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A fast algorithm for the linear programming problem constrained with the Weighted power mean – Fuzzy Relational Equalities (WPM-FRE)

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ABSTRACT

In this paper, a linear programming problem is investigated in which the feasible region is formed as a special type of fuzzy relational equalities (FRE). In this type of FRE, fuzzy composition is considered as the weighted power mean operator (WPM). Some theoretical properties of the feasible region are derived and some necessary and sufficient conditions are also presented to determine the feasibility of the problem. Moreover, two procedures are proposed for simplifying the problem. Based on some structural properties of the problem, an algorithm is presented to find the optimal solutions and finally, an example is described to illustrate the algorithm.

Keyword: Fuzzy relational equalities, mean operators, weighted power mean, fuzzy compositions, linear programming

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1 Introduction

In this paper, we study the following linear optimization model whose constraints are formed as a fuzzy system defined by the weighted power mean operator:

$$\begin{array}{l} \min \ cx \\ \quad A\varphi x = b \\ \quad x \in [0,1]^n \end{array} \tag{1}$$

where $I = \{1,2,...,m\}$, $J = \{1,2,...,n\}$, $A = (a_{ij})_{m\times n}$, $0 \le a_{ij} \le 1$ ($\forall i \in I$ and $\forall j \in J$), is a fuzzy matrix, $b = (b_i)_{m\times 1}$,$0 \le b_i \le 1$ ($\forall i \in I$), is an m–dimensional fuzzy vector, and "φ" is the max-weighted power mean composition, that is, $\varphi(x,y) = (wx^p + (1-w)y^p)^{1/p}$. Furthermore, let $S(A,b)$ denote the feasible solutions sets of problem (1), that is, $S(A,b) = \{x \in [0,1]^n \ : \ A\varphi x = b\}$. Additionally, if a_i denotes the i'th row of matrix A, then problem(1) can be also expressed as follows:

$$\begin{array}{l} \min \ cx \\ \quad \varphi(a_i, x) = b_i \ , \ i \in I \\ \quad x \in [0,1]^n \end{array} \tag{2}$$

where the constraints mean $\varphi(a_i,x) = \max_{j\in J}\{\varphi(a_{ij},x_j)\} = b_i$ ($\forall i \in I$) and $\varphi(a_{ij},x_j) = \left(w a_{ij}^p + (1-w)x_j^p\right)^{1/p}$.

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [39]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [35]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [25,33].

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Actually, The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [32]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2,3,12,13,15,16,28,29,32].

Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [15,16,34,36,37,40,42,43, 45,48,51]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI) [12 – 14,17,18,26 ,50]. Li and Yang [26] studied a FRI with addition-min composition and presented an algorithm to search for minimal solutions. Ghodousian et al. [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \leq b^1$and $D\varphi x \geq b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where $\varphi$is an operator with (closed) convex solutions.

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1,8,11 – 16,23,27,30,38,41,46,50]. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. In [23] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider. Wu et al. [44] improved the method used by Fang and Li, by decreasing the search domain. The topic of the linear optimization problem was also investigated with max-product operation [20,31]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [31]. Also, in [20] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [46] and max-t-norm composition [15,16,21,27,41].

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [6,10,15,16,18,24,30,47]. For example, Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients.

The optimization problem subjected to various versions of FRI could be found in the literature as well [12 – 14,17,18,49,50]. Xiao et al. [50] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.

In this paper, an algorithm is proposed to find all the optimal solutions of problem (1). Firstly, we describe some structural details of WPM-FREs such as the theoretical properties of WPM-fuzzy equalities and necessary and sufficient conditions for the feasibility of the problem. Then, the feasible region is completely determined by a finite number of convex cells. Additionally, some simplification processes are introduced to reduce the problem. Finally, an algorithm is presented to solve the main problem.

The remainder of the paper is organized as follows. Section 2 gives some basic results on the WPM-fuzzy equalities. Also, some feasibility conditions are derived. In section 3, the feasible region is characterized in terms of a finite number of closed convex cells. In section 4, some simplification rules are presented. These rules convert the problem into an equivalent one that is easier to solve. The optimal solution of the problem is described in Section 5 and, finally in section 6 an example is presented to illustrate the algorithm.

2 Basic properties of WPM – FREs

In this section, the structural properties of each fuzzy equation $\varphi(a_i, x) = b_i$ is investigated and its solutions are found. Let $S(a_i, b_i)$ denote the feasible solutions set of i'th equation, that is, $S(a_i, b_i) = \{x \in [0,1]^n \ : \ \varphi(a_i, x) = b_i\}$. So, $S(A, b) = \bigcap_{i \in I} S(a_i, b_i)$.

Lemma 1. Let $i \in I$, $j_0 \in J$ and $a_{ij_0} > \frac{b_i}{\sqrt[p]{w}}$. Then, $S(a_i, b_i) = \emptyset$.

Proof. Since φ is an increasing function on $[0,1]^2$ in both variables, we note that $\varphi(a_{ij_0}, x_j) > \varphi(b_i \big/ \sqrt[p]{w}, x_j) = \left(b_i^p + (1-w)x_{j_0}^p\right)^{1/p} \geq b_i$. Thus, for each $x \in [0,1]^n$ we have $\varphi(a_i, x) = \max\limits_{j \in J}\{\varphi(a_{ij}, x_j)\} \geq \varphi(a_{ij_0}, x_{j_0}) > b_i$. Hence, $x \notin S(a_i, b_i)$, $\forall x \in [0,1]^n$. □

Lemma 2. Let $a_{ij_0} \leq \frac{b_i}{\sqrt[p]{w}}$ for some $i \in I$ and $j_0 \in J$. If $b_i^p \geq 1 - w$ and $a_{ij_0} < \left(\frac{b_i^p + w - 1}{w}\right)^{1/p}$, then $\varphi(a_{ij_0}, x_{j_0}) < b_i$, $\forall x_{j_0} \in [0,1]$.

Proof. Since $b_i^p \geq 1 - w$, then $[(b_i^p + w - 1)/w]^{1/p} \geq 0$. Now, the result follows from the relations $\varphi(a_{ij_0}, x_{j_0}) < \varphi(\, [(b_i^p + w - 1)/w]^{1/p}, 1\,) = b_i$. □

Lemma 3. Let $a_{ij_0} \leq \frac{b_i}{\sqrt[p]{w}}$ for some $i \in I$ and $j_0 \in J$. Also, suppose that either $b_i^p < 1 - w$ or $a_{ij_0} \geq \left(\frac{b_i^p + w - 1}{w}\right)^{1/p}$. Then, $x_{j_0} = \left(\frac{b_i^p - w\,a_{ij_0}^p}{1-w}\right)^{1/p}$ is the unique solution to the equality $\varphi(a_{ij_0}, x_{j_0}) = b_i$.

Proof. It is easy to verify that $\varphi(a_{ij_0}, x_{j_0}) = b_i$. Now, since φ is an increasing function, we have $\varphi(a_{ij_0}, x_j) > b_i$ if $x_j > [(b_i^p - w\,a_{ij_0}^p)/(1-w)]^{1/p}$ and $\varphi(a_{ij_0}, x_j) < b_i$ if $x_j < [(b_i^p - w\,a_{ij_0}^p)/(1-w)]^{1/p}$. □

From Lemmas 1, 2 and 3, the following theorem is resulted that gives a necessary and sufficient condition for the feasibility of the set $S(a_i, b_i)$.

Theorem 1. For a fixed $i \in I$, $S(a_i, b_i) \neq \emptyset$ if and only if
(a) $a_{ij} \leq b_i \big/ \sqrt[p]{w}$, $\forall j \in J$.
(b) There exist some $j_0 \in J$ such that either $b_i^p < 1 - w$ or $a_{ij_0} \geq [(b_i^p + w - 1)/(w)]^{1/p}$.

Definition 1. For an arbitrary fixed $i \in I$, let $J^-(i) = \left\{ j \in J \ : \ a_{ij} > b_i \big/ \sqrt[p]{w} \right\}$. Additionally, define $J^\infty(i) = \left\{ j \in J \ : \ b_i^p \geq 1 - w \ , \ a_{ij} < [(b_i^p + w - 1)/w]^{1/p} \right\}$ and $J(i) = J - \{J^-(i) \bigcup J^\infty(i)\}$. According to Theorem 1, the following corollary is directly attained. This corollary characterizes all the feasible solutions of $S(a_i, b_i)$.

Corollary 1. $x' \in S(a_i, b_i)$if and only if $J^-(i) = \emptyset$, $J(i) \neq \emptyset$ and
(a) $x'_j \in [0, 1]$, $\forall j \in J^\infty(i)$.
(b) $x'_j \leq [(b_i^p - w a_{ij}^p)/(1 - w)]^{1/p}$, $\forall j \in J(i)$.
(c) There exist at least some $j_0 \in J(i)$ such that $x'_{j_0} = [(b_i^p - w a_{ij_0}^p)/(1 - w)]^{1/p}$.

Definition 2. Suppose that $S(a_i, b_i) \neq \emptyset$(hence, $J^-(i) = \emptyset$ from Corollary 1). Define $\overline{X}(i) \in [0, 1]^n$ such that

$$\overline{X}(i)_j = \begin{cases} \left(\frac{b_i^p - w a_{ij}^p}{1 - w} \right)^{1/p} & , if \ j \in J(i) \\ 1 & , if \ j \in J^\infty(i) \end{cases}$$

Theorem 2. Suppose that $S(a_i, b_i) \neq \emptyset$. Then, $\overline{X}(i)$ is the maximum solution of $S(a_i, b_i)$.

Proof. Since $S(a_i, b_i) \neq \emptyset$, then $J^-(i) = \emptyset$. Based on Corollary 1, $\overline{X}(i) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$. So, from Corollary 1, $x'_j \leq [(b_i^p - w a_{ij}^p)/(1 - w)]^{1/p}$, $\forall j \in J(i)$, and $x'_j \leq 1$, $\forall j \in J^\infty(i)$. Therefore, $x'_j \leq \overline{X}(i)_j$, $\forall j \in J$. □

Definition 3. Let $i \in I$ and $S(a_i, b_i) \neq \emptyset$. For each $j \in J(i)$, define $\underline{X}(i, j) \in [0, 1]^n$ such that

$$\underline{X}(i, j)_k = \begin{cases} \left(\frac{b_i^p - w a_{ij}^p}{1 - w} \right)^{1/p} & , k = j \\ 0 & , otherwise \end{cases}$$

Remark 1. Suppose that $S(a_i, b_i) \neq \emptyset$ and $j \in J(i)$. Then, from Definitions 2 and 3, we have $\overline{X}(i)_j = \underline{X}(i, j)_j$.

Theorem 3. Suppose that $S(a_i, b_i) \neq \emptyset$ and $j_0 \in J(i)$. Then, $\underline{X}(i, j_0)$ is a minimal solution of $S(a_i, b_i)$.

Proof. From Corollary 1, $\underline{X}(i, j_0) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$, $x' \leq \underline{X}(i, j_0)$and $x' \neq \underline{X}(i, j_0)$. So, $x'_j \leq \underline{X}(i, j_0)_j$, $\forall j \in J$ and $x' \neq \underline{X}(i, j_0)$. Therefore, $x'_j = 0$, $\forall j \in J - \{j_0\}$, and $x'_{j_0} < [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$. Hence, from Lemmas 1, 2 and 3 we have $\varphi(a_i, x') = \max\left\{\max_{j\in J-\{j_0\}}\{\varphi(a_{ij}, x'_j)\}, \varphi(a_{ij_0}, x'_{j_0})\right\} = \varphi(a_{ij_0}, x'_{j_0}) < b_i$ that contradicts $x' \in S(a_i, b_i)$. □

The following theorem shows that $S(a_i, b_i)$ can be stated in terms of the unique maximum solution and a finite number of minimal solutions.

Theorem 4. $S(a_i, b_i) = \bigcup_{j\in J(i)}[\underline{X}(i,j), \overline{X}(i)]$.

Proof. Let $x' \in S(a_i, b_i)$. From Theorem 2, $x' \le \overline{X}(i)$. Furthermore, there exist at least some $j_0 \in J(i)$ such that $x'_{j_0} = [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p}$ (Corollary 1). Thus, from Definition 3 we have $\underline{X}(i, j_0) \le x'$. Hence, $x' \in [\underline{X}(i,j_0), \overline{X}(i)]$. Conversely, let $x' \in \bigcup_{j\in J(i)}[\underline{X}(i,j), \overline{X}(i)]$. Therefore, $\varphi(a_{ij}, x'_j) \le \varphi(a_{ij}, \overline{X}(i)_j) \le b_i$, $\forall j \in J$. Moreover, there exists some $j_0 \in J(i)$ such that $x' \in [\underline{X}(i,j_0), \overline{X}(i)]$. So, Remark 1 implies $x'_{j_0} = \underline{X}(i,j_0)_{j_0} = \overline{X}(i)_{j_0}$ and therefore, $\varphi(a_{ij_0}, x'_{j_0}) = b_i$. Thus, we have

$$\varphi(a_i, x') = \max_{j\in J}\{\varphi(a_{ij}, x'_j)\} = \max\left\{\max_{j\in J-\{j_0\}}\{\varphi(a_{ij}, x'_j)\}, \varphi(a_{ij_0}, x'_{j_0})\right\} = \varphi(a_{ij_0}, x'_{j_0}) = b_i$$

which implies that $x' \in S(a_i, b_i)$. □

3 Feasible region of Problem (1)

In this section, a necessary and sufficient condition is derived to determine the feasibility of the main problem.

Definition 4. Let $\overline{X}(i)$ be as in Definition 2, $\forall i \in I$. We define $\overline{X} = \min_{i\in I}\left\{\overline{X}(i)\right\}$.

Definition 5. Let $e : I \to \bigcup_{i\in I} J(i)$ so that $e(i) \in J(i)$, $\forall i \in I$, and let E be the set of all vectors e. For the sake of convenience, we represent each $e \in E$ as an m–dimensional vector $e = [j_1, j_2, ..., j_m]$ in which $j_k = e(k)$, $k = 1, 2, ..., m$.

Definition 6. Let $e = [j_1, j_2, ..., j_m] \in E$. We define $\underline{X}(e) \in [0,1]^n$ such that $\underline{X}(e)_j = \max_{i\in I}\left\{\underline{X}(i, e(i))_j\right\} = \max_{i\in I}\left\{\underline{X}(i, j_i)_j\right\}$, $\forall j \in J$.

The following theorem indicates that the feasible region of problem 1 is completely found by a finite number of closed convex cells.

Theorem 5. $S(A, b) = \bigcup_{e\in E}[\underline{X}(e), \overline{X}]$.

Proof. Since $S(A, b) = \bigcap_{i\in I} S(a_i, b_i)$, from Theorem 4 we have

$S(A,b) = \bigcap_{i\in I} \bigcup_{j\in J_i} [\underline{X}(i,j), \overline{X}(i)]$. So, $S(A,b) = \bigcup_{e\in E} \bigcap_{i\in I} [\underline{X}(i,e(i)), \overline{X}(i)]$ (see Definitions 5 and 6), i.e.,
$S(A,b) = \bigcup_{e\in E} \left[\max_{i\in I} \{\underline{X}(i,e(i))\}, \min_{i\in I} \left\{\overline{X}(i)\right\}\right]$. Now, the result follows from Definitions 4 and 6. □

The following Corollary gives a simple necessary and sufficient condition for the feasibility of $S(A,b)$.

Corollary 2. $S(A,b) \neq \emptyset$ iff $\overline{X} \in S(A,b)$.

4 Simplification techniques

In practice, there are often some components of matrix A that have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. For this reason, various simplification processes have been proposed by researchers. We refer the interesting reader to [13] where a brief review of such these processes is given. Here, we present two simplification techniques based on the weighted power mean operator.

Definition 7. If a value changing in an element, say a_{ij}, of a given fuzzy relation matrix A has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.

Corollary 3. Suppose that $\varphi(a_{ij_0}, x_{j_0}) < b_i, \forall x \in S(A,b)$. In this case, it is obvious that $\max_{j=1}^{n} \left\{\varphi(a_{ij}, x_j)\right\} = b_i$ is equivalent to $\max_{\substack{j=1 \\ j\neq j_0}}^{n} \left\{\varphi(a_{ij}, x_j)\right\} = b_i$, that is, "resetting a_{ij_0} to zero" has no effect on the solutions of problem (1) (since component a_{ij_0} only appears in the i'th constraint of problem (1). Therefore, if $\varphi(a_{ij_0}, x_{j_0}) < b_i, \forall x \in S(A,b)$, then "resetting a_{ij_0} to zero" is an equivalence operation.

Lemma 4 (first simplification). Suppose that $j_0 \in J^{\infty}(i)$, for some $i \in I$ and $j_0 \in J$. Then, "resetting a_{ij_0} to zero" is an equivalence operation.
Proof. The proof is directly resulted from Lemma 2. □

Lemma 5 (second simplification). Suppose that $j_0 \in J(k)$, where $k \in I$ and $j_0 \in J$. If there exists some $i \in I$ $(i \neq k)$ such that $j_0 \in J(i)$ and $b_i^p - b_k^p < w(a_{ij_0}^p - a_{kj_0}^p)$, then "resetting a_{kj_0} to zero" is an equivalence operation.

Proof. We show that $\varphi(a_{kj_0}, x_{j_0}) < b_k, \forall x \in S(A,b)$. Consider an arbitrary feasible solution $x \in S(A,b)$. Since $x \in S(A,b)$, it turns out that $\varphi(a_{kj_0}, x_{j_0}) > b_k$ never holds. So, assume that $\varphi(a_{kj_0}, x_{j_0}) = b_k$. Since $j_0 \in J(k)$, from lemma 3 we conclude that $x_{j_0} = [(b_k^p - w a_{kj_0}^p)/(1-w)]^{1/p}$. On the other hand, inequality $b_i^p - b_k^p < w(a_{ij_0}^p - a_{kj_0}^p)$ implies that $[(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p} < [(b_k^p - w a_{kj_0}^p)/(1-w)]^{1/p}$. So, according to Definitions 3 and 4, $\overline{X}_{j_0} \le [(b_i^p - w a_{ij_0}^p)/(1-w)]^{1/p} < x_{j_0}$. Therefore, $x \notin \bigcup_{e \in E} [\underline{X}(e), \overline{X}]$ that means $x \notin S(A,b)$ (Theorem 5). □

5 Resolution of Problem (1)

It can be easily verified that $\overline{X}$ is the optimal solution for
min $\left\{Z_1 = \sum_{j=1}^n c_j^- x_j \,:\, A\varphi x = b\,,\, x \in [0,1]^n\right\}$, and the optimal solution for
min $\left\{Z_2 = \sum_{j=1}^n c_j^+ x_j \,:\, A\varphi x = b\,,\, x \in [0,1]^n\right\}$ is $\underline{X}(e^*)$ for some $e^* \in E$, where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \min\{c_j, 0\}$ for $j = 1,2,...,n$ [9,13,19,28]. According to the foregoing results, the following theorem shows that the optimal solution of Problem (1) can be obtained by the combination of $\overline{X}$ and $\underline{X}(e^*)$.

Theorem 6. Suppose that $S(A,b) \neq \emptyset$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems Z_1 and Z_2, respectively. Then $c^T x^*$ is the lower bound of the optimal objective function in (1), where $x^* = [x_1^*, x_2^*, ..., x_n^*]$ is defined as follows:

$$x_j^* = \begin{cases} \overline{X}_j & c_j < 0 \\ \underline{X}(e^*)_j & c_j \ge 0 \end{cases}$$

for $j = 1,2,...,n$.

Proof. For a general case, see the proof of Theorem 4.1 in [13]. □

Corollary 4. Suppose that $S(A,b) \neq \emptyset$. Then, x^* as defined in Theorem 5, is the optimal solution of problem (1).

Proof. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \le x_j^* \le \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E} [\underline{X}(e), \overline{X}] = S(A,b)$. □

6 Numerical example

Consider the following linear programming problem constrained with a fuzzy system defined by the weighted power mean operator:

$$\min\ Z = -7.6582x_1 - 2.029x_2 + 6.6277x_3 - 6.3x_4 + 0.0157x_5 - 7.4737x_6 + 7.2926x_7$$

$$\begin{bmatrix} 0.6763 & 0.8969 & 0.8403 & 0.3000 & 0.0710 & 0.0758 & 0.3529 \\ 0.3362 & 0.2721 & 0.1956 & 0.3396 & 0.0101 & 0.2557 & 0.1193 \\ 0.1637 & 0.5426 & 0.2534 & 0.3701 & 0.4916 & 0.5761 & 0.2454 \\ 0.5161 & 0.1330 & 0.9090 & 0.1477 & 0.3827 & 0.7212 & 0.2452 \\ 0.2319 & 0.8371 & 0.1275 & 0.8609 & 0.5201 & 0.6163 & 0.0654 \end{bmatrix} \varphi x = \begin{bmatrix} 0.8657 \\ 0.6520 \\ 0.6926 \\ 0.8833 \\ 0.8350 \end{bmatrix}$$
$$x \in [0,1]^7$$

where $|I| = 5$, $|J| = 7$ and $\varphi(x,y) = (wx^p + (1-w)y^p)^{1/p}$ in which $w = 3/4$ and $p = 3$. Moreover, $Z_1 = -7.6582x_1 - 2.029x_2 - 6.3x_4 - 7.4737x_6$ and $Z_2 = 6.6277x_3 + 0.0157x_5 + 7.2926x_7$. For each $i \in I$, we have $J^-(i) = \emptyset$. Also, $J(1) = \{2,3\}$, $J(2) = \{1,4\}$, $J(3) = \{2,5,6\}$, $J(4) = \{3\}$ and $J(5) = \{2,4\}$. Therefore, by Theorem 1, $S(a_i,b_i) \neq \emptyset$, $\forall i \in I$. According to Definition 2, the maximum solutions of $S(a_i,b_i) \neq \emptyset$, $\forall i \in I$, are attained as follows:

$$\overline{X}(1) = [1\ ,\ 0.7552\ ,\ 0.9341\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1]$$
$$\overline{X}(2) = [0.9982\ ,\ 1\ ,\ 1\ ,\ 0.9970\ ,\ 1\ ,\ 1\ ,\ 1]$$
$$\overline{X}(3) = [1\ ,\ 0.9471\ ,\ 1\ ,\ 1\ ,\ 0.9908\ ,\ 0.9107\ ,\ 1]$$
$$\overline{X}(4) = [1\ ,\ 1\ ,\ 0.7955\ ,\ 1\ ,\ 1\ ,\ 1\ ,\ 1]$$
$$\overline{X}(5) = [1\ ,\ 0.8286\ ,\ 1\ ,\ 0.7456\ ,\ 1\ ,\ 1\ ,\ 1]$$

Hence, by Definition 4, we have

$$\overline{X} = [0.9982\ ,\ 0.7552\ ,\ 0.7955\ ,\ 0.7456\ ,\ 0.9908\ ,\ 0.9107\ ,\ 1].$$

Also, by Definition 3 and Theorem 3, for example, the minimal solutions of $S(a_1,b_1)$are obtained as follows:

$$\underline{X}(1,2) = [0\ ,\ 0.7552\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0], \underline{X}(1,3) = [0\ ,\ 0\ ,\ 0.9341\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0]$$

Therefore, by Theorem 4, $S(a_1,b_1) = [\underline{X}(1,2), \overline{X}(1)] \bigcup [\underline{X}(1,3), \overline{X}(1)]$.
According to Corollary 2, since $\overline{X} \in S(A,b)$, then the problem is feasible. On the other hand, from Definition 6, we have $|E| = 24$. Therefore, the number of all vectors $e \in E$ is equal to 24. However, each solution $\underline{X}(e)$ generated by vectors $e \in E$ is not necessary a feasible minimal solution.
Additionally, we have $J^{\infty}(1) = \{1,4,5,6,7\}$, $J^{\infty}(2) = \{2,3,5,6,7\}$, $J^{\infty}(3) = \{1,3,4,7\}$, $J^{\infty}(4) = \{1,2,4,5,6,7\}$ and $J^{\infty}(5) = \{1,3,5,6,7\}$. So, from the first simplification technique (Lemma 4), "resetting all the components a_{ij} ($i \in I$ and $j \in J^{\infty}(i)$) to zeros" are equivalence operations. Also, by Lemma 5 (second simplification), we can change the value of components a_{13}, a_{24}, a_{32} and a_{52} to zeros. For example, since $3 \in J(1) \bigcup J(4)$ and $0.0404 = b_4^p - b_1^p < w(a_{43}^p - a_{13}^p) = 0.1183$, Lemma 5 implies $a_{13} = 0$.
By applying the simplification methods, $|E|$ is decreased from 24 to 2. Therefore, the simplification processes reduced the number of the minimal candidate solutions from

24 to 2, by removing 22 points $\underline{X}(e)$. Indeed, the feasible region has 2 minimal solutions as follows:

$$e_1 = [2,1,6,3,4] \Rightarrow \underline{X}(e_1) = [0.9982\ ,\ 0.7552\ ,\ 0.7955\ ,\ 0.7456\ ,\ 0\ ,\ 0.9107\ ,\ 0]$$
$$e_2 = [2,1,5,3,4] \Rightarrow \underline{X}(e_2) = [0.9982\ ,\ 0.7552\ ,\ 0.7955\ ,\ 0.7456\ ,\ 0.9908\ ,\ 0\ ,\ 0]$$

By comparison of the values of the objective function for the minimal solutions, $\underline{X}(e_1)$ is optimal for Z_2 (i.e., $e^* = e_1$). Thus, from Theorem 6,
$x^* = [0.9982\ ,\ 0.7552\ ,\ 0.7955\ ,\ 0.7456\ ,\ 0\ ,\ 0.9107\ ,\ 0]$ and then $Z^* = c^T x^* = -15.4085$.

7 Conclusion

In this paper, we proposed an algorithm to solve the linear optimization model constrained with weighted power mean fuzzy relational equalities (WPM-FRE). The feasible solutions set of each WPM-FRE was obtained and their feasibility conditions were described. Based on the foregoing results, the feasible region of the problem is completely resolved. It was shown that the feasible solutions set can be write in terms of a finite number of closed convex cells. Moreover, two simplification operations (depending on the max-WPM composition) were proposed to accelerate the solution of the problem. Finally, a method was introduced for finding the optimal solutions of the problem.

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