Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Edge-tenacity in Networks

DARA MOAZZAMI*[1,2]

[1]University of Tehran, College of Engineering, Department of Engineering Science.
[2]Golpayegan University of Technology, Department of Basic Science.

ABSTRACT

Numerous networks as, for example, road networks, electrical networks and communication networks can be modeled by a graph. Many attempts have been made to determine how well such a network is "connected" or stated differently how much effort is required to break down communication in the system between at least some nodes.
Two well-known measures that indicate how "reliable" a graph is are the "Tenacity" and "Edge-tenacity" of a graph. In this paper we present results on the tenacity and edge-tenacity, $T_e(G)$, a new invariant, for several classes of graphs. Basic properties and some bounds for edge-tenacity, $T_e(G)$, are developed. Edge-tenacity values for various classes of graphs are calculated and future work and concluding remarks are summarized .

ARTICLE INFO

Article history:
Received 30, August 2016
Received in revised form 18, February 2017
Accepted 30 March 2017
Available online 01, June 2017

Keyword: Edge-tenacity, network vulnerability.

AMS subject Classification: 05C78.

1 Introduction

The concept of graph tenacity was introduced by Cozzens, Moazzami and Stueckle in [2,3], as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. the

*dmoazzami@ut.ac.ir

motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the the concept was introduced in 1992. Cozzens et al. introduced two measures of network vulnerability termed the tenacity,T(G), and the Mix-tenacity, $T_m(G)$, of a graph. The tenacity T(G) of a graph G is defined as

$$T(G) = \min_{A \subset V(G)} \{\frac{\mid A \mid + \tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of G-A and $\omega(G-A)$ is the number of components of G-A.
The Mix-tenacity T_m of a graph G is defined as

$$T_m = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of G-A and $\omega(G-A)$ is the number of components of G-A.
we called this parameter Mix-tenacity. It seems Mix-tenacity is a better name for this parameter. T(G) and $T_m(G)$ turn out to have interesting properties. After the pioneering work of Cozzens, Moazzami, and Stueckle several groups of researchers have investigated tenacity, and related problems. The edge-tenacity $T_e(G)$ of a graph G was defined as

$$T_e(G) = \min_{F \subset E(G)} \{\frac{\mid F \mid + \tau(G-F)}{\omega(G-F)}\}$$

where the minimum is taken over all edge cutset F of G. We define G-F to be the graph induced by the edges of $E(G) - F$, $\tau(G-F)$ is the number of edges in the largest component of the graph induced by G-F and $\omega(G-F)$ is the number of components of $G-F$. A set $F \subset E(G)$ is said to be a T_e-set of G if

$$T_e(G) = \frac{|F|+\tau(G-F)}{\omega(G-F)}$$

Each component has at least one edge.

In [10] and [11] Piazza et al. used the above parameter as edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and $\tau(G-A)$ the number of vertices of a largest component. It may be observed that in the definition of T_m, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. This may not seem very satisfactory. This motivated the authors to introduce a new measure of vulnerability in this paper. This new measure of vulnerability involves edges only and thus is called the Edge-Integrity.
The concept of tenacity of a graph G was introduced in [1,2], as a useful measure of the "vulnerability" of G. In [5], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able

to distinguish between graphs that intuitively should have different levels of vulnerability. In [4,5,6,7,8,9,10,13,17,18], they studied more about this new invariant.
For our purposes a graph G is an ordered pair (V,E), where V is a finite set of elements called vertices and E is a finite set of elements called edges. Let A be a subset of V(G). The neighborhood of A, N(A), consists of all vertices of G adjacent to at least one vertex of A. A graph G is connected if every two vertices in G are joined by a path. Given a graph G, the graph G^2 has $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or the distance from u to v is 2. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\alpha(G)$ or α. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number $\alpha'(G)$ or α'. The line graph of a graph G, denoted L(G), is the graph whose vertices are the edges of G, and two vertices are adjacent whenever the corresponding edges of G are adjacent.
The vertex connectivity, $\kappa = \kappa(G)$, of a finite, undirected, connected, simple graph G (without loops or multiple edges) is the minimum number of vertices whose removal results in a disconnected graph or results in the trivial graph K_1. Graph G is called n-connected if $\kappa \geq n$. Analogously, the edge-connectivity, $\lambda = \lambda(G)$, of a finite, undirected, connected simple graph G is the minimum number of edges whose removal results in a disconnected or trivial graph K_1. A graph G is called n-edge- connected if $\lambda(G) \geq n$.
A collection of vertices in V(G) is called a cutset if their removal disconnects G, and a collection of edges in V(G) is called an edge-cutset if their removal disconnects G.
In this paper we prove a number of basic results about tenacity. We can prove the following propositions:
Proposition 1: If G is a spanning subgragh of H, then $T(G) \leq T(H)$.
Proposition 2: For any graph G, $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.
Proposition 3: If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$, where n is the number of vertices in G.
Proposition 4: If $m \leq n$ then $T(K_{m,n}) = \frac{m+1}{n}$.

Lemma 1: If A is a minimal T-set for the graph G then, for each vertex v of A, the induced subgraph $< V(G) - A + v >$ has fewer components than does G-A.
Proof: Let $A' = A - v$. If G-A' has at least as many components as G-A, then $\mid A' \mid = \mid A \mid -1$ and $\tau(G - A') \leq \tau(G - A) + 1$. Therefore $\frac{|A'|+\tau(G-A')}{\omega(G-A)} = \frac{|A|-1+\tau(G-A')}{\omega(G-A)} \leq \frac{|A|-1+\tau(G-A)+1}{\omega(G-A)} = T(G)$, contrary to our choice of A.

Theorem 1: Let $G = G_1 + G_2$, where $\mid V(G) \mid = n$, $\mid V(G_i) \mid = p_i$, $T(G) = T$ and $T(G_i) = T_i$ for $i = 1, 2$. Then if $G \neq K_n$ we have

$$min\{\frac{[n + \tau(G_1 - A_1)]T_1}{p_1 + \tau(G_1 - A_1}, \frac{[n + \tau(G_2 - A_2)]T_2}{p_2 + \tau(G_2 - A_2)}\} < T \leq min\{\frac{n - \alpha_1 + 1}{\alpha_1}, \frac{n - \alpha_2 + 1}{\alpha 2}\},$$

where α_i is the independence number of G_i, and A_i is a disconnecting set of G_i for $i = 1, 2$.

Proof: Because of the structure of G, the graph cannot be disconnected without removing one of $V(G_1)$ or $V(G_2)$. Having removed the appropriate set, we can then disconnect the graph by disconnecting the remaining graph, either G_1 or G_2. Candidates for T are of the form $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ or $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}$ where $n_i =\mid A_i \mid$ for $i = 1, 2$. Then $T = min\{\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}, \frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}\}$, where the minimum is taken over all A_1 and A_2 as defined. Also $T_1 \leq \frac{n_1+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ which implies $\omega(G_1 - A_1) \leq \frac{n_1+\tau(G_1-A_1)}{T_1}$. Thus $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)} \geq \frac{[n_1+p_2+\tau(G_1-A_1)]T_1}{n_1+\tau(G_1-A_1)}$. Similary,$\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)} \geq \frac{[n_2+p_1+\tau(G_2-A_2)]T_2}{n_2+\tau(G_2-A_2)}$. Thus $T \geq min\{[1+\frac{p_2}{n_1+\tau(G_1-A_1)}]T_1, [1+\frac{p_1}{n_2+\tau(G_2-A_2)}]T_2\}$. Also we know that $n_1 < p_1$ and $n_2 < p_2$, therefore $T > min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1)}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\}$. From Proposition 3, we observe that two candidates for T are $\frac{(p_1-\alpha_1)+1+p_2}{\alpha_1}$ and $\frac{(p_2-\alpha_2)+p_1}{\alpha_2}$, which yield $T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}\}$, $\frac{n-\alpha_2+1}{\alpha_2}\}$.

The effect of removing a vertex is considered first. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower T by too much.

Theorem 2: For any nontrivial noncomplete graph G on n vertices and any vertex v, $T(G - v) \geq T(G) - \frac{1}{2}$.
Proof: Let $G' = G - v$. If $G' = K_{n-1}$, then $T(G') = \infty$, and the theorem holds. Hence, assume $G' \neq K_{n-1}$. Let A' be a T-set for G', and let $\mid A' \mid= m$,
then $T(G') = \frac{m+\tau(G'-A')}{\omega(G'-A')}$. Now define $A = A' \cup \{v\}$. Clearly A is a disconnecting set for G and so $T(G) \leq \frac{|A|+\tau(G-A)}{\omega(G-A)}$. But $\mid A \mid= m + 1$ and $G - A = G' - A'$, so $T(G) \leq \frac{m+1+\tau(G'-A')}{\omega(G'-A')} = \frac{m+\tau(G'-A')}{\omega(G'-A')} + \frac{1}{\omega(G'-A')} = T(G') + \frac{1}{\omega(G'-A')} \leq T(G') + \frac{1}{2}$, since $\omega(G' - A') \geq 2$. Hence $T(G) \leq T(G') + \frac{1}{2}$.

We next obtain some bounds on the tenacity of a graph.

Proposition 5: If G is connected, then $T(G) \geq \frac{1}{\Delta(G)}$.
Proof: K_n is a special case, otherwise the removal of any vertex of a connected graph G yields at most $\Delta(G)$ components. Similarly, the removal of any n vertices yields at most $n\Delta(G)$ components. Then, from the definition we have $T(G) \geq \frac{n+1}{n\Delta(G)} \geq \frac{1}{\Delta(G)}$.

Theorem 3: If G is connected and a noncomplete $K_{1,3}$-free graph then $T(G) > \frac{\kappa(G)}{2}$.
Proof: Suppose G is a noncomplete, $K_{1,3}$-free graph with connectivity $\kappa(G)$. Let A be a T-set, and let $W_1, W_2, \cdots, W_m$ be the components of G-A. Since G has a connectivity $\kappa(G)$, it is $\kappa(G)$-connected and so there exist $\kappa(G)$ internally disjoint paths from $u_i \in W_i$ to $u_j \in W_j$ for all $1 \leq i, j \leq m$ with $i \neq j$. each of these paths must contain a vertex of A. Then for each i there are at least $\kappa(G)$ edges coming from W_i to distinct vertices of A. Thus in all there are at least m$\kappa(G)$ edges from G-A to A counting at most one from any component W_i to a particular vertex of A.

Suppose $v \in A$ is adjacent to vertices u_1, u_2, and u_3 in distinct components of G-A. Then, since $\{u_1, u_2, u_3\}$ is an independent set the graph induced by $\{v, u_1, u_2, u_3\}$ is a $K_{1,3}$, a contradiction. Hence we can conclude any vertex of A is adjacent to at most two components of G-A. Thus, there are at most 2 $\mid A \mid$ edges coming from G-A to vertices of A, counting at most one edge from any component to a particular vertex of A. Hence $m\kappa(G) \leq 2 \mid A \mid$, or $\frac{m\kappa(G)}{2} \leq \mid A \mid$. Therefore $\frac{m\kappa(G)}{2} < \mid A \mid +1$, or $\frac{|A|+1}{m} > \frac{\kappa(G)}{2}$. Thus $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)} \geq \frac{|A|+1}{m} > \frac{\kappa(G)}{2}$.

The toughness of a graph G was introduced by Chvátal in [1], who observed the relationship between this parameter and the existence of Hamilton cycles in the given graph, and several results regarding this invariant were obtained. The original approach to toughness is as follows. A connected graph G is called t-tough if $t\omega(G - A) \leq \mid A \mid$ for any subset A of V(G) with $\omega(G - A) > 1$, . If G is not complete, then there is a largest t such that G is t-tough; this number is the toughness of G and denoted by t(G). Thus $t(G) = min\{\frac{|A|}{\omega(G-A)}\}$, where A is a cutset of G. Since a complete graph has no cutset A, we set $t(K_n) = \infty$ for all $n \geq 1$.

Without attempting to obtain the best possible result, we can prove the following relation between T(G) and t(G). This result gives us a number of corollaries.

Theorem 4: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.
Proof: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.
Corollary 1: For any graph G, $T(G^2) > \kappa(G)$.
Corollary 2: Let G be a non-empty graph and let m be the largest integer such that $K_{1,m}$ is an induced subgraph of G. Then $T(G) \geq \frac{\kappa(G)}{m} + \frac{1}{\alpha(G)}$.

Theorem 5: Let G be a graph with n vertices and $G \neq K_n$, then $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.
Proof: We observe that at least one of G or $\overline{G}$ is connected. Suppose $\overline{G}$ is not connected. We proved (Proposition 5) that $T(G) \geq \frac{1}{\Delta(G)} \geq \frac{1}{n-1}$ for any graph G. Thus, $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$. Now suppose G is not connected but $\overline{G}$ is connected. Again by Proposition 5, we have $T(\overline{G}) \geq \frac{1}{n-1}$. Therefore $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.

Theorem 6: Let G be a graph with $0 < T(G) < \infty$, and let $\lambda(G) = \lambda$, then $T(L(G)) > \frac{\lambda}{2}$.
Proof: Assume there exist vertex cutsets A for L(G) such that A is a t-set. By Theorem 4, $T(L(G)) > t(L(G))$. Let E be those edges of G which are incident to vertices of A. Then E is an edge-cutset of G. Thus we have $t(L(G)) = min\{\frac{|A|}{\omega(L(G)-A)}\} \geq min\{\frac{|E|}{\omega(G-E)}\} = t'(G)$, where A is a cutset and E is an edge cutset of G.
Using the result of Chvátal [1] we have $t'(G) = min\{\frac{|E|}{\omega(G-E)}\} = \frac{\lambda}{2}$. Therefore $T(L(G)) > \frac{\lambda}{2}$.

The binding number of a graph G was defined by Woodall in [14] as

$$bind(G) = \min_{A}\{\frac{| N(A) |}{| A |}\}$$

where $\phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)$. In [15,16] the binding number was called the melting-point of the graph. the reason for the name "binding number" is that, roughly speaking, if bind(G) is large, then the vertices of G are bound tightly together, in the sense that G has many edges fairly well distributed.

Theorem 7: For any graph G, $T(G) \geq bind(G) - 1$.
Proof: Let bind(G) = c. If $c < 1$, then $c - 1 < 0$ and the result follows since T(G) is nonnegative. Consider $c \geq 1$. Suppose that A is a subset of V(G) such that $\omega = \omega(G - A) \geq 2$. We want to prove that $\frac{|A|+1}{\omega} > (c - 1)$. If each of the ω components of G-A has at least two vertices, let S consist of the vertices in all the components except the smallest, so that

$$| S | \geq \frac{| V(G) - A | (\omega - 1)}{\omega} \geq \frac{2\omega(\omega - 1)}{\omega} = 2(\omega - 1) \geq \omega.$$

If, on the other hand, V(G)-A contains an isolated vertex, let $S = V(G) - A$. So that $| S |=| V(G) - A | \geq \omega$. In either case $N(S) \neq V(G)$, and since $bind(G) = c \geq 1$,

$$| S | + | A | +1 >| S | + | A | \geq | N(S) | \geq c | S | .$$

It follows that $| A | +1 > (c - 1) | S | \geq (c - 1)\omega$. Therefore $\frac{|A|+1}{\omega} > c - 1$, so $T > c - 1$.

In [8] we showed the Hamiltoinan Properties of tenacity. The results follows for a graph G:
1) $1 < \frac{\kappa(G)}{\alpha(G)} < \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$
2) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1.$
Graphs satisfying the second inequality are not Hamiltonian-connected. Graphs satisfying the first inequality are Hamiltonian-connected.
3) $1 + \frac{n+1}{\alpha(G)} \leq \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$
4) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1 + \frac{n+1}{\alpha(G)}$
If G satisfies the forth inequality it is not n-Hamiltonian. If G satisfies the third inequality then G is n-Hamiltonian.

In [6], we compared integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.
Edge-Tenacity
The edge-tenacity $T_e(G)$ of a graph G was defined as

$$T_e(G) = \min_{F \subset E(G)} \{\frac{| F | + \tau(G-F)}{\omega(G-F)}\}$$

where the minimum is taken over all edge cutset F of G. We define G-F to be the graph induced by the edges of $E(G) - F$, $\tau(G-F)$ is the number of edges in the largest component of the graph induced by G-F and $\omega(G-F)$ is the number of components of $G-F$. A set $F \subset E(G)$ is said to be a T_e-set of G if

$$T_e(G) = \frac{|F|+\tau(G-F)}{\omega(G-F)}$$

Each component has at least one edge. In this paper we introduce a new invariant edge-tenacity, for graphs. it is another vulnerability measure. we present several properties and bounds on the edge-tenacity. we also compute the edge-tenacity of some classes of graphs.

Basic Properties and Bounds

In this section we develop basic properties and bounds related to edge-tenacity.

Proposition 6 Let G be a graph, and H, a subgraph of G. Then

$$T_e(H) \leq T_e(G)$$

Proof. Let H be a subgraph of G, and F is a minimum subset of E(G) which achieves $T_e(G)$ then $\omega(H-F) \geq \omega(G-F)$. Thus $\frac{1}{\omega(H-F)} \leq \frac{1}{\omega(G-F)}$. Also we have

$$| F | + \tau(H-F) \leq | F | + \tau(G-F)$$

Thus

$$\frac{| F | + \tau(H-F)}{\omega(H-F)} \leq \frac{| F | + \tau(G-F)}{\omega(G-F)}$$

Therefore

$$T_e(H) \leq T_e(G)$$

our next result, besides being of interest in its own right is also found to be useful in several proofs later.

Theorem 10 If F is a minimum subset of E(G) which achieves $T_e(G)$, then every edge of F connects vertices from different components of G-F.

Proof. Suppose $e = uv$ is an edge in F. If u and v lie in the same component of G-F, then if $F_1 = F - e$, we have $\omega(G-F_1) = \omega(G-F)$ and

$$\begin{aligned} T_e(G) &\leq \frac{|F_1|+\tau(G-F_1)}{\omega(G-F_1)} \\ &= \frac{|F_1|+\tau(G-F_1)}{\omega(G-F)} \\ &= \frac{|F|-1+\tau(G-F_1)}{\omega(G-F)} \\ &\leq \frac{|F|-1+\tau(G-F)+1}{\omega(G-F)} \\ &= T_e(G) \end{aligned}$$

This shows that F_1 achieves $T_e(G)$ which is a contradiction since F_1 is a proper subset of F.

Theorem 11 Suppose that $\lambda = \lambda(G) \geq 1$ the edge-connectivity of G and $e =\mid E(G) \mid$. Then

$$T_e(G) \geq min(e, \lceil \frac{\lambda}{16e}(8e - \lambda) \rceil)$$

Proof. Suppose that $F \subset E(G)$ is such that G-F has k components. Then $\tau(G - F) \geq \frac{e-|F|}{k}$. For $k = 1$ $\tau(G - F)+ \mid F \mid \geq e- \mid F \mid + \mid F \mid= e$. if $k > 1$, then $\mid F \mid\geq \frac{k\lambda}{2}$, so that

$$\frac{\tau(G - F)+ \mid F \mid}{k} \geq \frac{\frac{e-|F|}{k} + \mid F \mid}{k} \geq \frac{1}{k}(\frac{e}{k} + \frac{(k-1)\lambda}{2})$$

The quantity on the right hand side of this inequality is minimized for $k = \frac{4e}{\lambda}$. Hence the result follows.

Conclusion

In this paper we have investigated the tenacity and some properties of edge-tenacity of graphs. Work in progress related to this paper includes characterization of graphs for edge-tenacity and tenacity values are the same, and finding a formula for the edge-tenacity of hypercube, complete bipartite graph, and certain other classes of graphs [12].

Acknowledgement

This work supported by University of Tehran. My special thanks go to the University of Tehran, College of Engineering and Department of Engineering Science for providing all the necessary facilities available to me for successfully conducting this research.

References

[1] Chvátal, V. , Tough graphs and Hamiltonian circuits, **Discrete Math.5** (1973), 215-228.

[2] Cozzens. M.B., D. Moazzami. , S. Stueckle, The tenacity of the Harary Graphs, J. Combin. Math. Combin. Comput. 16 (1994), 33-56.

[3] Cozzens. M.B. , D. Moazzami, S. Stueckle, The tenacity of a graph, Graph Theory, Combinatorics, and Algorithms (Yousef Alavi and Allen Schwenk eds.) Wiley, New York, (1995), 1111-1112.

[4] Choudum. S. A. and N. Priya, Tenacity of complete graph products and grids, Networks 34 (1999), no. 3, 192-196.

[5] Choudum. S. A. and N. Priya, Tenacity-maximum graphs, J. Combin. Math. Combin. Comput. 37 (2001), 101-114.

[6] Moazzami. D. , Vulnerability in Graphs - a Comparative Survey, J. Combin. Math. Combin. Comput. 30 (1999), 23-31.

[7] Moazzami. D. , Stability Measure of a Graph - a Survey, Utilitas Mathematica, 57 (2000), 171-191.

[8] Moazzami. D. , On Networks with Maximum Graphical Structure, Tenacity T and number of vertices p, J. Combin.Math. Combin. Comput. 39 (2001).

[9] Moazzami. D., A note on Hamiltonian properties of tenacity, Poceedings of the International conference," Paul Erdös and his Mathematics" Budapest, July 4 - 11, (1999), 174-178.

[10] Piazza. B., F. Roberts, S. Stueckle, Edge-tenacious networks, Networks 25 (1995), no. 1, 7-17.

[11] Piazza. B. , S. Stueckle, A lower bound for edge-tenacity, Proceedings of the thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999) Congr. Numer. 137 (1999), 193-196.

[12] Salehian. B. and D. Moazzami, The edge-tenacity: A new measure of tenacity. Under Prepreation.

[13] Salehian. B. and D. Moazzami, Measures of Vulnerabilitty-the tenacity Families. Preprint.

[14] Woodall. D.R. , The binding number of a graph and its Anderson number, **J. Combin. Theory B 15** (1973), 225-255.

[15] Woodall. D.R. , Problems 1 to 3, in **Combinatorics (Proc. 1972 Oxford Combinatorial Conference)** (D.J.Welsh and D.R. Woodall, eds.), Institute of Mathematics and Applications, Southend-Sea, Essex, England, 1972, 359-360.

[16] Woodall. D.R. , Abstract No. 20 (The melting-point of a graph, and its Anderson number), **Graph Theory News letter 1 (NO. 4)** (1972).

[17] Zhang. S. , Cycles in weighted graphs and related topics, PhD Thesis, University of Twente, 2002.

[18] Li. Y. , S. Zhang, X. Li and Y. Wu, Relationships between tenacity and some other vulnerability parameters, Basic Sci. J. Tex. Univ. Vol. 17, No.1 (2004) 1-4.