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An Alternative Proof for a Theorem of R.L. Graham Concerning CHEBYSHEV Polynomials

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ABSTRACT

In this paper, an alternative proof is provided for a theorem of R.L.Graham concerning Chebyshev polynomials. While studying the properties of a double star, R.L.Graham [2] proved a theorem concerning Chebyshev polynomials of the first kind $T_n(x)$. The purpose of this paper is to provide an alternative proof for his theorem. Our method is based on the divisibility properties of the natural numbers. One may observe that the Chebyshev polynomials evaluated at integers considered by R.L.Graham match with the solutions of the Pell's equation for a general, square-free $D \in N$.

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1 Introduction

Chebyshev polynomials of the first kind $T_n(x)$ are defined by $T_0(x) = 1$, $T_1(x) = x$, $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$ for $n \geq 0$ (see for e.g., W.Magnus, F.Oberhettinger and R.P.Soni [3]). *Chebyshev polynomials of the second kind* $U_n(x)$ are defined by $U_0(x) = 1$,

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$U_1(x) = 2x$, $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$ for $n \geq 0$. While studying the properties of a double star, R.L.Graham [2] proved a theorem concerning Chebyshev polynomials of the first kind $T_n(x)$. The purpose of this paper is to provide an alternative proof for his theorem. Our method is based on the divisibility properties of the natural numbers.

2 Statement and proof

Theorem 1. (R.L.Graham [2])
A, B, C are integral solutions of the equation $(A^2 - 1)(B^2 - 1) = C^2$ with $A, B > 0$ if and only if $A = T_m(x)$, $B = T_n(x)$ for some choice of integers $x > 0, m, n \geq 0$.
Proof.
The basic tool to be employed is the prime factorization of a natural number. Suppose A, B, C are three integers such that $(A^2 - 1)(B^2 - 1) = C^2$. Consider the prime factorization of C. Let $C = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k} \longrightarrow (1)$
where p_1, p_2, $\cdots$, p_k are distinct primes and i_1, i_2, $\cdots$, $i_k \geq 1$. Then
$C^2 = p_1^{2i_1} p_2^{2i_2} ?_k^{2i_k} \longrightarrow (2)$
$E = A^2 - 1, \longrightarrow (3)$ and
$F = B^2 - 1. \longrightarrow (4)$ Since the squares of two integers do not differ by 1, neither E nor F can be a square.
Now $EF = C^2$. $\longrightarrow$ (5) If gcd $(E, F) = 1$, then Equation (5) would imply that each element in the set $\{E, F\}$ is a square, which is a contradiction.Hence gcd $(E, F) \neq 1$.
Now $EF = p_1^{2i_1} p_2^{2i_2} \cdots p_k^{2i_k} \longrightarrow (6)$ Each prime p_j in the right side of Equation (**??**) divides EF and so it divides E or F or both. Let us classify the prime factors of C into three mutually disjoint categories as follows:

(i) Primes $\alpha_1, \alpha_2, ?\alpha_u$ which divide both E and F.

(ii) Primes $\beta_1, \beta_2, ?\beta_v$ which divide E but not F

(iii) Primes $\gamma_1, \gamma_2, ?\gamma_w$ which divide F but not E.

Since gcd $(E, F) \neq 1$, the set in category (i) has at least one element.

Consider any prime β in category (ii). It divides E but not F. This implies that all the powers of β which divide C^2 must contribute the factors of E. Therefore, the factor corresponding to β in the factorization of E appears with even exponent. With the same reasoning for category (iii), the factor corresponding to any γ in the factorization of F appears with even exponent.

Consider the primes $\alpha_1, \alpha_2, ?\alpha_u$ in category (i). If each one of them were to appear with even exponent in E and F, then each one of E and F is a square, which is a contradiction.

Let α be a prime appearing with odd exponent in E. Since α appears with even exponent in EF, it follows that α appears with odd exponent in F and vice versa. Let $\alpha_1, \alpha_2, \cdots, \alpha_\mu$ be the primes appearing with odd exponents in E and F, not necessarily with the same exponents in E and F. Let $\alpha_{\mu+1}, \cdots, \alpha_u$ be the primes appearing with even exponents in E and F, not necessarily with the same exponents in E and F. Then E and F have prime factorizations of the forms
$E = \alpha_1^{2g_1+1}\alpha_2^{2g_2+1}\ldots\alpha_\mu^{2g_\mu+1}\alpha_{\mu+1}^{2h_{\mu+1}}\ldots\alpha_u^{2h_u}\beta_1^{2j_1}\beta_2^{2j_2}\ldots\beta_v^{2j_v}, \longrightarrow (7)$ *and*
$F = \alpha_1^{2\delta_1+1}\alpha_2^{2\delta_2+1}\ldots\alpha_\mu^{2\delta_\mu+1}\alpha_{\mu+1}^{2\lambda_{\mu+1}}\ldots\alpha_u^{2\lambda_u}\gamma_1^{2\sigma_1}\gamma_2^{2\sigma_2}\ldots\gamma_w^{2\sigma_w}. \longrightarrow (8)$
where the numbers appearing in the exponents are integers. Consequently, E and F reduce to the following expressions:
$E = \alpha_1\alpha_2\ldots\alpha_\mu\Big(\alpha_1^{g_1}\ \alpha_2^{g_2}\ \ldots\ \alpha_\mu^{g_\mu}\ \ \alpha_{\mu+1}^{h_{\mu+1}}\ \ldots\ \alpha_u^{h_u}\ \beta_1^{j_1}\ \beta_2^{j_2}\ \ldots\ \beta_v^{j_v}\Big)^2, \longrightarrow (9)$
$F = \alpha_1\alpha_2\ldots\alpha_\mu\ \Big(\alpha_1^{\delta_1}\ \alpha_2^{\delta_2}\ \ldots\ \alpha_\mu^{\delta_\mu}\ \ \alpha_{\mu+1}^{\lambda_{\mu+1}}\ \ldots\ \alpha_u^{\lambda_u}\ \gamma_1^{\sigma_1}\ \gamma_2^{\sigma_2}\ \ldots\ \gamma_w^{\sigma_w}\Big)^2. \longrightarrow (10)$ Hence $E = Dy_r^2, \longrightarrow (11)$
$F = Dy_s^2 \longrightarrow (12)$
where
$D = \alpha_1\alpha_2\ldots\alpha_\mu, \longrightarrow (13)$

$y_r = \alpha_1^{g_1}\ \alpha_2^{g_2}\ \ldots\ \alpha_\mu^{g_\mu}\ \ \alpha_{\mu+1}^{h_{\mu+1}}\ \ldots\ \alpha_u^{h_u}\ \beta_1^{j_1}\ \beta_2^{j_2}\ \ldots\ \beta_v^{j_v}, \longrightarrow (14)$

$y_s = \alpha_1^{\delta_1}\ \alpha_2^{\delta_2}\ \ldots\ \alpha_\mu^{\delta_\mu}\ \ \alpha_{\mu+1}^{\lambda_{\mu+1}}\ \ldots\ \alpha_u^{\lambda_u}\ \gamma_1^{\sigma_1}\ \gamma_2^{\sigma_2}\ \ldots\ \gamma_w^{\sigma_w}. \longrightarrow (15)$
Since $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ are distinct primes, D is a square-free natural number. Now we have

$A^2 - 1 = Dy_r^2$ and $B^2 - 1 = Dy_s^2$.

Therefore $A^2 - Dy_r^2 = 1$ and $B^2 - Dy_s^2 = 1$.

Consequently, $A + y_r\sqrt{D}$ and $A + y_s\sqrt{D}$ are solutions of the Pell's equation $L^2 - DM^2 = 1. \longrightarrow (16)$
Let $x+y\sqrt{D}$ denote the fundamental solution of the Pell's equation (see for e.g., L.J.Mordell [4]). Then $A + y_r\sqrt{D}$ and $A + y_s\sqrt{D}$ are obtained as integral powers of $x + y\sqrt{D}$.

E.I.Emerson [1] derived the following recurrence relations for the solutions $x_r + y_r\ \sqrt{D}$ of Equation 16:

$$x_0 = 1, x_1 = x, x_{r+2} = 2x\ x_{r+1} - x_r,$$
$$y_0 = 0, y_1 = y, y_{r+2} = 2x\ y_{r+1} - y_r,$$

The relationship between Chebyshev polynomials of the first kind $T_n(x)$ and second kind $U_n(x)$ has been dealt with in [3]. The two kinds of polynomials are related as follows:

$$\begin{aligned}
T_1^2(x) - (x^2-1)U_0^2(x) &= x^2 - (x^2-1).1^2 = 1,\\
T_2^2(x) - (x^2-1)U_1^2(x) &= (2x^2-1)^2 - (x^2-1).(2x)^2\\
&= (4x^4-4x^2+1) - (4x^4-4x^2) = 1,\\
T_3^2(x) - (x^2-1)U_2^2(x) &= (4x^3-3x)^2 - (x^2-1).(4x^2-1)^2\\
&= (16x^6-24x^4+9x^2) - (x^2-1)(16x^4-8x^2+1)\\
&= (16x^6-24x^4+9x^2) - (16x^6-24x^4+9x^2-1) = 1, etc.
\end{aligned}$$

In general, $T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1$.

It is seen that the sequence $\{x_r\}$ obtained from the solutions of Equation (16) is identical with the sequence of Chebyshev polynomials of the first kind $T_n\,(x)$. It follows that the sequence $\{y_r\}$ obtained from the solutions of Equation 16 is related to the sequence of Chebyshev polynomials of the second kind $\{U_n\,(x)\,\}$ by
$D\ y_r^2 = (x^2-1)\,U_{r-1}^2(x). \longrightarrow (17)$
This implies that D is the square-free part of x^2-1. From (13) and (17), it is seen that $\alpha_1\alpha_2\cdots\alpha_\mu$ equals the square-free part of (x^2-1). Thus the value of D in (16) depends on the value $T_1(x) = x$ coming from the Chebyshev polynomials of the first kind. Therefore $A^2-1 = (x^2-1)\,U_{m-1}^2(x)$ and $B^2-1 = (x^2-1)\,U_{n-1}^2(x)$ for some $m, n \in N$.

i.e. $A^2 - (x^2-1)\,U_{m-1}^2(x) = 1$ and $B^2 - (x^2-1)\,U_{n-1}^2(x) = 1$.
Consequently there exist $m, n \geq 0$ such that $A = T_m\,(x)$, $B = T_n\,(x)$.

For the converse, suppose $A = T_m\,(x)$, $B = T_n\,(x)$ for some choice of integers $x > 0$, $m, n \geq 0$. Then we have $T_m^2(x) - (x^2-1)\,U_{m-1}^2(x) = 1$ and $T_n^2(x) - (x^2-1)\,U_{n-1}^2(x) = 1$. Therefore $A^2 = (x^2-1)\,U_{m-1}^2(x) + 1$ and $B^2 = (x^2-1)\,U_{n-1}^2(x) + 1$. Hence $(A^2-1)(B^2-1) = \left((x^2-1)\,U_{m-1}\,(x)\,U_{n-1}(x)\right)^2$. Take $C = (x^2-1)\,U_{m-1}\,(x)\,U_{n-1}(x)$. Then A, B, C satisfy the equation $(A^2-1)(B^2-1) = C^2$.

This completes the proof.

For the double star problem, one requires $A \neq B$.

3 Solutions considered by R.L.Graham

Among the solutions of the equation $(A^2-1)(B^2-1) = C^2$, R.L.Graham [2] has mentioned the following four solutions: $(A, B, C) = (2, 26, 45), (3, 17, 48), (5, 49, 240),$
$(3, 99, 280)$.

These solutions emanating from the proof of R.L.Graham and the present proof are illustrated below.

Solution 1. A = 2, B = 26, C = 45.

From Chebyshev polynomials of the first kind: $T_0(x) = 1, T_1(x) = 2, T_2(x) = 7, T_3(x) = 26$. Select $x = 2, A = T_1(x), B = T_3(x)$. Square-free part of $(x^2 - 1) = 3$.

From Pell's equation:$1^2 - 3.0^2 = 1, 2^2 - 3.1^2 = 1, 7^2 - 3.4^2 = 1, 26^2 - 3.15^2 = 1$. Select $D = 3, A = 2, B = 26$.

Solution 2. A = 3, B = 17, C = 48.

From Chebyshev polynomials of the first kind: $T_0(x) = 1, T_1(x) = 3, T_2(x) = 17, T_3(x) = 99$. Select $x = 3, A = T_1(x), B = T_2(x)$. Square-free part of $(x^2 - 1) = 2$.

From Pell's equation: $1^2 - 2.0^2 = 1, 3^2 - 2.2^2 = 1, 17^2 - 2.12^2 = 1, 99^2 - 2.70^2 = 1$. Select $D = 2, A = 3, B = 17$.

Solution 3. A = 5, B = 49, C = 240.

From Chebyshev polynomials of the first kind: $T_0(x) = 1, T_1(x) = 5, T_2(x) = 49, T_3(x) = 485$. Select $x = 5, A = T_1(x), B = T_2(x)$. Square-free part of $(x^2 - 1) = 6$.

From Pell's equation: $1^2 - 6.0^2 = 1, 5^2 - 6.2^2 = 1, 49^2 - 6.20^2 = 1, 485^2 - 6.198^2 = 1$. Select $D = 6, A = 5, B = 49$.

Solution 4. A = 3, B = 99, C = 280.

From Chebyshev polynomials of the first kind: $T_0(x) = 1, T_1(x) = 3, T_2(x) = 17, T_3(x) = 99$. Select $x = 3, A = T_1(x), B = T_3(x)$. Square-free part of $(x^2 - 1) = 2$.

From Pell's equation:$1^2 - 2.0^2 = 1, 3^2 - 2.2^2 = 1, 17^2 - 2.12^2 = 1, 99^2 - 2.70^2 = 1$. Select $D = 2, A = 3, B = 99$.

One may observe that the Chebyshev polynomials evaluated at integers considered by R.L.Graham match with the solutions of the Pell's equation for a general, square-free $D \in N$.

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