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Boolean Rings Based on Multirings

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Abstract

The purpose of this paper is to construct Boolean rings from multirings. In this regards, a method to construct a multigroup(multiring) on a given non-empty set, are introduced and its properties has been investigated. Also, an equivalence relation on a multiring are introduced and it is extended to an smallest strongly regular equivalence relation, such that its quotient space be a commutative Boolean ring with identity. Finally, the transitivity of this relation based on complete parts are proved.

Keywords: Multigroup; Multiring; Fundamental relation; Boolean ring.

Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th}Congress of the Scandinavian Mathematicians [1].

Hyperstructures have many applications to several sectors of both pure and applied sciences, especially in atomic physics and in harmonic analysis and complex hypernetworks [2,3]. The notation of hyperring is one of important hyperstructures and was introduced by Krasner [4], who used it as a technical tool in his study on the approximation of valued fields. Hyperring is a structure generalizing that of a ring, but where the addition is an composition, however a hypercompositions, that is, in hyperrings the sum and the product of two elements is not an element but a subset.

Fundamental relations are one of the main tools in algebraic hyperstructures theory by which hyperstructures are converted to structures as it is a bridge between of algebraic structures and algebraic hyperstructures. It is the smallest equivalence relation on a hyperstructure such that the quotient of hyperstructure via this relation is a corresponding (fundamental) structure. The fundamental relations on (semi) hypergroups have been studied by many authors, for example see Corsini [5], Hamidi [6,7], Freni [8], Leoreanu-Fotea et al. [9]. The fundamental relation on a hyperring was introduced by Vougiouklis [10]. Davvaz et al.[11- 13] defned a new strongly regular equivalence relation on a hyperring, and they proved that the quotient of hyperring on this relation is a commutative ring. In [14], R. Ameri, et al. generalized the work of Freni to hyperrings and introduce a new relation θ on a given (semi) hyperring R and showed θ^* is strongly regular relation and a quotient $\frac{R}{\theta^*}$ is a commutative (semi) ring which θ^* is the transitive closure of the relation θ.

The notions of multigroups, multirings and their corresponding reduced versions introduced by Marshall in [15,16] and provide a convenient framework to study the reduced theory of quadratic forms and spaces of

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orderings. A multiring is just a ring with a multivalued addition and spaces of signs which are also known as abstract real spectra, objects which arise naturally in the study of constructible sets in real geometry [17,18], are shown to be multirings of a particular sort. All of these objects are very natural and very useful, although they are not at all widely known. For more study see [19,20].

In this paper, we try to generalize the concept of rings to multirings and to describe some of their properties. We further, investigated the results of multirings and tried to compare them with some types of hyperrings. Indeed, we worked on the construction of multigroups and multirings based on main properties of groups and commutative rings. The main purpose of this study was to make a connection between multirings and commutative rings. So we introduce a novel strongly regular relation on multirings such that the quotient of multirings on this relation would be a commutative ring. Moreover, we sought and compared the differences and similarities between of famous fundamental relations on multirings and our novel rings were obtained from the quotient of multirings on this strongly regular relation while, on others, a strongly regular relation was not a Boolean ring.

Preliminaries

In this section, we are going to review some definitions and results [1,3,10,15,16], which are required in the following.

Let R be a non-empty set and $P^*(R) = \{S \mid \emptyset \neq S \subseteq R\}$. Every map $+: R \times R \to P^*(R)$ is called a hyperoperation and for all x and y of R, $+(x,y)$ is called the hyperproduct of x and y. A binary hyperstructure $(R,+)$ is called a hypergroupoid and $A + B = \bigcup_{a\in A, b\in B} a + b$, where $\emptyset \neq A, B \subseteq R$. Recall that a hypergroupoid $(R,+)$ is called a semihypergroup if $\forall\, x,y,z \in R$, $(x + y) + z = x + (y + z)$ and a semihypergroup $(R,+)$ is called a hypergroup if satisfes in the reproduction axiom, i.e. $\forall\, x \in R$, $x + R = R + x = R$. A commutative hypergroup $(R,+)$ ($\forall\, x,y \in R$, $x + y = y + x$) is called a canonical hypergroup, provided that (i) it has a scalar identity 0 ($\forall\, x \in R$, $0 + x = x + 0 = \{x\}$), (ii) every element has a unique inverse, ($\forall\, x \in R$, there exists a unique $-x \in G$, such that $0 \in x + (-x) \cap (-x) + x$), (iii) $x \in y + z$ implies $y \in x + (-z)$ and $z \in -y + x$ and we will denote it by $(R,+,-,0)$. A system $(R,+,\cdot,-,0,1)$ is called a multiring if (i) $(R,+,-,0)$ is a canonical hypergroup (commutative multigroup), (ii) $(R,\cdot,1)$ is a commutative monoid ("$\cdot$" is a binary operation on R which is commutative and associative and $x \cdot 1 = x$, $\forall\, x \in R$), (iii) $x \cdot 0 = 0$, $\forall\, x \in R$, (iv) $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$. A system $(R,+,\cdot,-,0,1)$ is called a general hyperring if (i) $(R,+,-,0)$ is a hypergroup, (ii) $(R,\cdot,1)$ is a semihypergroup (iii) $\forall\, x,y,z \in R$ $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. A map $f : R \to R'$ is said to be a multiring homomorphism if, for all $x,y \in R$, we have (i) $f(x + y) \subseteq f(x) + f(y)$, (ii) $f(x \cdot y) = f(x) \cdot f(y)$, (iii) $f(-x) = -f(x)$, (iv) $f(0) = 0$ and $f(1) = 1$.

Let $(R,+,\cdot)$ be a hyperring and ρ be an equivalence relation on R. Letting $\frac{R}{\rho} = \{\rho(r) | r \in R\}$, be the set of all equivalence classes of R with respect ρ. Define hyperoperations $\oplus$ and $\otimes$ as follows:

$\rho(a) \oplus \rho(b) = \{\rho(c) \mid c \in \rho(a) + \rho(b)\}$ and

$\rho(a) \otimes \rho(b) = \{\rho(c) \mid c \in \rho(a).\rho(b)\}$

In [10] it was proved that $(\frac{R}{\rho},\oplus,\otimes)$ is a ring if and if only ρ is strongly regular. The smallest equivalence relation, γ^*on R such that $(\frac{R}{\gamma^*},\oplus,\otimes)$ is a ring is called fundamental relation. Let U denote the set of all finite sum of finite products of elements of R. Define relation γ on R by

$$a\gamma b \Longleftrightarrow \exists u \in U : \{a,b\} \in u$$

In [11] Davvaz, et.al defined a fundamental relation α on every hyperring as follow: $x\alpha y \Leftrightarrow \exists n,k_1,k_2,\ldots,k_n \in \mathbb{N}, \exists \tau \in S_n$ and $x_{i1},x_{i2},\ldots,x_{ik} \in R, \exists \tau_1 \in S_{k1}, \tau_2 \in S_{k2},\ldots,\tau_n \in S_{kn}$ such that $x \in \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}$ and $y \in \sum_{i=1}^{n} A_{\tau i}$, where $A_i = \prod_{j=1}^{k_i} x_{i\tau_i(j)}$. If α^* is the transitive closure of α, then α^* is the smallest strongly regular relation on R such that R/α^* is a commutative ring. Fundamental relation plays an important role in theory of algebraic hyperstructure. (for more see [10,14]).

Construction of multigroups and multirings

In this section, we are to introduce the concept of Boolean multigroup and Boolean multiring and for the given arbitrary set, constructed at least a multigroup and a multiring. We introduce some example of multirings that are not hyperring.

Theorem 1. Let R be a nonempty set and $|R| \geq 4$. Then there exists a binary hyperoperation "$+$" on R, $0 \in R$ and a function $-: R \to R$ such that $(R,+,-,0)$ is a commutative multigroup.

Proof. Let R be an arbitrary set, fixed $a_0 = 0 \in R$

and $C_3 = \{a_1, a_2, a_3\} \subseteq R$. Now for all $a_i, a_j \in R$, we define a hyperoperation " $+$ " on R as follows:

$$a_i + a_j = \begin{cases} R\backslash C_3 & i = j \neq 0, \\ R\backslash(C_3 \cup \{0\}) & i \neq j \geq 4, \\ C_3\backslash\{a_i, a_j\} & 1 \leq i \neq j \leq 3, \\ \{a_i\} & 1 \leq i \leq 3 \text{ and } j \geq 4 \end{cases}$$

where for all $a_i, a_j \in R$, $a_i + a_j = a_j + a_i$ and $a_i + 0 = \{a_i\}$. By a manipulation it is easy to verify that $(R, +, -, 0)$ is a commutative multigroup.

Example 2. Let $R = \{0, a_1, a_2, a_3, a_4, a_5\}$. Then $(R, +, -, 0)$ is a commutative multigroup as in Theorem 1, as follows:

$+$	0	a_1	a_2	a_3	a_4	a_5
0	0	a_1	a_2	a_3	a_4	a_5
a_1	a_1	T	a_3	a_2	a_1	a_1
a_2	a_2	a_3	T	a_1	a_2	a_2
a_3	a_3	a_2	a_1	T	a_3	a_3
a_4	a_4	a_1	a_2	a_3	T	T'
a_5	a_5	a_1	a_2	a_3	T'	T

where, $\mathrm{T} = \{0, a_4, a_5\}$ and $\mathrm{T}' = \{a_4, a_5\}$.

Defnition 3.
Let $(R, +, \cdot, -, 0, 1)$ be a multiring. Then

(i) multigroup $(R, +, -, 0)$ is called a Boolean multigroup, if for all $x \in R$, we have $0 \in x + x$,

(ii) multiring R is said to be a Boolean multiring, if for all $x \in R$, we have $x \cdot x = x$.

In the following, we present some examples of Boolean multigroups and Boolean multirings.

Example 4.
(i) Let $R = \{0, b, c, d, e, f, g, h, k\}$. Consider the hyperoperation $+$ on R as follows:

+	0	b	c	d	e	f	g	h	k
0	0	b	c	d	e	f	g	h	k
b	b	{0,k}	h	g	f	e	d	c	b
c	c	h	{0,k}	f	g	d	e	b	c
d	d	g	f	{0,k}	h	c	b	e	d
e	e	f	g	h	{0,k}	b	c	d	e
f	f	e	d	c	b	{0,k}	h	g	f
g	g	d	e	b	c	h	{0,k}	f	g
h	h	c	b	e	d	g	f	{0,k}	h
k	k	b	c	d	e	f	g	h	0

It is easy to see that check that $(R, +, -, 0)$ is a commutative Boolean multigroup.

(ii) Let $R = \{0, 1, b, c, d\}$. Then $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring as follows:

(iii)

+	0	1	b	c	d
0	0	1	b	c	d
1	1	{0,d}	c	b	1
b	b	c	{0,d}	1	b
c	c	b	1	{0,d}	c
d	d	1	b	c	{0,d}

,

·	0	1	b	c	d
0	0	0	0	0	0
1	0	1	b	c	d
b	0	b	b	d	d
c	0	c	d	c	d
d	0	d	d	d	d

Since $b \cdot (c + d) = \{d\} \subset \{0, d\} = b \cdot c + b \cdot d$, we get that R is not a hyperring.

(iv) Let $R = \{0, a, 1\}$. Then $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring as follows:

+	0	1	a
0	0	1	a
1	1	R	{1,a}
a	a	{1,a}	R

,

·	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

Since $a \cdot (a + 1) = \{a\} \subset a \cdot a + a \cdot 1 = R$, we get that it is not a hyperring.

(v) Let $R = \{0, a, 1\}$. Then $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring as follows:

+	0	1	a
0	0	1	a
1	1	{0,a}	1
a	a	1	{0,a}

,

·	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

Since $a \cdot (a + 1) = \{a\} \subset a \cdot a + a \cdot 1 = \{0, a\}$, we get that it is not a hyperring.

(vi) Let $R = \{0, 1, a, b\}$. Then $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring as follows:

+	0	1	a	b
0	0	1	a	b
1	1	R	{1,a}	{1,b}
a	a	{1,a}	R	{a,b}
b	b	{1,b}	{a,b}	R

,

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	b

$b \cdot (b + 1) = \{b\} \subset b \cdot b + b \cdot 1 = R$, implies that R is not a hyperring.

Example 5. (i) Let $R = \{0, 1, a, b\}$. Then $(R, +, \cdot, -, 0, 1)$ is a multiring as follows:

+	0	1	a	b
0	0	1	a	b
1	1	R	S	S
a	a	S	R	S
b	b	S	S	R

,

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	b
b	0	b	b	b

where $S = R \setminus \{0\}$. Since $a \cdot (a + b) = \{a, b\} \subset R = a \cdot a + a \cdot b$, we get that $(R, +, \cdot, -, 0, 1)$ is not a hyperring.

(ii) Let $R = \{0, 1, a, b\}$. Then $(R, +, \cdot, -, 0, 1)$ is a multiring as follows:

+	0	1	a	b
0	0	1	a	b
1	1	R	S	S
a	a	S	R	S
b	b	S	S	R

,

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	a

where $S = R \setminus \{0\}$. Since $a \cdot (a + b) = \{a\} \subset R = a \cdot a + a \cdot b$, we get that $(R, +, \cdot, -, 0, 1)$ is not a hyperring.

In the following, a method is presented to construct a multiring on an arbitrary set.

Theorem 6. Let R be a nonempty set and $|R| \geq 4$. Then there exist a binary hyperoperation " $+$ ", a binary operation " $\cdot$ ", on R, $0, 1 \in R$ and a function $-: R \to R$ such that $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring.

Proof. Let $a_0 = 0$ and $a_1 = 1$ be fixed in R. By Theorem 1, there exists a binary hyperoperation " $+$ " on R, $0 \in R$ and a function $-: R \to R$ such that $(R, +, -, 0)$ is a multigroup. Now for all $a_i, a_j \in R$, we define a hyperoperation " $\cdot$ " on R as follows:

$$a_i \cdot a_j = \begin{cases} 0 & i = 0, \\ a_j & i = 1, \\ a_i & i = j, \\ a_4 & i = k, j \geq k + 1, 2 \leq k, \end{cases}$$
$$and\ for\ all\ 0 \leq i \neq j, a_i \cdot a_j = a_j \cdot a_i.$$

Some modifications and computations show that $(R, +, \cdot, -, 0, 1)$ is a multiring. Since $a_2 \cdot (a_1 + a_1) = a_2 \cdot (R \setminus C_3) = \{0, a_4\}$ and $a_2 \cdot a_1 + a_2 \cdot a_1 = R \setminus C_3$, we get that $a_2 \cdot (a_1 + a_1) \subset a_2 \cdot a_1 + a_2 \cdot a_1$ and conclude that $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring while is not a hyperring.

Corollary 7. (i) For any cardinal number ζ, there exists a Boolean multigroup of order ζ.

(ii) For any cardinal number ζ, there exists a Boolean multiring $(R, +, \cdot, -, 0, 1)$ of order ζ.

Example 8.

Let $R = \{0, 1, a_2, a_3, a_4, a_5, a_6\}$. Then $(R, +, \cdot, -, 0, 1)$ is a Boolean multiring as follows:

+	0	1	a_2	a_3	a_4	a_5	a_6
0	0	1	a_2	a_3	a_4	a_5	a_6
1	1	T	a_3	a_2	1	1	1
a_2	a_2	a_3	T	1	a_2	a_2	a_2
a_3	a_3	a_2	1	T	a_3	a_3	a_3
a_4	a_4	1	a_2	a_3	T	T'	T'
a_5	a_5	1	a_2	a_3	T'	T	T'
a_6	a_6	1	a_2	a_3	T'	T'	T

,

$\cdot$	0	1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	0	0	0
1	0	1	a_2	a_3	a_4	a_5	a_6
a_2	0	a_2	a_2	a_4	a_4	a_4	a_4
a_3	0	a_3	a_4	a_3	a_4	a_4	a_4
a_4	0	a_4	a_4	a_4	a_4	a_4	a_4
a_5	0	a_5	a_4	a_4	a_4	a_5	a_4
a_6	0	a_6	a_4	a_4	a_4	a_4	a_6

where $\mathbf{T} = \{\mathbf{0}, \mathbf{a_4}, \boldsymbol{a_5}, \boldsymbol{a_6}\}$ and $\mathbf{T}' = \mathbf{T} \setminus \{\mathbf{0}\}$.

Fundamental relation on multirings

In this section, we introduced a new relation κ^* on multirings which is an equivalence relation on multirings. We proved that κ^* is the smallest strongly regular on multirings such that the quotient of any multirings on κ^* is a Boolean ring. Moreover, it was shown that κ^* is a fundamental relation on multirings. In this section there exists some example such that shows $\kappa^* \neq \beta^*, \kappa^* \neq \gamma^*$ and $\kappa^* \neq \alpha^*$.

Defnition 9. Let $(R, +, \cdot, -, 0, 1)$ be a multiring, $x, y \in R$ and $n \in \mathbb{N}$. Define $x\kappa_{n,m}y$ if and only if there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} z_i$, $y \in \sum_{i=1}^{n} {z_i}^{k_i}$ where $k_i \in \{1, m\}$ and for all $u \in R$, $u^m = \underbrace{u \cdot u \cdot \ldots \cdot u}_{(m-times)}$ Clearly, $\kappa_{1,1} = \Delta = \{(x, x) | x \in R\}$ and so $\kappa = \bigcup_{\kappa \in \{1,m\}} \bigcup_{n \geq 1} (\kappa_{n,m} \cup \kappa_{n,m}^{-1})$ is a reflexive and symmetric relation.

Let κ^* be the transitive closure of κ (the smallest transitive relation that contains κ and will show by $\kappa^* = \bar{\kappa}$). Then in the following theorem we show that κ^* is a strongly regular relation.

Theorem 10. Let R be a multiring. Then κ^* is a strongly regular relation on R.

Proof. Let $x, y, z \in R$ and $(x, y) \in \kappa^*$. Then there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} z_i$, $y \in \sum_{i=1}^{n} {z_i}^{k_i}$ where $k_i \in \{1, m\}$. Now consider $z = z_{n+1}$ for all $u \in x + z$ and for all $\in y + z$, we have $u \in \sum_{i=1}^{n+1} z_i$, $v \in \sum_{i=1}^{n+1} {z_i}^{k_i}$ where $k_i \in \{1, m\}$. It follows that $(u, v) \in \kappa_{n+1,m}$ and so $(u, v) \in \kappa^*$. Since $(R, +, -, 0)$ is commutative multigroup, we get that the relation κ^* is a strongly regular relation respect to hyperoperation " $+$ ". In addition, since " $\cdot$ " is an operation on R, we get that $|z \cdot x| = |z \cdot y| = 1$. It follows that for all $1 \leq i \leq n$ there exists $1 \leq j \leq n$ such that $(z \cdot z_i^{k_i})^{k_i} = z \cdot z_j$ or $(z \cdot z_j)^{k_i} = z \cdot z_i^{k_i}$. Hence $(u, v) \in \kappa^*$ and the commutativity of operation " $\cdot$ " implies that κ^* is a strongly regular relation respect to hyperoperation " $\cdot$ ". Therefore κ^* is a strongly regular relation on R.

Example 11. (i) Let $R \cong \mathbb{Z}_4 \cup \{\sqrt{2}\}$. Then $(R, +_{\sqrt{2}}, \cdot_{\sqrt{2}}, -, \bar{0}, \bar{1})$ is a multiring as follows:

$+_{\sqrt{2}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\sqrt{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\sqrt{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$	$\bar{2}$
$\bar{3}$	$\bar{3}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\sqrt{2}$	$\sqrt{2}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\{\bar{0}, \sqrt{2}\}$

,

$\cdot_{\sqrt{2}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\sqrt{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\sqrt{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\sqrt{2}$
$\sqrt{2}$	$\bar{0}$	$\sqrt{2}$	$\bar{0}$	$\sqrt{2}$	$\bar{0}$

If $m \geq 2$, then simple computations show that $\kappa^*(\bar{0}) = \{\bar{0}, \bar{2}, \sqrt{2}\}$, $\kappa^*(\bar{1}) = \{\bar{1}, \bar{3}\}$ and so $R/\kappa^* \cong \mathbb{Z}_2$ while $R/\gamma^* \cong R/\kappa^* \cong \mathbb{Z}_4$ as follows:

$+$	$\alpha^*(0)$	$\alpha^*(1)$	$\alpha^*(2)$	$\alpha^*(3)$
$\alpha^*(0)$	$\alpha^*(0)$	$\alpha^*(1)$	$\alpha^*(2)$	$\alpha^*(3)$
$\alpha^*(1)$	$\alpha^*(1)$	$\alpha^*(2)$	$\alpha^*(3)$	$\alpha^*(0)$
$\alpha^*(2)$	$\alpha^*(2)$	$\alpha^*(3)$	$\alpha^*(0)$	$\alpha^*(1)$
$\alpha^*(3)$	$\alpha^*(3)$	$\alpha^*(0)$	$\alpha^*(1)$	$\alpha^*(2)$

,

$\cdot$	$\alpha^*(0)$	$\alpha^*(1)$	$\alpha^*(2)$	$\alpha^*(3)$
$\alpha^*(0)$	$\alpha^*(0)$	$\alpha^*(0)$	$\alpha^*(0)$	$\alpha^*(0)$
$\alpha^*(1)$	$\alpha^*(0)$	$\alpha^*(1)$	$\alpha^*(2)$	$\alpha^*(3)$
$\alpha^*(2)$	$\alpha^*(0)$	$\alpha^*(2)$	$\alpha^*(0)$	$\alpha^*(2)$
$\alpha^*(3)$	$\alpha^*(0)$	$\alpha^*(3)$	$\alpha^*(2)$	$\alpha^*(1)$

$+$	$\kappa^*(\bar{0})$	$\kappa^*(\bar{1})$
$\kappa^*(\bar{0})$	$\kappa^*(\bar{0})$	$\kappa^*(\bar{1})$
$\kappa^*(\bar{1})$	$\kappa^*(\bar{1})$	$\kappa^*(\bar{0})$

,

$\cdot$	$\kappa^*(\bar{0})$	$\kappa^*(\bar{1})$
$\kappa^*(\bar{0})$	$\kappa^*(\bar{0})$	$\kappa^*(\bar{0})$
$\kappa^*(\bar{1})$	$\kappa^*(\bar{0})$	$\kappa^*(\bar{1})$

Clearly R/κ^* is a Boolean ring, while $R/\gamma^*$and R/α^* are not Boolean ring.

(ii) Consider the multiring R, which has been defined in Example 8. Since for every $m \in \mathbb{N}$ and for all $0 \le i \le 6$ we have $a_i^m = a_i$, we get that, $\kappa^*(0) = \{0, a_4, a_5, a_6\}$, $\kappa^*(1) = \{1\}$, $\kappa^*(a_2) = \{a_2\}$, $\kappa^*(a_3) = \{a_3\}$ and so R/κ^* is a Boolean ring of order 4 as follows:

$+$	$\kappa^*(0)$	$\kappa^*(1)$	$\kappa^*(a_2)$	$\kappa^*(a_3)$
$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(1)$	$\kappa^*(a_2)$	$\kappa^*(a_3)$
$\kappa^*(1)$	$\kappa^*(1)$	$\kappa^*(0)$	$\kappa^*(a_3)$	$\kappa^*(a_2)$
$\kappa^*(a_2)$	$\kappa^*(a_2)$	$\kappa^*(a_3)$	$\kappa^*(0)$	$\kappa^*(1)$
$\kappa^*(a_3)$	$\kappa^*(a_3)$	$\kappa^*(a_2)$	$\kappa^*(1)$	$\kappa^*(0)$

,

$\cdot$	$\kappa^*(0)$	$\kappa^*(1)$	$\kappa^*(a_2)$	$\kappa^*(a_3)$
$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(0)$
$\kappa^*(1)$	$\kappa^*(0)$	$\kappa^*(1)$	$\kappa^*(a_2)$	$\kappa^*(a_3)$
$\kappa^*(a_2)$	$\kappa^*(0)$	$\kappa^*(a_2)$	$\kappa^*(a_2)$	$\kappa^*(0)$
$\kappa^*(a_3)$	$\kappa^*(0)$	$\kappa^*(a_3)$	$\kappa^*(0)$	$\kappa^*(a_3)$

Example 12.
Let $R = \{0, 1, 2, 3, 4\}$. Then $(R, +, \cdot, -, 0, 1)$ is a hyperring as follows:

$+$	0	1	2	3	4
0	0	{1,4}	2	3	{1,4}
1	{1,4}	2	3	0	2
2	2	3	0	{1,4}	3
3	3	0	{1,4}	2	0
4	{1,4}	2	3	0	2

,

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	{1,4}	2	3	{1,4}
2	0	2	0	2	2
3	0	3	2	{1,4}	3
4	0	{1,4}	2	3	{1,4}

Let $m = 2$. Then computations show that $\kappa^*(0) = \{0, 2\}$, $\kappa^*(1) = \{1, 3, 4\}$ and $R/\kappa^* = \{\kappa^*(0), \kappa^*(1)\}$. Hence $(R/\kappa^*, +, \cdot, -, \kappa^*(0), \kappa^*(1))$ is a Boolean ring as follows:

$+$	$\kappa^*(0)$	$\kappa^*(1)$
$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(1)$
$\kappa^*(1)$	$\kappa^*(1)$	$\kappa^*(0)$

,

$\cdot$	$\kappa^*(0)$	$\kappa^*(1)$
$\kappa^*(0)$	$\kappa^*(0)$	$\kappa^*(0)$
$\kappa^*(1)$	$\kappa^*(0)$	$\kappa^*(1)$

Now, it is easy to see that $\gamma^*(0) = \{0\}$, $\gamma^*(1) = \{1, 4\}$, $\gamma^*(2) = \{2\}$, $\gamma^*(3) = \{3\}$ and $R/\gamma^* = \{\gamma^*(0), \gamma^*(1), \gamma^*(2), \gamma^*(3)\}$. Hence $(R/\gamma^*, +, \cdot, -, \gamma^*(0), \gamma^*(1))$ is a (non Boolean) ring as follows:

$+$	$\gamma^*(0)$	$\gamma^*(1)$	$\gamma^*(2)$	$\gamma^*(3)$
$\gamma^*(0)$	$\gamma^*(0)$	$\gamma^*(1)$	$\gamma^*(2)$	$\gamma^*(3)$
$\gamma^*(1)$	$\gamma^*(1)$	$\gamma^*(2)$	$\gamma^*(3)$	$\gamma^*(0)$
$\gamma^*(2)$	$\gamma^*(2)$	$\gamma^*(3)$	$\gamma^*(0)$	$\gamma^*(1)$
$\gamma^*(3)$	$\gamma^*(3)$	$\gamma^*(0)$	$\gamma^*(1)$	$\gamma^*(2)$

,

$\cdot$	$\gamma^*(0)$	$\gamma^*(1)$	$\gamma^*(2)$	$\gamma^*(3)$
$\gamma^*(0)$	$\gamma^*(0)$	$\gamma^*(0)$	$\gamma^*(0)$	$\gamma^*(0)$
$\gamma^*(1)$	$\gamma^*(0)$	$\gamma^*(1)$	$\gamma^*(2)$	$\gamma^*(3)$
$\gamma^*(2)$	$\gamma^*(0)$	$\gamma^*(2)$	$\gamma^*(0)$	$\gamma^*(2)$
$\gamma^*(3)$	$\gamma^*(0)$	$\gamma^*(3)$	$\gamma^*(2)$	$\gamma^*(1)$

In other words, since R is a commutative hyperring, we get that $\alpha^* = \gamma^*$ and so $R/\alpha^* \cong R/\gamma^* \ncong R/\kappa^*$.

Corollary 13. Let R be a multiring.
(i) Necessarily $\kappa \neq \bar{\kappa}$,
(ii) Necessarily $\kappa^* \neq \beta^*$,
(iii) Necessarily $\kappa^* \neq \gamma^*$,
(iv) Necessarily $\kappa^* \neq \alpha^*$.

Theorem 14. Let $n \in \mathbb{N}$. Then
(i) there exists Boolean group R such that $|R| = 4^n$.
(ii) there exists Boolean ring R such that $|R| = 4^n$.

Proof. Let $n \in \mathbb{N}$ and R be non-empty set. (i) If $|R| = n + 1$, then by Theorem 1, there exists a binary hyperoperation "$+$" on R, $0 \in R$ and a function $-: R \to R$ such that $(R, +, -, 0)$ is a commutative multigroup. Since $C_3 = \{a_1, a_2, a_3\}$ we get that $\beta^*(0) = R \setminus C_3$, $\beta^*(a_1) = \{a_1\}$, $\beta^*(a_2) = \{a_2\}$, $\beta^*(a_3) = \{a_3\}$ and so R/β^* is a Boolean group of order 4. If $R' = \underbrace{R/\beta^* \times R/\beta^* \times \cdots \times R/\beta^*}_{n-times}$ then one can see that R' is a Boolean group.

(ii) By item (i) and Theorem 6, there exist a binary hyperoperation "$+$", binary operation "$\cdot$", on R, $0, 1 \in R$ and a function $-: R \to R$ such that $(R, +, \cdot, -, 0, 1)$ is a multiring. Since $C_3 = \{a_1, a_2, a_3\}$, we get that $\kappa^*(a_0) = \kappa^*(0) = R \setminus C_3$, $\kappa^*(a_1) = \kappa^*(1) = \{a_1\}$, $\kappa^*(a_2) = \{a_2\}$, $\kappa^*(a_3) = \{a_3\}$ and so $R/\kappa^*$is a Boolean ring of order 4. If $R' = \underbrace{R/\kappa^* \times R/\kappa^* \times \cdots \times R/\kappa^*}_{n-times}$ then one can see that R' is a Boolean ring.

Theorem 15. Let $m \in \mathbb{N}$. If $m = 1$, then $\beta^* = \kappa^*$ or $\gamma^* = \kappa^*$.

Proof. Let $x, y \in R$ and $(x, y) \in \kappa^*$. If $m = 1$, then there exists $z_1, z_2, \ldots, z_n \in R$ such that $\{x, y\} \in \sum_{i=1}^{n} z_i$ Thus $(x, y) \in \beta^*$ Moreover, since R is a commutative multiring, we get that $\beta^* = \gamma^*$.

Example 16. Consider the multiring R which is defined in Example 11. If $m = 1$, then $R/\kappa^* \cong R/\beta^* \cong R/\gamma^*$.

Corollary 17. $\beta^* \subseteq \kappa^*$ and $\gamma^* \subseteq \kappa^*$.

Theorem 18. Let R be a multiring. Then R/κ^* is a Boolean ring.

Proof. Since by Theorem 10, κ^* is a strongly regular equivalence relation on R, then for all x, y we have $|\kappa^*(x) + \kappa^*(y)| = 1$ and $|\kappa^*(x) \cdot \kappa^*(y)| = 1$. Now, R is a multiring, it follows that R/κ^* is a commutative ring. In addition, for all $x \in R$, $x \in x + 0$ and $x^2 \in x^2 + 0$ implies that $\kappa^*(x) = \kappa^*(x^2) = (\kappa^*(x))^2$. Thus R/κ^* is a Boolean ring.

Theorem 19. Let R be a multiring. Then κ^* is the smallest strongly regular equivalence relation on R, such that R/κ^* is a Boolean ring.

Proof. Since κ^* is a strongly regular equivalence relation on R, by Theorem 18, we get that R/κ^* is a Boolean ring. Now, we show that it is the smallest. Let θ be a strongly regular equivalence relation on R, such that R/θ is a Boolean ring. Let $\varphi : R \to R/\theta$ be the canonical homomorphism and $(x, y) \in \kappa^*$. Thus there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} z_i$ and $y \in \sum_{i=1}^{n} {z_i}^{k_i}$, where $k_i \in \{1, m\}$. Since θ is a strongly regular equivalence relation on R, we have $\varphi(x) = \sum_{i=1}^{n} \varphi(z_i)$ and $\varphi(y) \in \sum_{i=1}^{n} \varphi({z_i}^{k_i}) = \sum_{i=1}^{n} (\varphi(z_i))^{k_i}$. Since R/θ is a Boolean ring, we get that $\sum_{i=1}^{n} \varphi(z_i) = \sum_{i=1}^{n} \varphi({z_i}^{k_i})$ and so $\varphi(x) = \varphi(y)$.

It follows that $(x, y) \in \theta$ and so $\kappa^* \subseteq \theta$. Therefore, κ^* is a smallest strongly regular equivalence relation on R, such that R/κ* is a Boolean ring.

Transitivity of κ via the κ-parts

In this section, we have determined some necessary and sufficient conditions so that the relation κ would be transitive.

Definition 20. Let M be a non-empty subset of a multiring R. M is called a κ-part if for every $n, m \in \mathbb{N}$ and $(z_1, z_2, \ldots, z_n) \in R^n$, we have

$\sum_{i=1}^{n} z_i \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} {z_i}^{k_i} \subseteq M$,

where $k_i \in \{1, m\}$.

Example 21. (i) Let $R = \mathbb{Z}_5 \cup \{\sqrt{2}\}$. Then $(R, +_{\sqrt{2}}, \cdot_{\sqrt{2}}, -, \bar{0}, \bar{1})$ is a multiring as follows:

$+_{\sqrt{2}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\sqrt{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\sqrt{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$	$\bar{2}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{4}$	$\bar{4}$	$\{\bar{0}, \sqrt{2}\}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\sqrt{2}$	$\sqrt{2}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\{\bar{0}, \sqrt{2}\}$

,

$\cdot_{\sqrt{2}}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\sqrt{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\sqrt{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$	$\sqrt{2}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\sqrt{2}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\sqrt{2}$
$\sqrt{2}$	$\bar{0}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\bar{0}$

Let $M = \{\bar{0}, \sqrt{2}\}$. Then $M \cap (\bar{2} + \bar{3}) \neq \emptyset$ but $\bar{2}^2 + \bar{3} \nsubseteq M$, so M is not a κ-part, while it is a β-part. Routine computations show that $M = R$ is only a κ-part of multiring R.

(ii) Consider the multiring which has been defined in Example 8. Let $k \in \mathbb{N}$. It is easy to see that for all $0 \leq i \leq 6$, we have $a_i^k = a_{\mathrm{i}}$. Hence for every $m \in \mathbb{N}$, $1 \leq i \leq 6$ and $k_i \in \{1, m\}$ we get that $\sum_{i=1}^{6} a_i^{k_i} = \sum_{i=1}^{6} a_{\mathrm{i}}$. So if $T \cap \sum_{i=1}^{6} a_{\mathrm{i}} \neq \emptyset$ implies that $\sum_{i=1}^{6} a_i^{k_i} \subseteq T$ therefore, $M = T$ is a κ-part of multiring R.

Lemma 22. Let M be a non-empty subset of a multiring R. Then, the following conditions are equivalent:

(i) M is a κ-part of R,

(ii) $x \in M, x \,\kappa\, y$ imply $y \in M$,

(iii) $x \in M, x \,\kappa^* y$ imply $y \in M$.

Proof. $(i) \Longrightarrow (ii)$. Let $x \in M$, $y \in R$ and $x\kappa y$. Then there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} z_i$ and $y \in \sum_{i=1}^{n} {z_i}^{k_i}$ where $k_i \in \{1, m\}$. Since M is a κ-part and $\sum_{i=1}^{n} z_i \cap M \neq \emptyset$ we have $y \in M$.

$(ii) \Longrightarrow (iii)$. Let $x \in M$, $y \in R$ and $x \,\kappa^* y$. Then there exist $n \in \mathbb{N}$, $a_i \in R$ such that $x \,\kappa\, a_1 \,\kappa\, a_2 \,\kappa \ldots \kappa\, a_n \kappa\, y$. Now $x \in M$ and $x \,\kappa\, a_1$, then by the item (ii), we obtain that $a_1 \in M$. Since for every $1 \leq i \leq n$, $a_i \kappa a_{i+1}, a_1 \in M$, using the item (ii), we conclude $\mathrm{a_n} \in$ M and so y $\in$ M.

(iii) $\Longrightarrow$ (i). Let $n \in \mathbb{N}$ and $(z_1, z_2, \ldots, z_n) \in R^n$. If $\sum_{i=1}^{n} z_i \cap M \neq \emptyset$, then there exists $y \in \sum_{i=1}^{n} z_i \cap M$. Now, for all $t \in \sum_{i=1}^{n} {z_i}^{k_i}$ where $k_i \in \{1, m\}$, $y \in \sum_{i=1}^{n} z_i$ implies that $t\, \kappa^* y$. Hence $y \in M$, $t\, \kappa^* y$ and the item (iii), imply that $t \in M$.

Example 23. Let $R = \{0, 1, a_2, a_3, a_4, a_5\}$. Then $(R, +, \cdot, -, 0, 1)$ is a multiring as follows:

$+$	0	1	a_2	a_3	a_4	a_5
0	0	1	a_2	a_3	a_4	a_5
1	1	T	a_3	a_2	1	1
a_2	a_2	a_3	T	1	a_2	a_2,
a_3	a_3	a_2	1	T	a_3	a_3
a_4	a_4	1	a_2	a_3	T	T'
a_5	a_5	1	a_2	a_3	T'	T

$\cdot$	0	1	a_2	a_3	a_4	a_5
0	0	0	0	0	0	0
1	0	1	a_2	a_3	a_4	a_5
a_2	0	a_2	a_4	a_4	a_4	a_4
a_3	0	a_3	a_4	a_3	a_4	a_4
a_4	0	a_4	a_4	a_4	a_4	a_4
a_5	0	a_5	a_4	a_4	a_4	a_5

where $T = \{0, a_4, a_5\}$ and $T' = T \setminus \{0\}$. Obviously $\kappa^*(0) = T$, $\kappa^*(1) = \{1\}$, $\kappa^*(a_2) = \{a_2\}$ and $\kappa^*(a_3) = \{a_3\}$. Clearly T is a κ-part, $\{1\}, \{a_2\}, \{a_3\}$ and R are κ-parts. But $T = \{0, a_4, a_5\}$ is a γ-Part, $\{1\}, \{a_2\}$ and $\{a_3\}$ are γ-parts.

Theorem 24. Let R be a multiring. Then, the following conditions are equivalent:

(i) κ is a transitive relation

(ii) for any $x \in R$, $\kappa^*(x)$ is a κ-part.

Proof. $(i) \Longrightarrow (ii)$. Let $x \in R$, $n \in \mathbb{N}$ and $(z_1, z_2, \ldots, z_n) \in R^n$. If $\sum_{i=1}^{n} z_i \cap \kappa^*(x) \neq \emptyset$ then for every $y \in \sum_{i=1}^{n} {z_i}^{k_i}$ where $k_i \in \{1, m\}$, there exists $t \in \sum_{i=1}^{n} z_i \cap \kappa^*(x)$ and so $y\, \kappa\, t$. Hence $x\, \kappa\, t\, \kappa\, y$ and by the item (i), transitivity of κ implies that $y \in \kappa^*(x)$.

$(ii) \Longrightarrow (i)$. Let $x\, \kappa\, y$ and $y\, \kappa\, z$. Then there exist $n, n' \in \mathbb{N}$, $z_1, z_2, \ldots, z_n \in R$, $z_1', z_2', \ldots, z_n' \in R$, $m, m' \in \mathbb{N}$ such that $x \in \sum_{i=1}^{n} z_i$, $y \in \sum_{i=1}^{n} {z_i}^{k_i}$, $y \in \sum_{i=1}^{n'} z_i'$ and $z \in \sum_{i=1}^{n'} {z_i'}^{k_i'}$, where $k_i, k_i' \in \{1, m\}$. Since $\kappa^*(x)$ is a κ-part, $x \in \kappa^*(x) \cap \sum_{i=1}^{n} z_i$ and $y \in \sum_{i=1}^{n} {z_i}^{k_i}$ then

$$\sum_{i=1}^{n} {z_i}^{k_i} \subseteq \kappa^*(x)$$
$$\Rightarrow y \in \sum_{i=1}^{n'} z_i' \cap \kappa^*(x)$$
$$\Rightarrow \sum_{i=1}^{n'} {z_i'}^{k_i'} \subseteq \kappa^*(x) \quad (\kappa^*(x) \; is \; a \; \kappa - part)$$
$$\Rightarrow z \in \kappa^*(x).$$

Now, $z \in \kappa(z)$ and $z\, \kappa^* x$, then by Lemma 22, $x \kappa z$.

Example 25. Consider the multiring R which has been defined in Example 23. We saw that T is a κ-Part, $T_1 = \{a_1\}$, $T_2 = \{a_2\}$, $T_3 = \{a_3\}$ and R are only κ-parts.

Let R be a multiring and $x, y \in R$. We denote by β_+ and $\beta_{\cdot}^{(m)}$ the following binary relations:

$x \beta_+ y$ if and only if $\exists a_1, a_2, \ldots, a_n \in R$ such that $\{x, y\} \subseteq \sum_{i=1}^{n} a_i$

$x \beta_{\cdot}^{(m)} y$ if and only if $\exists m \in \mathbb{N}$ such that $x = y^m$ or $y = x^m$.

and denote the transitive closures of the relations β_+ and $\beta_{\cdot}^{(m)}$ by β_+^* and $\beta_{\cdot}^{(m,*)}$. Clearly for all $a \in R$ we have, $\beta_+^*(a) \subseteq \kappa^*(a)$ and $\beta_{\cdot}^{(m,*)}(a) \subseteq \kappa^*(a)$. Define $\varphi_+ : R \to R/\beta_+^*$ by $\varphi_+(a) = \beta_+^*(a)$, $\varphi_{\cdot} : R \to R/\beta_{\cdot}^{(m,*)}$ by $\varphi_{\cdot}(a) = \beta_{\cdot}^{(m,*)}(a)$ and $\varphi : R \to R/\kappa^*$ by $\varphi(a) = \kappa^*(a)$. We denote by w_+, w the kernels of φ_+ and φ, respectively and define

$w_+ = \{a \in R \mid \beta_+^*(a) = \beta_+^*(0),$

$w = \{a \in R \mid \kappa^*(a) = \kappa^*(0)\}.$

Corollary 26. Let $m \in \mathbb{N}$ and R be a multiring. Then

(i) $\beta_{\cdot}^{(1,*)}$ is a regular relation on R,

(ii) for all $x \in R$, $\beta_{\cdot}^{(m,*)}(x) = \beta_{\cdot}^{(m,*)}(x^m)$.

Example 27. Condider the multiring R in Example 8. Routine computations show that for all $m \in \mathbb{N}$ and $1 \leq i \leq 6$, we have $\beta_{\cdot}^{(m,*)}(a_i) = \{a_i\}$. Then $R/\beta_{\cdot}^{(m,*)} \cong R$ is a multiring and so for all $m \in \mathbb{N}$, $\beta_{\cdot}^{(m,*)}$ is a regular relation on R.

Let $\beta_{\cdot}^{(m,**)}$ be the regular closure of $\beta_{\cdot}^{(m,*)}$ (the smallest regular relation such that contains $\beta_{\cdot}^{(m,*)}$).

Theorem 28. Let R be a multiring and $m \in \mathbb{N}$. Then

(i) R/β_+^* is a commutative group,

(ii) $((R, +, \cdot)/\beta_{\cdot}^{(m,**)})/\beta_+^* \cong (R, +, \cdot)/\kappa^*$,

where

$$\beta_{\cdot}^{(m,**)}(a) \oplus \beta_{\cdot}^{(m,**)}(b) = \left\{\beta_{\cdot}^{(m,**)}(c) \middle| c \in \beta_{\cdot}^{(m,**)}(a) + \beta_{\cdot}^{(m,**)}(b)\right\}$$

and

$$\beta_{\cdot}^{(m,**)}(a) \odot \beta_{\cdot}^{(m,**)}(b) = \left\{\beta_{\cdot}^{(m,**)}(c) \middle| c \in \beta_{\cdot}^{(m,**)}(a) \cdot \beta_{\cdot}^{(m,**)}(b)\right\}$$

Proof. (i) Clearly R/β_+^* is a group. Since R is a multiring, then R is a commutative multigroup and so R/β_+^* is a commutative group.

(ii) Define a map

$\varphi : (R/\beta_{.}^{(m,**)})/\beta_+^* \to R/\kappa^*$ by $\varphi(\beta_+^*(\beta_{.}^{(m,**)}(x))) = \kappa^*(x)$. Let $\beta_+^*(\beta_{.}^{(m,**)}(x_1)) = \beta_+^*(\beta_{.}^{(m,**)}(x_2))$. Then there exist $t \in \beta_{.}^{(m,**)}(x_2)$ and $m \in \mathbb{N}$ such that $x_1 = t^m$ or $x_1^m = t$ and $\beta_+^*(x_1) = \beta_+^*(t)$. Hence there exist $m \in \mathbb{N}$ such that $x_1 = x_2^m$ or $x_1^m = x_2$ and $\beta_+^*(x_1) = \beta_+^*(x_2)$. It concludes that $\kappa^*(x_1) = \kappa^*(x_2)$ and so φ is a well-defined map.

Let $\kappa^*(x_1) = \kappa^*(x_2)$. Then there exist $z_1, z_2, \ldots, z_n \in R$ and $m \in \mathbb{N}$ such that $x_1 \in \sum_{i=1}^n z_i$ and $x_2 \in \sum_{i=1}^n {z_i}^{k_i}$ where $k_i \in \{1, m\}$. Thus $\beta_{.}^{(m,**)}(x_1) \in \beta_{.}^{(m,**)}(\sum_{i=1}^n z_i) = \sum_{i=1}^n \beta_{.}^{(m,**)}(z_i)$ and $\beta_{.}^{(m,**)}(x_2) \in \beta_{.}^{(m,**)}(\sum_{i=1}^n {z_i}^{k_i}) = \sum_{i=1}^n \beta_{.}^{(m,**)}({z_i}^{k_i})$. So by Corollary 26, $\{\beta_{.}^{(m,**)}(x_1), \beta_{.}^{(m,**)}(x_2)\} \subseteq \sum_{i=1}^n \beta_{.}^{(m,**)}(z_i)$.

Hence $\beta_+^*(\beta_{.}^{(m,**)}(x_1)) = \beta_+^*(\beta_{.}^{(m,**)}(x_2))$ and so φ is an one to one map. Clearly φ is an epimorphism and so it is an isomorphism.

Example 29. Consider the multiring R in Example 21. For $m = 2$, we have $\beta_{.}^{(m,**)}(\bar{0}) = \{\bar{0}, \sqrt{2}\}$, $\beta_{.}^{(m,**)}(\bar{1}) = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ and so we have the following tables:

$\oplus$	$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{1})$
$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{1})$
$\beta_{.}^{(2,**)}(\bar{1})$	$\beta_{.}^{(2,**)}(\bar{1})$	R

,

□	$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{1})$
$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{0})$
$\beta_{.}^{(2,**)}(\bar{1})$	$\beta_{.}^{(2,**)}(\bar{0})$	$\beta_{.}^{(2,**)}(\bar{1})$

and so

$((R,+,\cdot)/\beta_{.}^{(2,**)})/\beta_+^* \cong (R,+,\cdot)/\kappa^*$, where $|(R,+,\cdot)/\kappa^*| = 1$.

If $m = 3$, then $\beta_{.}^{(3,**)}(\bar{0}) = \{\bar{0}, \sqrt{2}\}$, $\beta_{.}^{(3,**)}(\bar{1}) = \{\bar{1}\}$, $\beta_{.}^{(3,**)}(\bar{2}) = \{\bar{2}, \bar{3}\}$ and $\beta_{.}^{(3,**)}(\bar{4}) = \{\bar{4}\}$. So we have

$\oplus$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{4})$
$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{4})$
$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{2})$	A	$\beta_{.}^{(3,**)}(\bar{0})$,
$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{2})$	A	B	C
$\beta_{.}^{(3,**)}(\bar{4})$	$\beta_{.}^{(3,**)}(\bar{4})$	$\beta_{.}^{(3,**)}(\bar{0})$	C	$\beta_{.}^{(3,**)}(\bar{2})$

□	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{4})$
$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{0})$
$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{1})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{4})$
$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{4})$	$\beta_{.}^{(3,**)}(\bar{2})$
$\beta_{.}^{(3,**)}(\bar{4})$	$\beta_{.}^{(3,**)}(\bar{0})$	$\beta_{.}^{(3,**)}(\bar{4})$	$\beta_{.}^{(3,**)}(\bar{2})$	$\beta_{.}^{(3,**)}(\bar{1})$

where $A = \beta_{.}^{(3,**)}(\bar{2}) \cup \beta_{.}^{(3,**)}(\bar{4})$, $B = \beta_{.}^{(3,**)}(\bar{0}) \cup \beta_{.}^{(3,**)}(\bar{1}) \cup \beta_{.}^{(3,**)}(\bar{4})$ and $C = \beta_{.}^{(3,**)}(\bar{2}) \cup \beta_{.}^{(3,**)}(\bar{1})$. It is easy to see that $((R,+,\cdot)/\beta_{.}^{(3,**)})/\beta_+^* \cong (R,+,\cdot)/\kappa^*$, where $|(R,+,\cdot)/\kappa^*| = 1$.

Theorem 30. Let R be a multiring. Then

(i) $w_+ \subseteq w$ and $w_+ + w_+ \subseteq w$,

(ii) $w + w \subseteq w$,

(iii) $Rw \subseteq w$,

Proof. (i) Let $x \in w_+$ Since $\beta_+^* \subseteq \kappa^*$, we get that $x \in w$. In addition, if $x \in w_+ + w_+$, then there exists $y, z \in w_+$ such that $\varphi_+(x) = \varphi_+(y) + \varphi_+(z) = \varphi_+(0)$. Hence $x \in w_+$ and by (i), $w_+ + w_+ \subseteq w$.

(ii) Let $x \in w + w$. Then there exists $y, z \in w$ such that $\kappa^*(x) = \kappa^*(y + z)$ and so $\varphi(x) = \varphi(y) + \varphi(z) = \varphi(0)$. Hence $x \in w$ and $w + w \subseteq w$.

(iii) Let $x \in Rw$. Then there exist $r \in R$ and $t \in w$ such that $x = rt$ and so $\varphi(x) = \varphi(r)\varphi(0) = \varphi(0)$. Hence $x \in w$ and so $Rw \subseteq w$.

Example 31. Consider the multiring R in Example 11. Routine computations show that $w_+ = \{\bar{0}, \sqrt{2}\}$, $w = \{\bar{0}, \bar{2}, \sqrt{2}\}$ and $Rw = w$.

Results and Discussion

The current paper considered the notion of multigroup, multirings and investigated some properties of them. It is introduced a strongly regular relation κ^*on multirings and is shown that:

(i) Boolean rings with identity are obtained via the fundamental relation on multirings.

(ii) $\kappa^* \neq \gamma^*$and $\kappa^* \neq \alpha^*$.

(iii) Boolean rings are obtained from quotient of multirings on κ^*.

(iv) Necessarily $k \neq \bar{k}$

(v) The conditions for transitivity of κ are obtained with respect to complete parts.

We hope that these results are helpful for furthers studies in multigroup, multirings and rings. In our future studies, we hope to obtain more results regarding fuzzy multiring and soft and rough multiring.

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