Iranian Journal of Management Studies (IJMS) 2022, 15(2): 287-303
RESEARCH PAPER

Determining the Exact Stability Region and Radius Through Efficient Hyperplanes

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(Received: November 19, 2020; Revised: May 20, 2021; Accepted: May 24, 2021)

Abstract
The main goal of this study was to address the sensitivity analysis fora specific efficient decision-making unit (DMU), which is under evaluation, by the variable returns to scale (VRS) technology to extend the efficiency stability region. Variations in inputs or outputs of any DMU can change the efficiency classification of that DMU as well as other DMUs, i.e., an efficient DMU can become inefficient and vice versa. This study considered the largest performance stability region for an extreme efficient DMU whose data could be changed in all directions of input/output space, including both directions of improving the situation and worsening the situation such that under these changes, the efficiency classification of all extreme DMUs would be preserved. We found the largest symmetric cell to the center of the extreme efficient DMU under evaluation, leading to an efficiency stability radius. In addition, data changes were only applied for the extreme efficient DMU, and the data for the other DMUs were assumed fixed. This stability region was determined by the defining hyperplanes of production possibility set (PPS) of VRS technology and the corresponding half-spaces. The suggested method is illustrated using real-world data.

Keywords: data envelopment analysis, sensitivity analysis, stability region, stability radius, defining hyperplane.

1. Introduction

Data Envelopment Analysis (DEA) is a linear programming method for estimating the efficiency frontier, assessing the performance of DMUs, and classifying them into efficient or inefficient groups(Charnes et al., 2013). The word in this arena started by Charnes et al. (1978)(CCR model) and was improved by other researchers, especially Banker et al. (1984) (BCC model). Nowadays, DEA entails a wide variety of applied research, and many topics with many successful applications and case studies have been reported in the DEA literature (Liu et al., 2013; Nurcan & Deniz Köksal, 2021; Zhou et al., 2016). Since DEA is data-based, any uncertainty in data such as measurement errors, incomplete data, and noisy inputs and outputs lead to problems in DEA performance appraisal. Changes in inputs or outputs of any DMU can alter the efficiency classification of that DMU as well as other DMUs, i.e., the status of an efficient DMU can change to inefficient and vice versa. A mostly asked question is "To what extent can perturbations in the input/output data be tolerated before the DEA efficiency is changed?" This issue has been argued as efficiency sensitivity (stability) analysis in DEA. Recently, efficiency robustness analysis and performance stability

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region determination have been studied in many articles with different approaches. Some of these studies have considered sensitivity to model changes or to augmentation or diminution of the number of DMUs (Wilson, 1995). Other research projects have focused on direct data changes and have examined the impacts of changes to the inputs and outputs on the efficiency (Abri et al., 2009; He et al., 2016;Hladík 2019; Neralić, 2004). An alternative to direct change approaches comes from indirect methods, which determine the maximum radius that maintains the efficiency or inefficiency (Jahanshahloo et al., 2005a; Jahanshahloo et al., 2011; Khoveyni & Eslami, 2021).

However, there are drawbacks to the methods discussed, one of which is that the existing methods for an extreme efficient DMU do not take into account data changes in all directions of input/output space. In addition, there is not an efficiency stability region for an extreme DMU in all directions of data space. Regarding the abovementioned instances, in this study we will obtain a new stability region in which the inputs of extreme efficient DMU decrease and its outputs increase simultaneously.Then we will enlarge the stability region for the efficient DMU whose data can be changed in all directions of input/output space, including both directions of improving the situation and worsening the situation such that the efficiency classification of all extreme efficient DMUs remains constant. Moreover, we will find the largest value of data changes for the extreme DMU, leading to an efficiency stability radius. In this context, the investigation questions of the study are:

- How to find the improvement stability region for an extreme efficient DMU so that the performance classification of all extreme DMUs is maintained?
- How to achieve the greatest region of performance stability for an efficient DMU in all directions of the input/output space such that the efficiency classification of all extreme DMUs is not disturbed?
- What is the maximum value of the input/output changes of the extreme efficient DMU in all directions so that the efficiency of all DMUs remains constant?

The answers to these questions will be investigated in the current study. The structure of this paper is as follows. In Section 2, the review of literature explains the sensitivity analysis background in DEA. Section 3 provides some basic DEA models and a brief explanation of the approaches to finding the defining hyperplanes of PPS. Section 4 introduces a method for determining the exact stability region and radius for a specific efficient DMU. Section 5 gives a numerical example. Section 6 illustrates the numerical results of the application of the procedure to the real-world case of European banks. In the last section, the conclusions and suggestions for future studies are presented.

2. Literature Review

An important issue in the context of DEA which has been considered by many researchers is sensitivity analysis, and numerous studies have focused on this topic. Some of these studies investigate the sensitivity of efficiency results by selecting different DEA models (Ahn & Seiford, 1993). Some other studies have concentrated on the problem size and have considered the efficiency stability for a DMU under evaluation compared to the number of DMUs and the number of inputs and outputs (Wilson, 1995). From another perspective, performance sensitivity has been considered according to the data entry errors for DMUs(Sexton et al., 1986). Several analytically mathematical methods have examined the results of direct data changes on the efficiency classification (Charnes & Neralić, 1990; Neralić, 2004). Research on analytical methods of DEA sensitivity analysis was first proposed by Charnes et al. (1985). They suggested an algorithm that dealt with optimal basis matrix of the LP model and examined the change in a single output for an efficient DMU in which the optimal basis matrix after the perturbation remained optimal. Charnes and Neralić (1990) introduced the sensitivity analysis of the additive model for an efficient unit and provided sufficient conditions for simultaneous variations of all inputs and all outputs such that its obtained efficiency classification did not change. Charnes et al. (1992) and Charnes et al. (1996)proposed metric approaches in the super-efficiency models for each DMU. In this approach, the DMU under evaluation was deleted from the reference set and the input/output changes were considered as variables. The objective of the LP model applied was to find the minimum input/output changes such that the DMU under evaluation could be placed in the convex hull of other DMUs. They obtained a stable cell where the data changes occurred by applying norm $l_1$and norm l_∞. This method resulted in sufficient conditions for input/output changes. Zhu (1996) improved the work of Charnes et al. (1992) to identify allowable variations in every input and output for each DMU before a change in status occurs for the DMU under evaluation.

Thompson et al. (1994) and Thompson et al. (1996) used the strong complementary slackness condition (SCSC) multipliers and the interior point algorithm for preserving the efficiency of extreme efficient DMU in the multiplier DEA model in a situation where the data for all efficient DMUs and all inefficient DMUs simultaneously changed in opposite directions but at the same ratio. Seiford and Zhu (1998a, 1998b) utilized the modified DEA models to find the stability radius for an efficient DMU by the worst-case scenario where the efficiency of the test DMU was deteriorating while the efficiencies of the other DMUs were improving. Zhu (2001) employed the super-efficiency model for preserving a DMU's efficiency classification when various proportional and absolute changes were applied to all DMUs. Neralić (2004) examined sufficient conditions for simultaneous efficiency preservation of all efficient DMUs and inefficiency preservation of all inefficient DMUs in the additive model of DEA using an approximate inverse of the perturbed optimal basis matrix.

Jahanshahloo et al. (2004) proposed DEA interval models to determine the stability radius of each unit in the presence of interval data such that the efficiency classification remains unchanged. Jahanshahloo et al. (2005a) presented a new method to sensitivity analysis of a unit under evaluation. They developed a stability region for DMU_o by using the supporting hyperplanes that pass through DMU_o and the new frontier that is constructed by elimination DMU_o from the observations set. Jahanshahloo et al. (2005b) obtained variation ranges for inputs and outputs of efficient and inefficient DMUs for maintaining classification of DMUs by molding the two models proposed by Cooper et al. (2001) into one model. Boljunčić (2006)employed an iterative procedure and achieved data changes using the optimal simplex tableau and applying parametric programming. Abri et al. (2009)used a super-efficiency model in DEA to determine the stability radius of efficient DMUs and quasi- efficient DMUs.

Mozaffari et al. (2009)introduced MOLP methods (weighted-sums and STEM interactive method) for preserving a DMU's efficiency classification when simultaneous changes of all interval data are applied to the test DMU. Singh (2010)provided multi-parametric sensitivity analysis by classifying the perturbation parameters as ‘‘focal’’ and ‘‘non-focal’’ and found critical regions for the efficient DMU under evaluation. Jahanshahloo et al. (2011)examined the sensitivity analysis of the inefficient DMUs.Their technique achieved an “exact necessary change region”where the efficiency score of a specific inefficient DMU changes to a defined efficiency score.

Wen et al. (2011) declared DEA fuzzy models based on credibility measures for preserving a DMU's efficiency classification with fuzzy data. Khalili-Damghani and Taghavifard (2013)suggested Fuzzy Two-Stage DEA (FTSDEA) models to determine stability radius within which the classification of each DMU (first and second sub-DMU) remains unchanged in the presence of data variations in a fuzzy environment. Agarwal et al. (2014) proposed a "New Slack Model"(NSM) to examine the robustness of DEA efficiency scores by changing the reference set of the inefficient DMUs. Tohidi et al. (2014) obtained the stability region of efficient and inefficient units with integer data. Daneshvar et al. (2014)presented an algorithmic approach using facet analysis for preserving the efficiency of an anchor DMU under evaluation. Banker et al. (2015)employed the stochastic DEA (SDEA) model and perturbation of optimal basis matrix inverse to maintain the efficiency classification. Khodabakhshi et al. (2015)examined simultaneous variations in all data of an efficient DMU in input relaxation super-efficiency models in DEA that preserve efficiency.

Sarfi et al. (2015) obtained a stability region in T_{v} by utilizing the defining hyperplanes such that if a DMU is added in this region, all of the extreme efficient DMUs are still remained on frontier. He et al. (2016) used percentage changes of all interval data simultaneously for all DMUs to determine the stability radius for DMU under evaluation. When an additional DMU needs to be added to the set of the observed DMUs, Zamani and Borzouei (2016) presented a stability region using the defining hyperplanes of PPS for the additional DMU so that the additional DMU becomes efficient and the classification of all DMUs remains unchanged. In addition, they obtained a region in which some specific efficient DMUs become inefficient. Employing a super-efficiency model based on input and output slacks, Dar et al. (2017) examined the performance classification sensitivity and the returns to scale (CRS, IRS, and DRS) of DMUs. Ghazi et al. (2018)obtained the improvement region for an inefficient DMU based on a value judgment using all the defining hyperplanes of PPS. Using a tight linear programming approximation based on a robust optimization viewpoint, Hladík (2019)appraised the efficiency scores, determined the stability region to preserve the efficiency classification of DMU under evaluation, and maintained the order of rankings.

The methodologies discussed above considered a deteriorating scenario for an extreme efficient DMU to find the efficiency stability radius or efficiency stability region for that DMU, meaning that they examined the data changes in a direction in which the position of extreme DMU is worsening. In contrast, the sensitivity analysis in DEA with an improving scenario for an extreme efficient DMU deserves study under the condition that the efficiency classification of other extreme DMUs remains unchanged (optimistic sight). Economically, in order to increase the productivity of the companies and organizations, it is important to obtain a region in which the inputs of the under-evaluation unit are reduced and its outputs are increased,because these rates of change have important economic and managerial implications.

Investigation to find the largest stability region is still an open question, because the larger the area is, the more reliable the obtained results will be. Using the defining hyperplanes of PPS, Jahanshahloo et al. (2005a) identified an area in which the data for specific efficient DMU_o change in the following directions: 1) the increase of inputs and the decrease of outputs, 2) the increase of inputs and the increase of outputs, and 3) the decrease of inputs and the decrease of outputs.

In this study, we aim to extend the stability region proposed by Jahanshahloo et al. (2005a) and determine the largest and most complete area in which data variations apply for DMU_o in the total input/output space and the data for the other DMUs are fixed. Our suggested method, in addition to the cases mentioned in the Jahanshahloo method, obtains a new region in which the inputs of efficient DMU_o decrease and its outputs increase simultaneously such that the efficiency classification of all extreme efficient DMUs remains constant. The new stability region is specified by the defining hyperplanes of PPS passing from the corner DMUs adjacent to the DMU_o under evaluation and the frontier of $DMU_j(j \neq o)$when the data of the remaining DMUs are kept at their current levels.

3. Preliminaries

Consider a set of n homogenous DMUs, $\{DMU_j;\ j = 1, \ldots, n\}$, where each DMU converts$m$ inputs into s outputs. Suppose that the observed input and output vectors of DMU_j are $X_j = (x_{1j}, \ldots, x_{mj})$ and $Y_j = (y_{1j}, \ldots, y_{sj})$, respectively, and let $X_j \geq 0, X_j \neq 0$ and $Y_j \geq 0, Y_j \neq 0$.The PPS of BCC model with VRS technology, which we are interested in this paper, is represented by T_v,and is defined as follows:

$$T_v = \left\{ (X\ ,\ Y\ ) \middle| X \geq \sum_{j=1}^{n} \mu_j X_j\ ,\ Y \leq \sum_{j=1}^{n} \mu_j Y_j\ , \sum_{j=1}^{n} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots,\ n \right\}$$

By eliminating(X_0, Y_0) from the set of T_v , the new PPS is as follows:

$$T'_v = \left\{ (X\ ,\ Y\ ) \middle| X \geq \sum_{j=1,\, j\neq 0}^{n} \mu_j X_j\ ,\ Y \leq \sum_{j=1,\, j\neq 0}^{n} \mu_j Y_j\ ,\ \sum_{j=1,\, j\neq 0}^{n} \mu_j = 1,\ \mu_j \geq 0,\ j = 1, \ldots,\ n,\ j \neq 0 \right\}$$

T'_v is the new empirical production possibility set constructed by the elimination of(X_0, Y_0) from the set of T_v.

Based on T_v, the envelopment form ofBCC model is as follows:

$$\begin{aligned} &\theta^{o*} = \min \theta^o \\ &\text{s. t.} \quad \sum_{j=1}^{n} \mu_j X_j \leq \theta^o X_o \\ &\sum_{j=1}^{n} \mu_j Y_j \geq Y_o \\ &\sum_{j=1}^{n} \mu_j = 1 \\ &\quad \mu_j \geq 0,\ j = 1, \ldots,\ n. \end{aligned} \tag{1}$$

Moreover, the multiplier form of BCC model is:

$$\begin{aligned} Max \quad & \beta^T Y_o + \varphi \\ s.t. \quad & \beta^T Y_j - \alpha^T X_j + \varphi \leq 0,\ j = 1, \ldots,\ n \\ & \alpha^T X_o = 1 \\ & \beta \geq 0,\ \alpha \geq 0. \end{aligned} \tag{2}$$

where $X_o$and $Y_o$are correspondingly the input and output vectors of $DMU_o(o \in \{1, \ldots, n\})$ under consideration. $DMU_o(X_o,\ Y_o)$ is called efficient DMU if the optimal value of the objective function of the BCC model, $\theta^{o*} = \beta^{*T} Y_o + \varphi^*$, is equal to one; otherwise DMU_o is inefficient.

Based on Charnes et al. (1991) assertion, a set of efficient DMUs can be partitioned into three groups: set E (extreme efficient DMUs), set E′(efficient but not extreme DMUs), and set F (weakly efficient DMUs). The E consists of the DMUs located at the vertices of the frontier; therefore, they cannot be represented as a linear combination (with nonnegative coefficients) of the remaining DMUs. In the case of VRS, instead of having a linear combination, we have a convex one. TheE′consists of efficient DMUs that are efficient at both input and output orientations and are not at the vertices, and the F consists of DMUs that are efficient in the input orientation and inefficient in the output orientation or vice versa.

To discriminate between these efficient DMUs, many methods have been suggested. We use the super-efficiency DEA model of Andersen and Petersen (1993) to identify the classification of DMU_o. That is,

$$\begin{aligned} &\theta_o^{\text{super}*} = \min \theta_o^{\text{super}} \\ &\text{s. t.} \quad \sum_{j=1,\, j\neq o}^{n} \mu_j X_j \le \theta_o^{\text{super}} X_o \\ &\sum_{j=1,\, j\neq o}^{n} \mu_j Y_j \ge Y_o \\ &\sum_{j=1,\, j\neq o}^{n} \mu_j = 1 \\ &\qquad \mu_j \ge 0,\ j \neq o,\ j = 1, \ldots, n. \end{aligned} \tag{3}$$

For the optimal solution $\theta_o^{\text{super}*}$:

a) If $\theta_o^{super*} > 1$ or (3) is infeasible, then $DMU_o \in$ set E.

b) If $\theta_o^{\text{super}*} = 1$, then $DMU_o \in$ set $E' \cup F$.

c) If $\theta_o^{\text{super}*} < 1$, then DMU_o is inefficient.

A method for producing efficient surfaces of the BCC model passing through the efficient DMU (X_o, Y_o) was first proposed by Huang et al. (1997), which is as follows:

$$\begin{aligned} &Min \qquad \varphi \\ &s.t. \qquad \beta^T Y_o - \alpha^T X_o = \varphi \\ &\qquad\qquad \beta^T Y_j - \alpha^T X_j \le \varphi,\ j = 1, \ldots, n \\ &\qquad\qquad \alpha^T e + \beta^T e = 1 \\ &\qquad\qquad \beta \ge 0,\ \alpha \ge 0. \end{aligned} \tag{4}$$

At an optimal solution $(\alpha^*, \beta^*, \varphi^*)$ to (4), all of those observed efficient DMU_js which satisfy their associated constraints as equalities also lie on the efficient facet contained in the hyperplane passing through (X_o, Y_o). Such DMUs, together with (X_o, Y_o), constitute a subset (but not necessarily all) of the facet members. By applying (4) for each DEA-efficient DMU in turn, all efficient facets and their member DMUs can possibly be identified. (For more details of the defining hyperplanes and properties, seeJahanshahloo et al., 2007; Jahanshahloo et al., 2010; Lotfi et al., 2011).

Considering the importance of extreme efficient DMUs that define the efficient frontier, the present study introduces a method to sensitivity analysis and determination of the stability region and radius for these DMUs bythe concept of the defininghyperplanes of PPS.

As expected, decreasing the inputs or increasing the outputs for an extreme efficient DMU does not disrupt its performance classification, but when these changes exceed a certain amount, the efficiency classification of other extreme efficient DMUs can change and become inefficient. Therefore, the purpose of this article is to obtain the largest performance stability region for an extreme efficient DMU whose data changing in all directions of input/output space, including both directions of improving the situation and worsening the situation, such that under these changes the efficiency classification of all extreme DMUs remains constant.

Definition 1. A region of allowable inputs/outputs changes in all directions of input/output space is called a stability region if and only if DMU_o remains extreme efficient after such changes occur in this region.Moreover,the efficiency classification of other extreme DMUs preserves under these changes.

Definition 2. Given DMU_o, the stability radius is the largest number, r^*, such that feasible perturbations to DMU_o in the R^{m+s} space strictly $<r^*$ preserve the efficiency classification of all extreme DMUs.

4. Stability Analysis of Extreme Efficient DMUs

In this study, we provide an exact stability region for extreme efficient DMU_o by the concept of the defining hyperplanes of PPS and the corresponding half-spaces. Jahanshahloo et al. (2005a) proposed an algorithm and obtained a part of the stability region of DMU_o using the defining supporting hyperplanes of PPS. As it is shown in Figure 1, Jahanshahloo et al.'s(2005a) method to determine the stability region of extreme efficient DMU_B reach only the R_1 area, which is an area in the following directions: 1) the increase of inputs and the decrease of outputs, 2) the increase of inputs and the increase of outputs, and 3) the decrease of inputs and the decrease of outputs, while the R_2 region is also the stability region of DMU_B which includes decreasing changes in inputs and increasing changes in outputs and is determined by the defining supporting hyperplanes passing from the corner DMUs adjacent to the DMU_B. It is also significant that these changes do not disrupt the performance status of extreme DMUs. This paper aims to obtain the complete stability region, namely $R_1 \cup R_2$ in Figure 1 for an extreme efficient DMU_o. This region is the most complete area in which data variations are applied for extreme DMU_o in the total input/output space, and the data for the other DMUs are assumed fixed. Moreover, under these changes, the classification of all extreme DMUs remains constant.

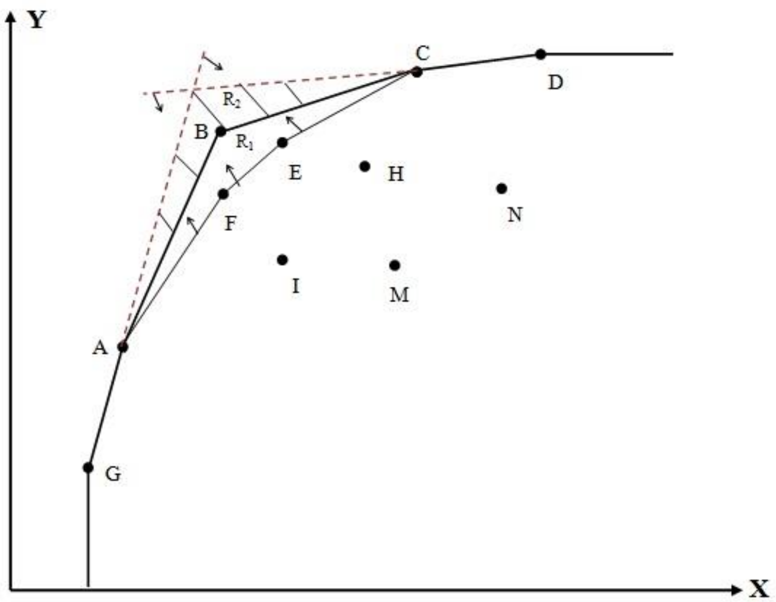

Figure 1. The Complete Stability Region of DMU_B

Suppose DMU_o, which is DMU under evaluation, is identified as an extreme efficient DMUby model (3). The purpose is to find an exact stability region for DMU_o provided that the efficiency classification of all extreme DMUs is preserved. To do this, we first identify all extreme DMUs and defining hyperplanes of T_v passing through DMU_o. Assumed that these hyperplanes are $H_r(r=1, \dots, t)$, we find all extreme DMUs that lie on $H_r(r=1, \dots, t)$. These DMUs are adjacent to DMU_o and satisfy some equations of $H_r(r=1, \dots, t)$. Let$\delta_1 = \{j_1, \dots, j_h\}$ be the set of extreme DMUs adjacent to DMU_o. We determine all defining supporting hyperplanes of PPS that are binding at the members of set δ_1 and then we single out those hyperplanes among them that are not passing from DMU_o. Assume that these hyperplanes are $H_1, H_2, \ldots, H_l$, which are defined as follows:

$$H_k = \left\{ (X, Y) \middle| \beta_k^{*T} Y - \alpha_k^{*T} X = \varphi_k^* \right\}, \qquad k = 1, \ldots, l \tag{5}$$

where $(\alpha^*, \beta^*, \varphi^*)$ is an optimal solution of the model (4), α is m-vector, and β is s-vector. Based on the hyperplane H_k mentioned in (5), the half-space H_k^- is defined as follows:

$$H_k^- = \left\{ (X, Y) \middle| \beta_k^{*T} Y - \alpha_k^{*T} X \le \varphi_k^* \right\}, \quad k = 1, \ldots, l \tag{6}$$

Now, we remove DMU_o from the set of observations and re-evaluate inefficient DMUs in the model (3) by the BCC model related to the new PPS (T'_v) as follows:

$$\begin{aligned} \text{Max} \quad & \beta^T Y_k + \varphi \\ s.t. \quad & \beta^T Y_j - \alpha^T X_j + \varphi \le 0, \qquad j = 1, \ldots, n, \; j \ne o \\ & \alpha^T X_k = 1 \\ & \beta \ge 0, \; \alpha \ge 0. \end{aligned} \tag{7}$$

Let $\delta_2 = \{ j_{t_1}, \ldots, j_{t_z} \}$ be the set of efficient DMUs in the model (7). By the application of the following model, we obtain all defining supporting hyperplanes of the new PPS (T'_v)that are binding at the members of set $\delta_1 \cup \delta_2$.

$$\begin{aligned} \text{Min} \quad & \varphi \\ s.t. \quad & \beta^T Y_q - \alpha^T X_q = \varphi \\ & \beta^T Y_j - \alpha^T X_j \le \varphi, \qquad j = 1, \ldots, n, \; j \ne o \\ & \alpha^T e + \beta^T e = 1 \\ & \beta \ge 0, \; \alpha \ge 0. \end{aligned} \tag{8}$$

Model (8) should be solved for each $q \in \{ j_1, \ldots, j_h, j_{t_1}, \ldots, j_{t_z} \} - \{o\}$.

Suppose that $H_{i_1}, H_{i_2}, \ldots, H_{i_f}$ are efficient hyperplanes of the new PPS that are defined as follows:

$$H_{i_k} = \left\{ (X, Y) \middle| \beta_{i_k}^{*T} Y - \alpha_{i_k}^{*T} X = \varphi_{i_k}^* \right\}, \qquad k = 1, \ldots, f \tag{9}$$

Corresponding to hyperplane H_{i_k}, the half-space $H_{i_k}^+$ is defined as follows:

$$H_{i_k}^+ = \left\{ (X, Y) \middle| \beta_{i_k}^{*T} Y - \alpha_{i_k}^{*T} X > \varphi_{i_k}^* \right\}, \quad k = 1, \ldots, f \tag{10}$$

Now according to the obtained half-spaces, we set:

$$S_1 = \bigcap_{k=1}^{l} H_k^-, \qquad S_2 = \bigcap_{k=1}^{f} H_{i_k}^+, \qquad S = S_1 \cap S_2 \tag{11}$$

Set S is the largest stability region for DMU_o, meaning that if the input/output vector (X_o, Y_o) changes in this region, DMU_o will remain extreme efficient.In addition, the performance classification of the other extreme efficient DMUs is maintained. The region S obtained in (11) is equivalent to the region $R_1 \cup R_2$ in the case of one input and one output in Figure 1. Therefore, the stability region obtained by our proposed method is larger and more complete than Jahanshahloo et al.'s(2005a) method. As a result, the stability region obtained by Jahanshahloo et al.'s(2005a) method is a subset of the stability region obtained by the method of this paper.

Theorem. Set $S = S_1 \cap S_2$ is the largest stability region for DMU$_o$.

Proof. Without loss of generality, we prove that the efficiency status of extreme efficient DMU$_o$ and other extreme DMUs remains unchanged in the set S. By contradiction, suppose that DMU$_o$(X_o, Y_o) becomes BCC inefficient. Hence, there is at least an efficient DMU$_k$ belonging to T'_v that dominates DMU$_o$. If DMU$_k \in \delta_1 \cup \delta_2$, then ${\beta_{i_k}^{*}}^T Y_o - {\alpha_{i_k}^{*}}^T X_o < \varphi_{i_k}^*$ that ${\beta_{i_k}^{*}}^T Y - {\alpha_{i_k}^{*}}^T X = \varphi_{i_k}^*$ is the passing hyperplane from DMU$_k$ and it is supporting hyperplane on T'_v. Therefore, DMU$_o \notin$ S$_2$ consequently DMU$_o \notin$ S, which is a contradiction. If DMU$_k \notin \delta_1 \cup \delta_2$, then for each DMU$_l \in \delta_1 \cup \delta_2$, we have:

$$\beta_{i_l}^{*T} Y_k - \alpha_{i_l}^{*T} X_k \leq \varphi_{i_l}^* . \tag{*}$$

On the other hand, DMU$_k$ dominates DMU$_o$. Therefore,

$$(-X_k, Y_k) \geq (-X_o, Y_o) \text{ and } (-X_k, Y_k) \neq (-X_o, Y_o) \tag{**}$$

According to (*) and (**), $\beta_{i_l}^{*T} Y_o - \alpha_{i_l}^{*T} X_o < \beta_{i_l}^{*T} Y_k - \alpha_{i_l}^{*T} X_k \leq \varphi_{i_l}^*$. Hence DMU$_o \notin$ S.

Now, by contradiction, suppose that one of the extreme efficient DMUs (excluding DMU$_o$), like DMU$_f$, becomes inefficient. Therefore, there is another (virtual) DMU of S such that $(-\sum_{j=1}^{n} \mu_j X_j, \sum_{j=1}^{n} \mu_j Y_j) \geq (-X_f, Y_f)$ and $(-\sum_{j=1}^{n} \mu_j X_j, \sum_{j=1}^{n} \mu_j Y_j) \neq (-X_f, Y_f)$.

Suppose that $\beta_k^{*T} Y_f - \alpha_k^{*T} X_f = \varphi_k^*$ is the defining hyperplane passing from DMU$_f$(excluding those passing from DMU$_o$). Hence, $\beta_k^{*T} \sum_{j=1}^{n} \mu_j Y_j - \alpha_k^{*T} \sum_{j=1}^{n} \mu_j X_j > \beta_k^{*T} Y_f - \alpha_k^{*T} X_f = \varphi_k^*$. This implies that $(\sum_{j=1}^{n} \mu_j X_j, \sum_{j=1}^{n} \mu_j Y_j)$ S$_1$, from here $(\sum_{j=1}^{n} \mu_j X_j, \sum_{j=1}^{n} \mu_j Y_j)$ S, consequently DMU$_f$ remains efficient.

4.1. Stability Radius

Now, having the border hyperplanes of region *S*, we can determine the largest symmetric cell to the center of the DMU$_o$ and to the radius r^* so that the same changes in the data within the symmetric cell maintain the current classification of DMU$_o$ and other DMUs. In order to do this, we obtain the minimum distance of DMU$_o$ from each of the border hyperplanes of region *S*.

By considering $\|.\|_p$, the minimum distance of *(X_o, Y_o)* from $\partial(S)$, which is denoted by $d_{.p}\left[(X_o, Y_o), \partial(S)\right]$, can be defined as:

$$d_{.p}\left[(X_o, Y_o), \partial(S)\right] = \min (X, Y) - (X_o, Y_o)_P \quad \text{s. t.} (X, Y) \in \partial(S) \tag{12}$$

where *(X_o, Y_o)* is the extreme efficient DMU under evaluation. In addition, $\partial(S)$ is the frontier of region *S* and consists of the mentioned hyperplanes in (5) and (9). For simplification purposes, we can show these hyperplanes in general form as follows:

$$H : P^{*T} Z - \varphi^* = 0 \qquad where Z = (X, Y), \quad P^* = (\beta^*, -\alpha^*) \tag{13}$$

By considering the Euclidean norm, the minimum distance of DMU$_o$ from each of these hyperplanes can be defined by:

$$r = \frac{\left| P^{*T} Z_o - \varphi^* \right|}{P^{*T}{}_2} where Z_o = (X_o, Y_o) \tag{14}$$

Assume that the obtained distances are $r_1, ..., r_l, r_{i_1}, ..., r_{i_f}$, then r^* which is defined as follows, is the stability radius of the largest symmetric cell to the center of DMU_o.

$$r^* = \min\{r_1, ..., r_l, r_{i_1}, ..., r_{i_f}\} \tag{15}$$

5. Numerical Example

The numerical example is taken from Lotfi et al. (2011) and is about the efficiency evaluation of eight DMUs with two inputs and one output. The input and output variables and the efficiency score of the BBC model are shown in Table 1. According to Table 2, units 3, 4, 5, 6, and 7 are efficient. We find all defining supporting hyperplanes of PPS using the model (4). These hyperplanes are shown in Table 2. Assume that DMU_4 is the under evaluation DMU, as is shown in Table 2, H_2, H_6 and H_7 are the hyperplanes passing through DMU_4, and $\delta_1 = \{DMU_3, DMU_5\}$ is the set of DMUs adjacent to DMU_4 located on H_2, H_6 and H_7. The hyperplanes which are binding at the members of set δ_1 but do not pass through DMU_4 are H_1, H_3, H_4, H_9. Corresponding to these hyperplanes, we have the following half-spaces:

$$H_1^- = \{(X, Y) | 6Y - 5X_1 - 2 \le 0\}, \qquad H_3^- = \{(X, Y) | -X_1 + 2 \le 0\}$$

$$H_4^- = \{(X, Y) | -X_2 + 1 \le 0\}, \qquad H_9^- = \{(X, Y) | 4Y - X_1 - 20 \le 0\}$$

Then we set S_1 as follows:

$$S_1 = H_1^- \cap H_3^- \cap H_4^- \cap H_9^-$$

Now we remove DMU_4 from the observation set and re-evaluate inefficient DMUs of the model (2), i.e., DMU_1, DMU_2 and DMU_8 by model (7), and find out that DMU_1 is an efficient DMU. Then we determine the defining hyperplanes of new PPS passing through $\delta_1 \cup \delta_2 = \{DMU_1, DMU_3, DMU_5\}$ by considering model (8). These hyperplanes are as follows:

$$H_{i_1} = \{(X, Y) | -X_1 - 2X_2 + 10 = 0\}, \quad H_{i_2} = \{(X, Y) | -X_1 + 2 = 0\}$$
$$H_{i_3} = \{(X, Y) | 4Y - X_1 - 20 = 0\}, \qquad H_{i_4} = \{(X, Y) | -X_2 + 1 = 0\}$$
$$H_{i_5} = \{(X, Y) | 6Y - 5X_1 - 2 = 0\}$$

Corresponding to these hyperplanes we have half-spaces below:

$$H_{i_1}^+ = \{(X, Y) | -X_1 - 2X_2 + 10 \ge 0\}, \qquad H_{i_2}^+ = \{(X, Y) | -X_1 + 2 \ge 0\}$$
$$H_{i_3}^+ = \{(X, Y) | 4Y - X_1 - 20 \ge 0\}, \qquad H_{i_4}^+ = \{(X, Y) | -X_2 + 1 \ge 0\}$$
$$H_{i_5}^+ = \{(X, Y) | 6Y - 5X_1 - 2 \ge 0\}$$

Then S_2 can be established as follows:

$$S_2 = H_{i_1}^+ \cup H_{i_2}^+ \cup H_{i_3}^+ \cup H_{i_4}^+ \cup H_{i_5}^+$$

According to (11), the largest stability region for DMU_4 is obtained as follows:

$$S = S_1 \cap S_2.$$

Table 1. Data and Efficiency Score of BCC Model

DMUs	**DMU_1**	**DMU_2**	**DMU_3**	**DMU_4**	**DMU_5**	**DMU_6**	**DMU_7**	**DMU_8**
Input1	4	7	8	4	2	10	12	10
Input2	3	3	1	2	4	1	1	1.5
Output	2	4	7	3	2	5	8	7
Efficiency	0.86	0.69	1	1	1	1	1	0.8

We can also determine the stability radius of DMU_4 by obtaining the minimum distance of DMU_4 from each frontier hyperplane of region *S*. These hyperplanes and the minimum distance of DMU_4 from them are as follows:

$$\{(X, Y)|-X_1-2X_2+10=0\} \xrightarrow{\min\|(X_4, Y_4)-(X, Y)\|_2} r_1=\frac{2}{\sqrt{5}}\simeq 0.89$$

$$\{(X, Y)|-X_1+2=0\} \xrightarrow{\min\|(X_4, Y_4)-(X, Y)\|_2} r_2=2$$

$$\{(X, Y)|4Y-X_1-20=0\} \xrightarrow{\min\|(X_4, Y_4)-(X, Y)\|_2} r_3=\frac{12}{\sqrt{17}}\simeq 2.91$$

$$\{(X, Y)|-X_2+1=0\} \xrightarrow{\min\|(X_4, Y_4)-(X, Y)\|_2} r_4=1$$

$$\{(X, Y)|6Y-5X_1-2=0\} \xrightarrow{\min\|(X_4, Y_4)-(X, Y)\|_2} r_5=\frac{4}{\sqrt{61}}\simeq 0.51$$

Consequently,

$$r^*=\min\{r_1, r_2, r_3, r_4, r_5\}=0.51$$

The results mentioned above indicate that the stability radius of DMU_4 is equal to $r^*=0.51$, meaning that if all inputs and outputs of DMU_4 change by maximum 0.51, this unit will remain efficient.

Table 2. All Defining Hyperplanes of PPS and the DMUs on These Hyperplanes

$H_1=\{(X,Y)\|6Y-5X_1-2=0\}$	$\{DMU_3, DMU_5\}$
$H_2=\{(X,Y)\|0.353Y-0.412X_1-0.235X_2+1.0=0\}$	$\{DMU_3, DMU_4, DMU_5\}$
$H_3=\{(X,Y)\|-X_1+2=0\}$	$\{DMU_5\}$
$H_4=\{(X,Y)\|-X_2+1=0\}$	$\{DMU_3, DMU_6, DMU_7\}$
$H_5=\{(X,Y)\|Y-8X_2=0\}$	$\{DMU_7\}$
$H_6=\{(X,Y)\|-X_1-X_2+6=0\}$	$\{DMU_4, DMU_5\}$
$H_7=\{(X,Y)\|-X_1-4X_2+12=0\}$	$\{DMU_3, DMU_4\}$
$H_8=\{(X,Y)\|Y-8=0\}$	$\{DMU_7\}$
$H_9=\{(X,Y)\|4Y-X_1-20=0\}$	$\{DMU_3, DMU_7\}$

6. Case Study: Banking Industry

In this section, we consider the real-world data set for the European banks in the year 2017. The related data[1] are presented in Table 3. This dataset consists of 10 DMUs (bank branches), where E={ DMU_3, DMU_4, DMU_9, DMU_{10}}.

Table 3. Top 10 Bank Branches of European Banks Ranking in the Year 2017

DMU	Bank name	Country	Capital (x_1)	Assets (x_2)	Profit (y_1)
1	HSBC Holdings	UK	138022.00	2374986.00	7112.00
2	Credit Agricole	France	88344.21	1813525.26	8162.11
3	BNP Paribas	France	86475.79	2186272.63	11800.00
4	Banco Santander	Spain	77588.27	1409605.26	11334.74
5	Barclays	UK	70330.86	1497686.42	5095.06
6	Groupe BPCE	France	59586.32	1300252.63	6705.26
7	Deutsche Bank	Germany	58406.32	1674258.95	852.63
8	Societe Generale	France	55374.74	1454990.53	6638.95
9	BBVA	Spain	52718.95	770374.74	6728.42
10	RBS	UK	49900.00	985995.06	5039.51

1. https://www.thebankerdatabase.com/index.cfm?fuseaction=home_page.data.

There are two inputs of capital and assets, and one output of profit. For DMU_9, we estimate stability region and stability radius. At first, we obtain the efficient hyperplanes of PPS passing through DMU_9. These hyperplanes are:

$$H_1 = \{(X, Y) | 884Y - 156X_1 + 2561747 = 0\}$$

$$H_2 = \{(X, Y) | -X_2 + 770374.74 = 0\}$$

$$H_3 = \{(X, Y) | -987X_1 - 13X_2 + 61980269 = 0\}$$

The set of DMUs adjacent to DMU_9 located on H_1, H_2, and H_3 is $\delta_1 = \{\mathrm{DMU}_4, \mathrm{DMU}_{10}\}$. In addition, the supporting hyperplanes that pass from the members of δ_1 but do not pass from DMU_9 are:

$$H_4 = \{(X, Y) | 950Y - 50X_1 - 6911208 = 0\}$$

$$H_5 = \{(X, Y) | 9994Y - 6X_2 - 104840402 = 0\}$$

Corresponding to these hyperplanes, we have the following half-spaces:

$$H_4^- = \{(X, Y) | 950Y - 50X_1 - 6911208 \le 0\}$$

$$H_5^- = \{(X, Y) | 9994Y - 6X_2 - 104840402 \le 0\}$$

Then,

$$S_1 = H_4^- \cap H_5^-$$

Now, by eliminating DMU_9 from the PPS and reevaluating inefficient DMUs of model (2) by model (7), we have $\delta_2 = \{\mathrm{DMU}_8\}$. New defining supporting hyperplanes of PPSthat are binding at the members of set $\delta_1 \mathrm{U} \delta_2 = \{\mathrm{DMU}_4, \mathrm{DMU}_{10}, \mathrm{DMU}_8\}$ are:

$$H_{i_1} = \{(X, Y) | 985Y - 15X_2 + 9472497 = 0\}$$

$$H_{i_2} = \{(X, Y) | 774Y - 226X_1 + 7382064 = 0\}$$

$$H_{i_3} = \{(X, Y) | -X_2 + 985995 = 0\}$$

$$H_{i_4} = \{(X, Y) | 824Y - 176X_1 - 0.8X_2 + 5362168 = 0\}$$

Then,

$$H_{i_1}^+ = \{(X, Y) | 985Y - 15X_2 + 9472497 \ge 0\}$$

$$H_{i_2}^+ = \{(X, Y) | 774Y - 226X_1 + 7382064 \ge 0\}$$

$$H_{i_3}^+ = \{(X, Y) | -X_2 + 985995 \ge 0\}$$

$$H_{i_4}^+ = \{(X, Y) | 824Y - 176X_1 - 0.8X_2 + 5362168 \ge 0\}$$

And

$$S_2 = H_{i_1}^+ \cup H_{i_2}^+ \cup H_{i_3}^+ \cup H_{i_4}^+$$

Consequently, the stability region of DMU_9 is established as follows:

$$S = S_1 \cap S_2$$

Now, according to the Euclidean norm, we estimate the stability radius of DMU_9 by calculating the minimum distance of DMU_9 from each frontier hyperplane of region *S*.

$$H_4=\{(X\ ,\ Y\ )|950Y\ -50X_1-6911208=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_1\simeq 3316.64$$

$$H_5=\{(X\ ,\ Y\ )|9994Y\ -6X_2-104840402=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_2\simeq 4224.45$$

$$H_{i_1}=\{(X\ ,\ Y\ )|985Y\ -15X_2+9472497=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_{i_1}\simeq 4613.06$$

$$H_{i_2}=\{(X\ ,\ Y\ )|774Y\ -226X_1+7382064=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_{i_2}\simeq 837.61$$

$$H_{i_3}=\{(X\ ,\ Y\ )|-X_2+985995=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_{i_3}\simeq 215620.26$$

$$H_{i_4}=\{(X\ ,\ Y\ )|824Y\ -176X_1-0.8X_2+5362168=0\}\xrightarrow{\min(X_9,\ Y_9)-(X\ ,\ Y\ )_2} r_{i_4}\simeq 1200.53$$

The minimum amount between r_1 and r_2 equals 3316.64, and this implies that the maximum reduction rate of inputs and the maximum increase rate of output for DMU_9 to reach the defining supporting hyperplanes of PPS which are binding at $\delta_1=\{\mathrm{DMU}_4,\mathrm{DMU}_{10}\}$ is equal to 3316.64, i.e., the maximum allowable rate ofdecrease in capital and assets as well as the maximum allowable increase rate of profit for BBVA Bank in Spain is 3316.64, and for larger amounts, the efficiency of Banco Santander Bank, which is adjacent to BBVA Bank, is disrupted and becomes inefficient. Moreover, the minimum rate among $\{r_{i_1},r_{i_2},r_{i_3},r_{i_4}\}$ equals 837.61.This means that the maximum increase rate of inputs and the maximum decrease rate of output for DMU_9 to reach the defining supporting hyperplanes ofthe new PPS (T'_v)which are passing from the members of set $\delta_1 \cup \delta_2=\{\mathrm{DMU}_4,\mathrm{DMU}_{10},\mathrm{DMU}_8\}$ is equal to 837.61.That is, the maximum allowable rate of increase in capital and assets as well as the maximum allowable rate of decrease in profit for BBVA Bank is 837.61, and for amounts greater than 837.61, BBVA Bank becomes inefficient. Therefore the stability radius of the largest symmetric cell to the center of DMU_9 is obtained by:

$$r^*=\min\{r_1,\ r_2,\ r_{i_1},\ r_{i_2},\ r_{i_3},\ r_{i_4}\}=837.61$$

Namely, if all inputs and outputs of BBVA Bank change by maximum 837.61, this bank will remain efficient and the efficiency status of other banks remains constant.

To assess the efficiency sensitivity analysis of organizations, the consideration of input/output changes is very significant. The proposed sensitivity analysis approach could be used as a " what-if " toolin managers' strategies to change the input or output of the DMU under consideration, in the sense that management is able to know what may happen to the efficiency of units if data variation occurs in all input/output space for an efficient unit. As a result, this subject can lead managers to better decisionmaking. In addition, this approach can be applied in each application of real-world problems such as hospitals, universities, schools, companies, etc.,since these rates of change have important economic and managerial implications for trade-off analysis and resource allocation.

7. Conclusion Remarks and Further Research

In this paper, a novel sensitivity analysis approach is presented. Most of the sensitivity analysis methods for an extreme efficient DMU have been performed based on worsening the position of the extreme efficient DMU.In contrast,it was required to study the way to find a region in which the status of the extreme DMU improves while the position of other extreme DMUs remains unchanged. Therefore, the main contributions of the current research can be mentioned as follows:

- The presented method is capable of finding the largest performance stability region for an extreme efficient DMU under changes in all directions of input/output space through efficient hyperplanes.
- It determines the largest symmetric cell to the center of the extreme efficient DMU and consequently the largest stability radius in all directions of input/output space.
- The efficacy of the proposed approach is illustrated by a numerical example.
- The implementation of the suggested method is done using a real-world case study from banking industry.

For future research, a similar argument could be put forward for CCR-efficient DMUs and obtaining a stability radius with norm l_1 and norm l_∞. The proposed sensitivity analysis could also be adapted to many other models, including fuzzy data, interval data, and ratio data to suggest the new methods of sensitivity analysis (see Hatami-Marbini & Toloo, 2019; Peykani et al., 2019).

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