$

Finally, the linear version of the proposed technology is presented as:

$$\begin{aligned} T = \{(x,c,v,w) \mid & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) x_n^k \le x_n, && n = 1,\dots,N \\ & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) c_m^k = c_m, && m = 1,\dots,M \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) v_s^k \ge v_s, && s = 1,\dots,S \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) w_h^k = w_h, && h = 1,\dots,H \\ & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) \ge 1, && k = 1,\dots,K \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) \le 1, \\ & \overline{\alpha^k} + \overline{\beta^k} \le \overline{\lambda^k} + \overline{\mu^k}, \\ & \overline{\lambda^k}, \overline{\mu^k}, \overline{\alpha^k}, \overline{\beta^k} \ge 0 \quad \} \end{aligned} \tag{8}$$

To analyze the performance of DMU_o, a directional distance function measure is presented as follows:

$$\begin{aligned} & \max \bar{\rho} \\ \text{s.t.} \quad & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) x_n^k \le x_n^o, && n = 1,\dots,N \\ & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) c_m^k = (1+\bar{\rho}) c_m^o, && m = 1,\dots,M \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) v_s^k \ge v_s^o, && s = 1,\dots,S \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) w_h^k = (1-\bar{\rho}) w_h^o, && h = 1,\dots,H \\ & \sum_{k=1}^{K}\left(\overline{\lambda^k} + \overline{\mu^k}\right) \ge 1, && k = 1,\dots,K \\ & \sum_{k=1}^{K}\left(\overline{\alpha^k} + \overline{\beta^k}\right) \le 1, \\ & \overline{\alpha^k} + \overline{\beta^k} \le \overline{\lambda^k} + \overline{\mu^k}, \\ & \overline{\lambda^k}, \overline{\mu^k}, \overline{\alpha^k}, \overline{\beta^k} \ge 0 . \end{aligned} \tag{9}$$

where widely used inputs are maximized, and undesirable outputs are minimized by implementing the direction vector $(d_c, d_v) = \left(c_m^o, -w_h^o\right)$. $\bar{\rho}$ indicates a measure of the inefficiency of the unit under consideration. If the optimal value of the objective function is equal to 0, DMU_o is identified as an efficient DMU. To illustrate, by considering $\bar{\rho}^*$ as the optimal solution of model (9), we have $\bar{\rho}^* \geq 0$. The unit under evaluation will be inefficient if $\bar{\rho}^* > 0$. Therefore, efficiency can be defined as $1-\bar{\rho}^*$.

In the following, the proposed model is tested for an application.

An Application

In this section, the efficiency of ten domestic sewage treatment plants is evaluated by using the introduced approach. The data was firstly used in Huang and Lee (2013). The utilization of domestic sewage treatment plants leads to environmental protection, and national funds are invested in these facilities for emission reduction and energy saving. Each plant contains two inputs and two outputs. To explain more, national financial funds are considered as normal input (x) , and untreated sewage is deemed as a widely used input (c). Meanwhile, recycled water is assumed as desirable output (v) and waste sludge is incorporated as undesirable output (w). The data set is presented in Table 3. By using the direction vector $(d_c, d_v) = \left(c_m^o, -w_h^o\right)$, model (9) is utilized to analyze the performance in the presence of the widely used input and the undesirable output. Columns 7 and 8 of Table 3 illustrate the obtained results. The inefficiency scores as $\bar{\rho}^*$ are provided in column 7. Also, the efficiency scores obtained as $1-\bar{\rho}^*$ are shown in column 8.

Table 3. The Data of 10 Domestic Sewage Treatment Plants in China

DMU	x	c	v	w	Huang efficiency	$\bar{\rho}^*$	New efficiency
1	10812.26	5913.94	3369.20	234.75	0.6959(6)	0.4627	0.5373(9)
2	7824.40	7477.90	4959.63	211.25	1(1)	0	1(1)
3	9235.62	3655.14	3147.82	208.14	0.7867(5)	0.4338	0.5662(7)
4	8688.29	6935.55	4729.15	198.70	1(1)	0.0619	0.9381(3)
5	7265.80	5448.90	4641.65	173.76	1(1)	0	1(1)
6	13555.64	7065.23	4874.93	275.28	0.5724(8)	0.2689	0.7311(5)
7	13539.87	7336.06	4820.23	243.24	0.5762(7)	0.1991	0.8009(4)
8	13874.94	3948.56	3532.66	256.36	0.5270(9)	0.4841	0.5159(10)
9	8424.13	3831.78	3013.56	184.50	0.8625(4)	0.3886	0.6114(6)
10	14280.32	4346.09	3977.30	265.18	0.5167(10)	0.4385	0.5615(8)

Source: Huang and Lee (2013) and Research finding.

As can be seen in columns 7 and 8, DMU_2 and DMU_5 are introduced as efficient units, and unit 8 with the score of 0.5159 is the most inefficient. Furthermore, the numbers in parentheses show the rankings of DMUs. As shown, plant 4 gained the third-ranking in this way.

To compare the results of the proposed approach with the existing approaches, Huang and Lee (2013) are employed. Huang and Lee (2013) developed an undesirable input-output two-phase DEA model in environmental efficiency and optimized the traditional DEA model. By computing Huang and Lee's model (2013), DMU_2, DMU_4 and DMU_5 were presented as the efficient units in the first phase, which are indicated in column 6 of Table 3. Although, by using the second phase, two DMUs of 2 and 5 were specified as efficient and DMU 4 got the third-ranking. Also, as can be seen in the parentheses, unit 10, as the most inefficient unit, has had the lowest ranking.

Findings of the proposed model in comparison with Huang and Lee (2013) show that there are differences in the efficiency scores and rankings. For instance, plant 1 with a score of 0.6959 obtained a ranking of 6 in Huang and Lee's approach (2013) while it with a score of 0.5373 achieved a ranking of 9 in the proposed approach. Also, in the introduced approach, efficiencies are obtained in one stage while the efficiency scores obtained from Huang and Lee (2013) are estimated in two phases. Therefore, the proposed approach is rational from the computational aspect. Moreover, it can discriminate the efficiency scores of units appropriately.

Conclusion

The idea of weak disposability is usually used to reduce pollutant outputs. Yet, in many applications, there are widely used inputs such as plastic wastes and rotten fruits in the process of recycling that should be increased. In this paper, a definition of weak disposability of inputs was provided and, then, a technology set was developed based on five axioms i.e. inclusion of observations, free disposability of desirable outputs and normal inputs, weak disposability of inputs, convexity, and minimum extrapolation. Then, to construct a comprehensive format, a definition of weak disposability of inputs-outputs was provided, in which normal and widely used inputs increased proportionally, and desirable and undesirable outputs decreased. Then the algebraic format of the technology set was proposed. To find a linear version of a proposed technology, a linearization method was introduced. Finally, the model was used for 10 domestic sewage treatment plants in China. Results showed that this model could calculate the directional distance efficiency in a reasonable way when weakly disposable undesirable inputs and outputs were present at the same time. Also, the introduced method is credible and appropriate from the computational perspective.

It is clear that in this study the performance of organizations with widely used inputs and undesirable outputs has been managed using the deterministic DEA approach. However, in an alternative consideration, it can be analyzed and addressed under the SFA at a provided period. Further studies might investigate the organizations' performance by integer-valued undesirable inputs and outputs. Also, the analysis of efficiency in the presence of weakly disposable imprecise input/output factors can be a concern for future studies.

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Appendix

Here, the proof of suppositions provided in section 4 is presented as follows:

A1) Inclusion of observations is obvious.

A2) Free disposability of normal inputs and desirable outputs:

If $(x,c,v)\in L$, then:

$$\sum_{k=1}^{K}\varphi^k x_n^k z^k \le x_n \text{ and } x_n^{'} \ge x_n \Rightarrow \sum_{k=1}^{K}\varphi^k x_n^k z^k \le x_n \le x_n^{'} \Rightarrow \sum_{k=1}^{K}\varphi^k x_n^k z^k \le x_n^{'}$$

$$\sum_{k=1}^{K} v_s^k z^k \ge v_s \text{ and } v' \le v_s \Rightarrow \sum_{k=1}^{K} v_s^k z^k \ge v_s \ge v_s^{'} \Rightarrow \sum_{k=1}^{K} v_s^k z^k \ge v_s^{'}$$

Hence, $(x',c,v')\in L$.

A3) Weak disposability of inputs:

We assume that $(x,c,v)\in L, \varphi \ge 1$. Then we have:

$$\sum_{k=1}^{K}\varphi^k x_n^k z^k \le x_n$$

$$\sum_{k=1}^{K}\varphi^k c_m^k z^k = c_m$$

By multiplying the equations by $\sigma \geq 1$, we have:

$$\sum_{k=1}^{K} \sigma \varphi^k x_n^k z^k \leq \sigma x_n$$

$$\sum_{k=1}^{K} \sigma \varphi^k c_m z^k \leq \sigma c_m$$

On the other hand, according to $\varphi^k \geq 1$ and $\sigma \geq 1$, we have $\varphi^k \sigma \geq 1$. By assumption of $\omega = \varphi^k \sigma$ we have $\omega \geq 1$. Hence, we conclude that $(\omega x, \omega c, v) \in L$.

A4) Convexity:

We assume that (x_1, c_1, v_1) and $(x_2, c_2, v_2) \in L$. Then we have:

$$\sum_{k=1}^{K} \varphi_1^k x_n^k z_1^k \leq x_1$$

$$\sum_{k=1}^{K} \varphi_2^k x_n^k z_2^k \leq x_1$$

By multiplying the first inequality by δ and the second by $(1-\delta)$, we have

$$\sum_{k=1}^{K} \delta \varphi_1^k x_n^k z_1^k \leq \delta x_1$$

$$\sum_{k=1}^{K} (1-\delta) \varphi_2^k x_n^k z_2^k \leq (1-\delta) x_2$$

By summation of the achieved inequality, we have:

$$\sum_{k=1}^{K} \delta \varphi_1^k x_n^k z_1^k + \sum_{k=1}^{K} (1-\delta) \varphi_2^k x_n^k z_2^k \leq \delta x_1 + (1-\delta) x_2$$

Let $\delta x_1 + (1-\delta) x_2 = \hat{x}$, Hence $\sum_{k=1}^{K} (\delta \varphi_1^k z_1^k + (1-\delta) \varphi_2^k z_2^k) x_n^k \leq \hat{x}$. By multiplying the left side of the inequality by $\frac{\left(\delta z_1^k + (1-\delta) z_2^k\right)}{\left(\delta z_1^k + (1-\delta) z_2^k\right)}$, we have:

$$\sum_{k=1}^{K} \frac{\left(\delta \varphi_1^k z_1^k + (1-\delta) \varphi_2^k z_2^k\right)}{\left(\delta z_1^k + (1-\delta) z_2^k\right)} \left(\delta z_1^k + (1-\delta) z_2^k\right) x_n^k \leq \hat{x}$$

By assumption of $\delta z_1^k + (1-\delta) z_2^k = \hat{z}_k$ and $\hat{\varphi}_k = \frac{\left(\delta \varphi_1^k z_1^k + (1-\delta) \varphi_2^k z_2^k\right)}{\left(\delta z_1^k + (1-\delta) z_2^k\right)}$, we have $\sum_{k=1}^{K} \hat{\varphi}_k \hat{z}^k x_n^k \leq \hat{x}$. On the other hand, by $\hat{z}_k$ and $\hat{\varphi}_k$, the last three constraints of the proposed technology are confirmed.

A5) Minimum extrapolation:

To prove that L is the minimal set, which follows the previous axioms, we assume that L' is a production possibility set that follows axioms. If $(x, c, v) \in L$ then:

$$\sum_{k=1}^{K} \varphi^k x_n^k z^k \leq x_n \qquad n = 1, \ldots, N$$

$$\sum_{k=1}^{K} \varphi^k c_m^k z^k = c_m \qquad m = 1, \ldots, M$$

$$\sum_{k=1}^{K} v_s^k z^k \geq v_s \qquad s = 1,\ldots,S$$

$$\sum_{k=1}^{K} z^k = 1 \qquad k = 1,\ldots,K$$

where $L \neq \phi$. Thus, for the intensity vector $z = \left(z^1,\ldots,z^k\right)$ we have:

$$\begin{pmatrix} x_z \\ c_z \\ v_z \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} \varphi^k x_n^k z^k \\ \sum_{k=1}^{K} \varphi^k c_m^k z^k \\ \sum_{k=1}^{K} v_s^k z^k \end{pmatrix} \in L'$$

This set of vectors dominates the vector $(x,c,v) \in T$. Hence, $(x,c,v) \in L'$.