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Vibration Analysis of Global Near-regular Mechanical Systems

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ABSTRACT

Some near-regular mechanical systems involve global deviations from their corresponding regular system. Despite extensive research on vibration analysis (eigensolution) of regular and local near-regular mechanical systems, the literature on vibration analysis of global near-regular mechanical systems is scant. In this paper, a method for vibration analysis of such systems was developed using Kronecker products and matrix manipulations. Specifically, the eigensolution of the corresponding regular mechanical system was inserted in the algorithm to further accelerate the solution. The developed method allowed reduction in computational complexity (i.e., O(n^2)) when compared to earlier methods. The application of the method was indicated using a simple example.

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1 Introduction

Vibration analysis and eigensolution of large-scale regular mechanical systems are performed using concepts from graph and group theories. The stiffness and mass matrices of such regular mechanical systems have canonical forms with specific repetitive or circulant block patterns allowing efficient eigensolution. Specifically, the power of these methods becomes more evident when analysis is to be repeated many times, for instance in dynamic design or non-linear analysis of regular mechanical systems. However, most mechanical systems do not exactly hold the pattern of a regular system such that the application of efficient eigensolutions are limited.
Some mechanical systems involve a small amount of irregularities wherein a small number of degrees of freedom are affected by the irregularity [5,7]. These mechanical systems with locally concentrated irregularities are called local near-regular systems, eigensolution of which has been obtained using modified numerical approaches [2]. Another group of mechanical systems involve global deviations (i.e., irregularity affects a large number of degrees of freedom) from their corresponding regular system and are called global near regular mechanical systems [1]. Eigensolution for these systems, however, has not been yet developed. Therefore, the objective of this paper was set to develop efficient methods for vibration analysis and eigensolution of mechanical systems with global near-regularity.

2 Vibration analysis of a global near-regular system

Developing a simple algorithm for the vibration analysis of global near-regular mechanical systems from available vibration solution of their corresponding regular mechanical systems is aimed.
Consider the given eigenproblem for a global near-regular system

$$Kx = \lambda Mx \tag{1}$$

where K is the stiffness matrix, x and λ are the eigenpairs, and M is the mass matrix. Assume ΔK and ΔM are the deviations of the stiffness and mass matrices of the near-regular system from the stiffness (K_{reg}) and mass (M_{reg}) matrices of the regular system.

$$K = K_{reg} + \Delta K \quad ; \quad M = M_{reg} + \Delta M \tag{2}$$

And

$$K_{reg}x_{reg} = \lambda_{reg}M_{reg}x_{reg} \tag{3}$$

where x_{reg} and λ_{reg} are eigenpairs of the regular system.
Substituting Eq. (2) in Eq. (1)

$$\left(K_{reg} + \Delta K\right) x = \lambda Mx \tag{4}$$

Assume the eigenvector x can be approximated using a linear combination of m linearly independent basis vectors $y_1, y_2, \cdots, y_m$

$$\begin{aligned} &x_{(n\times1)} = a_1y_1 + a_2y_2 + \cdots + a_my_m \qquad \text{or} \qquad x_{(n\times1)} = Y_{(n\times m)}A_{(m\times1)} \\ &\text{and} \qquad Y = [y_1, y_2, \cdots, y_m]_{(n\times m)} \qquad ; \qquad A^T = [a_1, a_2, \cdots, a_m]_{(1\times m)} \end{aligned} \tag{5}$$

where A is a vector of unknown coefficients. We set m to be much smaller than n (i.e., the number of degrees of freedom) and substitute Eq. (5) in Eq. (1)

$$\begin{aligned} Kx =&\lambda Mx \rightarrow KYA = \lambda MYA \rightarrow Y^TKYA = \lambda Y^TMYA \\ K_r = Y^TKY &\text{ and } M_r = Y^TMY \rightarrow K_rA = \lambda M_rA \end{aligned} \tag{6}$$

where K_r and M_r are reduced stiffness and mass matrices of dimension m. Therefore, rather than solving a system of dimension n in Eq. (1), the smaller system of dimension m should be solved. Upon finding the vector A in Eq. (6), x is calculated from Eq. (5). However, the basis vectors are not determined yet.
Consider Eq. (4) and pre-multiply it by K_{reg}^{-1}

$$(K_{reg} + \Delta K)\,x = \lambda Mx \rightarrow (1 + G)\,x = y_1 \tag{7}$$

where $G = K_{reg}^{-1}\Delta K$ and $y_1 = K_{reg}^{-1}\lambda Mx$. Therefore,

$$x = (1 + G)^{-1}\,y_1$$

But, we can write

$$\begin{aligned} (1 - G)\left(1 + G + G^2 + \cdots + G^r\right) = 1 - G^{r+1} \\ \left(1{+}G + G^2 + \cdots + G^r\right) = (1 - G)^{-1} - (I - G)^{-1}\,G^{r+1} \end{aligned}$$

Since $(I - G)^{-1}\,G^{r+1} = 0$ as $G^r \rightarrow 0$, we have:

$$G \rightarrow -G \;:\; (I + G)^{-1} = (I - G + G^2 - \cdots) \tag{8}$$

Thus,

$$x = (I + G)^{-1}y_1 = \left(I - G + G^2 - \right)y_1$$

Now assuming $\lambda = \lambda_{reg}$ and $x = x_{reg}$

$$y_1 = K_{reg}^{-1}\lambda Mx = K_{reg}^{-1}\lambda_{reg}Mx_{reg} \rightarrow y_1 = K_{reg}^{-1}R_{reg} \tag{9}$$

Where $R_{reg} = \lambda_{reg}Mx_{reg}$. Finally,

$$\begin{cases} y_i = -Gy_{i-1} \quad (i = 2:m) \\ y_1 = K_{reg}^{-1}R_{reg} \end{cases} \rightarrow x = \sum_{i=1}^{m} y_i \quad ; \; \lambda_j = \frac{x_j^TKx_j}{x_j^TMx_j} \tag{10}$$

For a regular mechanical system with given eigenpair matrices, Λ_{reg} and X_{reg}, decomposition of stiffness matrix would be $K_{reg} = X_{reg}^T \Lambda_{reg} X_{reg}$ and y_1 is solved as follows:

$$y_1 = \left(X_{reg}^T \Lambda_{reg} X_{reg}\right)^{-1} R_{reg} = X_{reg}^T \Lambda_{reg}^{-1} X_{reg} \lambda_{reg} M x_{reg} \tag{11}$$

With x and calculated from Eq. (10), we can update y_1 (i.e.,R_{reg}) and recalculate x and λ to achieve the desired accuracy. Also, same procedures should be conducted to calculate other eigenvalues and eigenvectors of the mechanical system.
Regardless of the required effort for defining the eigenpair matrices Λ_{reg} and X_{reg}, the solution of Eq. (10) includes matrix by vector multiplications of the computational complexity O(n^2). However, the method is efficient only if there is a simple way to obtain Λ_{reg} and X_{reg}.

3 Decomposition of stiffness matrix of regular mechanical system

For a regular mechanical system full decomposition of stiffness matrix can be obtained using Kronecker products and matrix manipulations. The two general groups of decomposable regular mechanical systems include repetitive and circulant matrices with the following block patterns:

$$\begin{aligned} M_n &= F_l(A_m, B_m, A_m) = I_l \otimes A_m + T_l \otimes B_m \\ M_n &= G_l(A_m, B_m, A_m) = (P_1)_l \otimes A_m + (P_2)_l \otimes B_m + (P_3)_l \otimes B_m \end{aligned} \tag{12}$$

where I is an identity matrix, $T = F(0, 1, 0)$, and P_1, P_2 and P_3 are permutation matrices. Using the following lemma, matrix M can be decomposed:
The sufficient condition for converting Hermitian matrices A_1 and A_2 into upper triangular matrices using one orthogonal matrix is [3]:

$$A_1 A_2 = A_2 A_1 \tag{13}$$

Let the matrix M be the sum of two (it can also be generalized to n Kronecker products) Kronecker products as $M = A_1 \otimes B_1 + A_2 \otimes B_2$. If F is the matrix that upper triangularizes the matrices A_1 and A_2, then $U = F \otimes I$ block upper triangularizes the matrix M, meaning that $U^t M U$ is block upper diagonalized. Thus, we can write:

$$\lambda_M = \bigcup_{i=1}^{l} eig(M_i) \quad ; \quad M_i = \lambda_i(A_1) B_1 + \lambda_i(A_2) B_2 \tag{14}$$

where matrices A_1 and A_2 are of dimension l, and matrices B_1 and B_2 are of dimension m. Since $IT = TI$ in Eq. (12) and considering that $\lambda_{T_l} = 2\cos\frac{k\pi}{l+1}$ [4], we will have

$$\lambda_M = \bigcup_{k=1}^{l} eig\left(2\cos\frac{k\pi}{l+1} B + A\right) \tag{15}$$

Permutation matrices also hold the following property:

$$P_i P_j = P_j P_i \tag{16}$$

The eigenpairs of the permutation matrix P_i are obtained as follows

$$\begin{gathered}\lambda = \left(e^{\frac{2\pi i}{l}}\right)^k = \omega^k \quad ; \quad i=\sqrt{-1} \quad ; \quad k=0:l-1 \\ \vartheta_k = \frac{1}{\sqrt{l}}(1,\omega_k,\omega_k^2,\cdots,\omega_k^{l-1})^T\end{gathered} \tag{17}$$

And

$$\lambda_M = \bigcup_{k=1}^{l} eig\left(2\left(e^{\frac{2\pi i}{l}}\right)^{k-1} B + A\right) \tag{18}$$

The eigenvectors of matrix M are calculated through $u \otimes v$ [6], where u is the eigenvector of A_2 (T in Eq. (12)) and v is the eigenvector of B_1. With the obtained eigenpairs, we can decompose M as follows:

$$M = V^T \lambda V \tag{19}$$

Now, with λ and V in hand (i.e., Λ_{reg} and X_{reg} in Eq. (11), Eq. (10) and Eq. (11) are solved.

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