Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

# Super Pair Sum Labeling of Graphs

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**ABSTRACT**

Let $G$ be a graph with $p$ vertices and $q$ edges. The graph $G$ is said to be a super pair sum labeling if there exists a bijection $f$ from $V(G) \cup E(G)$ to $\{0, \pm 1, \pm 2, \ldots, \pm(\frac{p+q-1}{2})\}$ when $p+q$ is odd and from $V(G) \cup E(G)$ to $\{\pm 1, \pm 2, \ldots, \pm(\frac{p+q}{2})\}$ when $p+q$ is even such that $f(uv) = f(u) + f(v)$. A graph that admits a super pair sum labeling is called a *super pair sum graph*. Here we study about the super pair sum labeling of some standard graphs.

**ARTICLE INFO**

*Article history:*
Received 08, February 2017
Received in revised form 24, July 2017
Accepted 30 October 2017
Available online 01, December 2017

*Keyword:* labeling, super pair sum labeling, super pair sum graph.

AMS subject Classification: 05C78.

## 1 Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let $G(V, E)$ be a graph with $p$ verticies and $q$ edges. For notations and terminology, we follow [2].

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Path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. $K_{1,m}$ is called a *star* and it is denoted by $S_m$. The bistar $B_{m,n}$ is the graph obtained from $K_2$ by identifying the central vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of $K_2$ respectively. $B_{m,m}$ is often denoted by $B(m)$. The corona of a graph $G$ on $p$ vertices $v_1, v_2, \ldots, v_p$ is the graph obtained from $G$ by adding $p$ new vertices $u_1, u_2, \ldots, u_p$ and the new edges $u_iv_i$ for $1 \leq i \leq p$. The corona of $G$ is denoted by $G \odot K_1$. The graph $P_n \odot K_1$ is called a *comb*. Let $S_m$ be a star with central vertex vertex $v_0$ and pendant vertices $v_1, v_2, \ldots, v_m$ and let $[P_{2n}; S_m]$ be the graph obtained from $2n$ copies of $S_m$ with vertices $v_{0_j}, v_{1_j}, \ldots, v_{m_j}$ $(1 \leq j \leq 2n)$ and joining $v_{0)j}$ and $v_{0_{j+1}}$ by means of an edge, $1 \leq j \leq 2n-1$. A caterpillar $T$ is a tree with a path $P_n = u_1, u_2, \ldots, u_n$ called *spine* with leaves (pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. It is noted that every spine vertex $u_i$ is attached to $X_i$ (possibly zero) number of leaves $b_{i_j}$ $(1 \leq j \leq X_i, 1 \leq i \leq n)$. The caterpillar is denoted as $T = S(X_1, X_2, \ldots, X_n)$. The graceful labelings of graphs was first introduced by Rosa in 1961 [1] and super vertex graceful labeling of some standard graphs was discussed in [5]. The concept of pair sum labeling was introduced and studied by R. Ponraj et al. [3, 4]. It motivates us to define a new concept called *super pair sum labeling of graphs.*
Let $G$ be a $(p, q)$ graph. A one-one map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\}$ or $\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\}$ according as $q$ is even or odd. A graph with a pair sum labeling defined on it is called a *pair sum graph.*
A graph $G$ with $p$ vertices and $q$ edges is said to have a super pair sum labeling if there exists a bijection $f$ from $V(G) \cup E(G)$ to $\{0, \pm 1, \pm 2, \ldots, \pm(\frac{p+q-1}{2})\}$ when $p+q$ is odd and from $V(G) \cup E(G)$ to $\{\pm 1, \pm 2, \ldots, \pm(\frac{p+q}{2})\}$ when $p+q$ is even such that $f(uv) = f(u) + f(v)$. A graph that admits a super pair sum labeling is called a *super pair sum graph.*
A super pair sum labeling of a graph $B(2)$ is given in Figure 1

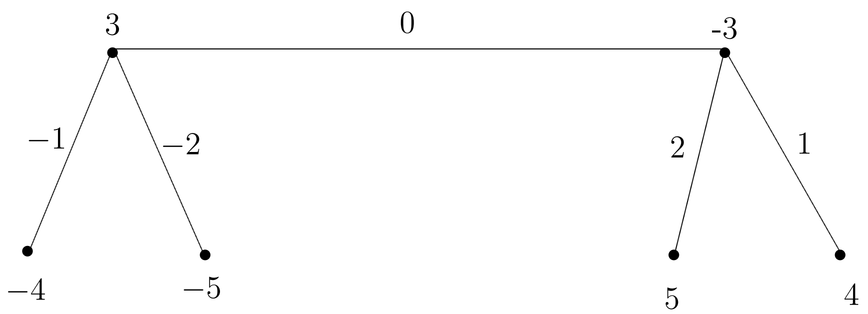

Figure 1: A super pair sum labeling of $B(2)$

In this paper, we prove that the graphs $P_n, K_{1,m}$, bistar $B_{m,n}$ for $m \geq 1$, $n \geq 1, [P_{2n}; S_m]$ for $n \geq 1, m \geq 1$, comb, $C_{2n}, K_{1,m} \cup K_{1,n}$ and banana tree $S(m, 0, 0, \ldots, 0), m \geq 1$ are super pair sum graphs.

## 2 Super Pair Sum Graphs

**Theorem 2.1.** *Any path is a super pair sum graph.*

*Proof.* Let $u_1, u_2, \dots, u_n$ be the vertices of the path $P_n$.
**Case i.** $n$ is odd.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \dots, \pm(n-1)\}$ as follows:

$$f(u_i) = \begin{cases} \frac{i+1-2n}{2} & \text{if } i \text{ is odd} \\ \frac{n+i-1}{2} & \text{if } i \text{ is even} \end{cases}$$

$$\text{and } f(u_i u_{i+1}) = \frac{2i-n+1}{2}, 1 \leq i \leq n-1.$$

Thus, $f$ is a super pair sum labeling.
**Case ii.** $n$ is even.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \dots, \pm(n-1)\}$ as follows:

$$f(u_i) = \begin{cases} \frac{3n+2i-4}{4} & 1 \leq i \leq \frac{n}{2}, i \text{ is odd and } n \equiv 2(mod\ 4) \\ & 2 \leq i \leq \frac{n}{2}, i \text{ is even and } n \equiv 0(mod\ 4) \\ \frac{n+2i-4}{4} & \frac{n+4}{2} \leq i \leq n-1, i \text{ is odd and } n \equiv 2(mod\ 4) \\ & \frac{n+4}{2} \leq i \leq n, i \text{ is even and } n \equiv 0(mod\ 4) \\ \frac{2-3n+2i}{4} & 2 \leq i \leq \frac{n-2}{2}, i \text{ is even and } n \equiv 2(mod\ 4) \\ & 1 \leq i \leq \frac{n-2}{2}, i \text{ is odd and } n \equiv 0(mod\ 4) \\ \frac{2-5n+2i}{4} & \frac{n+2}{2} \leq i \leq n, i \text{ is even and } n \equiv 2(mod\ 4) \\ & \frac{n+2}{2} \leq i \leq n-1, i \text{ is odd and } n \equiv 0(mod\ 4), \end{cases}$$

$$f(u_i u_{i+1}) = i, \text{ for } 1 \leq i \leq \frac{n-2}{2},$$

$$f(u_{\frac{n}{2}} u_{\frac{n+2}{2}}) = 0 \text{ and}$$

$$f(u_i u_{i+1}) = i-n, \text{ for } \frac{n+2}{2} \leq i \leq n-1.$$

Thus, $f$ is a super pair sum labeling and hence $P_n$ is a super pair sum graph. $\square$

For example, super pair sum labelings of $P_{11}, P_{12}$ and $P_{10}$ are shown in Figure 2.

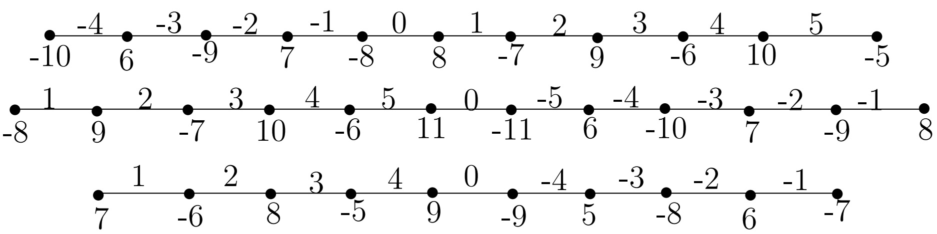

Figure 2: Super pair sum labelings of $P_{11}, P_{12}$ and $P_{10}$

**Theorem 2.2.** *Every star graph $S_m$ is a super pair sum graph for $m \geq 1$.*

*Proof.* Let $v_0, v_1, v_2, \ldots, v_m$ be the vertices of the star $K_{1,m}$ with $v_0$ as the central vertex.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm m\}$ as follows:

$$\begin{aligned} f(v_0) &= m, \\ f(v_i) &= i - 1 - m, 1 \leq i \leq m \text{ and} \\ f(v_0 v_i) &= i - 1, 1 \leq i \leq m. \end{aligned}$$

Thus, $f$ is a super pair sum labeling and hence $S_m$ is a super pair sum graph for $m \geq 1$. □

A super pair sum labeling of $S_8$ is shown in Figure 3.

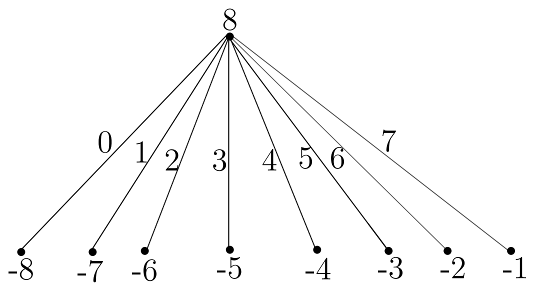

Figure 3: A super pair sum labeling of $S_8$

**Theorem 2.3.** *Bistar $B_{m,n}$ is a super pair sum graph for $m \geq 1, n \geq 1$.*

*Proof.* Let $V(K_2) = \{u, v\}$ and $u_i (1 \leq i \leq m), v_j (1 \leq j \leq n)$ be the vertices adjacent to $u$ and $v$ respectively. $B_{m,n}$ has $m+n+2$ vertices and $m+n+1$ edges.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm (m + n + 1)\}$ as follows:

$$\begin{aligned} f(u) &= -(m + n + 1), \\ f(v) &= m + n + 1, \\ f(u_i) &= m + n + 1 - i, 1 \leq i \leq m, \\ f(v_j) &= -(m + n + 1 - j), 1 \leq j \leq n, \\ f(uu_i) &= -i, 1 \leq i \leq m, \\ f(vv_j) &= i, 1 \leq j \leq n \text{ and} \\ f(uv) &= 0. \end{aligned}$$

Thus, $f$ is a super pair sum labeling and hence $B_{m,n}$ is a super pair sum graph for $m \geq 1, n \geq 1$. □

A super pair sum labeling of $B_{7,8}$ is shown in Figure 4.

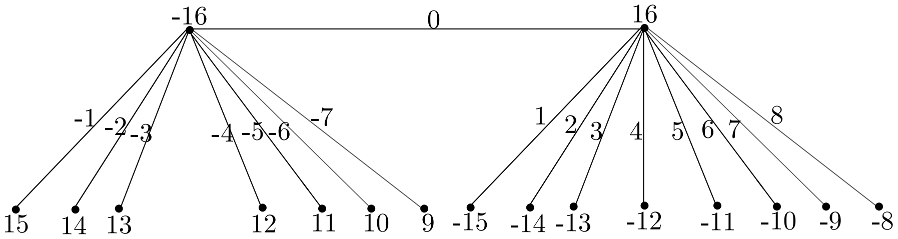


Figure 4: A super pair sum labeling of $B_{7,8}$

**Theorem 2.4.** *$[P_{2n}; S_m]$ is a super pair sum graph for $n \geq 1, m \geq 1$.*

*Proof.* Let $v_{0_j}, v_{1_j}, v_{2_j}, \ldots, v_{m_j}$ be the vertices in the $j^{th}$ copy of $S_m, 1 \leq j \leq 2n$. The number of vertices and edges of $[P_{2n}; S_m]$ are $2n(m+1)$ and $2n(m+1)-1$ respectively.
**Case i.** $n \equiv 0(mod\ 2)$.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm(2n(m+1)-1)\}$ as follows:

$$f(v_{0_j}) = \begin{cases} \frac{(m+1)(3n-j+1)}{2} - 1 & \text{if } j \text{ is odd, } 1 \leq j \leq n-1 \\ 1 - \frac{(m+1)(3n+j)}{2} & \text{if } j \text{ is even, } 2 \leq j \leq n, \end{cases}$$
$$f(v_{0_j}) = -f(v_{0_{2n+1-j}}), n+1 \leq j \leq 2n,$$
$$f(v_{i_j}) = \begin{cases} 1 - i - \frac{(m+1)(3n+j-1)}{2} & \text{if } j \text{ is odd, } 1 \leq j \leq n-1 \\ \frac{(m+1)(3n-j+2)}{2} - i - 1 & \text{if } j \text{ is even, } 2 \leq j \leq n, \end{cases}$$
$$f(v_{i_j}) = -f(v_{i_{2n+1-j}}), n+1 \leq j \leq 2n, 1 \leq i \leq m,$$
$$f(v_{0_j}v_{0_{j+1}}) = -j(m+1), 1 \leq j \leq n-1,$$
$$f(v_{0_n}v_{0_{n+1}}) = 0,$$
$$f(v_{0_j}v_{0_{j+1}}) = (2n-j)(m+1), n+1 \leq j \leq 2n-1, \text{ and}$$

$$f(v_{0_j}v_{i_j}) = \begin{cases} (1-j)(m+1) - i & 1 \leq j \leq n, 1 \leq i \leq m \\ (2n-j)(m+1) + i, & n+1 \leq j \leq 2n, 1 \leq i \leq m. \end{cases}$$

Thus, $f$ is a super pair sum labeling.
**Case ii.** $n \equiv 1(mod\ 2)$.

Define $f : V(G) \cup E(G) \to \{0, \pm1, \pm2, \ldots, \pm(2n(m+1)-1)\}$ as follows:

$$f(v_{0_j}) = \begin{cases} \frac{1-(m+1)(3n+j)}{2} & \text{if } j \text{ is odd, } 1 \le j \le n \\ \frac{(m+1)(3n-j+1)}{2} - 1 & \text{if } j \text{ is even, } 2 \le j \le n-1, \end{cases}$$
$$f(v_{0_j}) = -f(v_{0_{2n+1-j}}), n+1 \le j \le 2n,$$
$$f(v_{i_j}) = \begin{cases} \frac{(m+1)(3n-j+2)}{2} - i - 1 & \text{if } j \text{ is odd, } 1 \le j \le n \\ 1 - i - \frac{(m+1)(3n+j-1)}{2} & \text{if } j \text{ is even, } 2 \le j \le n-1, \end{cases}$$
$$f(v_{i_j}) = -f(v_{i_{2n+1-j}}), n+1 \le j \le 2n, 1 \le i \le m,$$
$$f(v_{0_j}v_{0_{j+1}}) = -j(m+1), 1 \le j \le n-1,$$
$$f(v_{0_n}v_{0_{n+1}}) = 0,$$
$$f(v_{0_j}v_{0_{j+1}}) = (2n-j)(m+1), n+1 \le j \le 2n-1 \text{ and}$$
$$f(v_{0_j}v_{i_j}) = \begin{cases} (1-j)(m+1) - i, & 1 \le j \le n, 1 \le i \le m, \\ (2n-j)(m+1) + i, & n+1 \le j \le 2n, 1 \le i \le m. \end{cases}$$

Thus, $f$ is a super pair sum labeling. Hence, $[P_{2n}; S_m]$ is a super pair sum graph. □

For example, super pair sum labelings of $[P_8; S_3]$ and $[P_6; S_4]$ are shown in Figure 5.

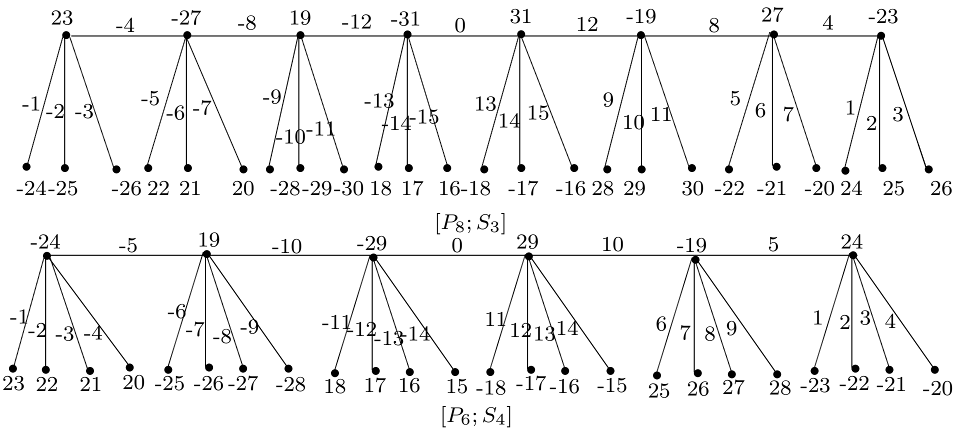


Figure 5: Super pair sum labelings of $[P_8; S_3]$ and $[P_6; S_4]$

**Theorem 2.5.** *Any comb is a super pair sum graph*

*Proof.* Let $G$ be the comb obtained from a path $P_n : v_1, v_2, \ldots, v_n$ by joining a vertex $u_i$ to $v_i (1 \le i \le n)$.
**Case i.** $n \equiv 1 (mod\ 4)$.
Define $f : V(G) \cup E(G) \to \{0, \pm1, \pm2, \ldots, \pm(2n-1)\}$ as follows:

$$f(v_i) = \begin{cases} 1 - 2i & \text{if } i \text{ is odd, } 1 \le i \le n \\ 2(n-i)+1 & \text{if } i \text{ is even, } 1 \le i \le n, \end{cases}$$

$$f(u_i) = \begin{cases} f(v_{i+1}) + 2, & 1 \le i \le n-1 \\ 1, & i = n, \end{cases}$$
$$f(v_i v_{i+1}) = 2n - 4i, 1 \le i \le n-1 \text{ and}$$
$$f(u_i v_i) = 2n - 4i + 2, 1 \le i \le n.$$

Then, $f$ is a super pair sum labeling.
**Case ii.** $n \equiv 3(mod\ 4)$.
Define $f : V(G) \cup E(G) \to \{0, \pm1, \pm2, \ldots, \pm(2n-1)\}$ as follows:

$$f(v_i) = \begin{cases} 2(n-i) + 1 & \text{if } i \text{ is odd, } 1 \le i \le n \\ 1 - 2i & \text{if } i \text{ is even, } 1 \le i \le n, \end{cases}$$
$$f(u_i) = \begin{cases} f(v_{i+1}) + 2, & 1 \le i \le n-1 \\ 1 - 2n, & i = n, \end{cases}$$
$$f(v_i u_{i+1}) = 2n - 4i, 1 \le i \le n-1 \text{ and}$$
$$f(u_i v_i) = 2n - 4i + 2, 1 \le i \le n.$$

Thus, $f$ is a super pair sum labeling. When $n$ is even and $m = 1$, the result follows from Theorem 2.4. Hence, any comb is a super pair sum graph. □

For example, super pair sum labelings of $P_9 \odot K_1$ and $P_{11} \odot K_1$ are shown in Figure 6.

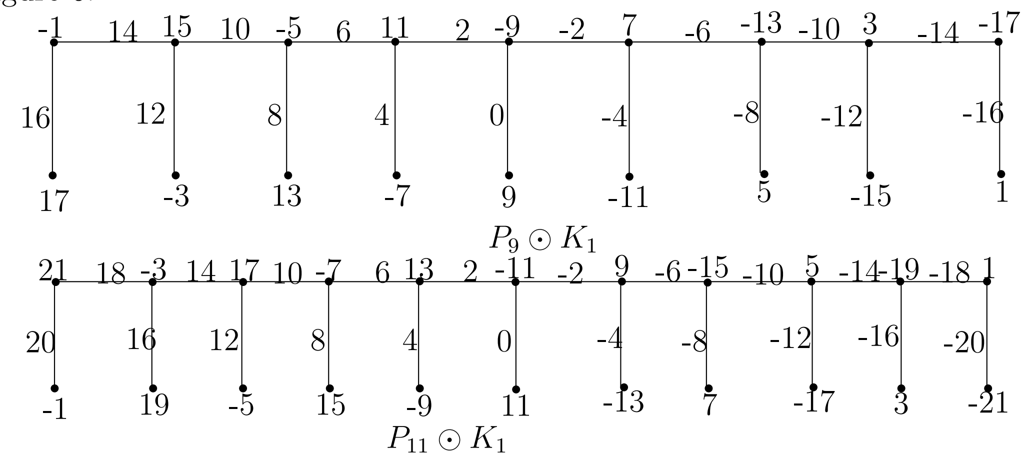


Figure 6: Super pair sum labelings of $P_9 \odot K_1$ and $P_{11} \odot K_1$

**Theorem 2.6.** *$C_{2n}$ is a super pair sum graph for $n \ge 1$.*

*Proof.* Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the cycle $C_{2n}$.
Define $f : V(G) \cup E(G) \to \{\pm1, \pm2, \ldots, \pm2n\}$ as follows:

$$f(v_1) = 1,$$
$$f(v_{2i+1}) = 1 - n - i, 1 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$
$$f(v_{2i}) = 2n - i, 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$
$$f(v_i) = -f(v_{i-n}), n + 1 \le i \le 2n,$$

$$\begin{aligned} f(v_1v_2) &= 2n, \\ f(v_iv_{i+1}) &= n-i+1, 2 \le i \le n-1, \\ f(v_nv_{n+1}) &= \frac{3n-2}{2}, \\ f(v_iv_{i+1}) &= -f(v_{i-n}v_{i+1-n}), n+1 \le i \le 2n-1 \text{ and} \\ f(v_{2n}v_1) &= -f(v_nv_{n+1}). \end{aligned}$$

Thus, $f$ is a super pair sum labeling and hence $C_{2n}$ is a super pair sum graph. □

For example, a super pair sum labeling of $C_{12}$ is shown in Figure 7.

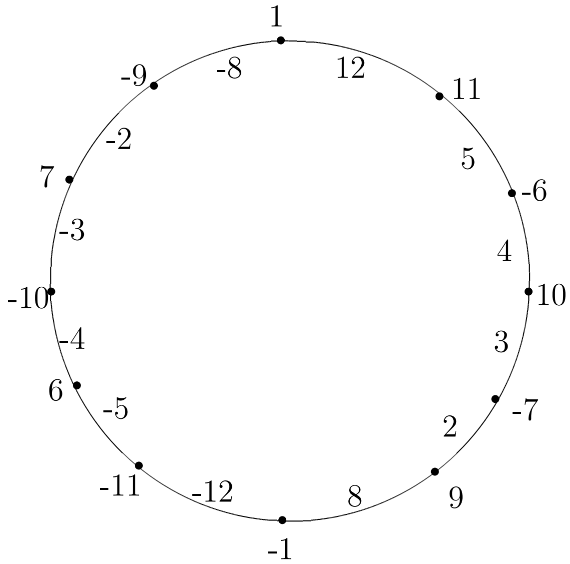


Figure 7: A super pair sum labeling of $C_{12}$

**Theorem 2.7.** $K_{1,m} \cup K_{1,n}$ *is a super pair sum graph.*

*Proof.* Let $u_0, u_1, u_2, \ldots, u_m$ be the vertices of $K_{1,m}$ and $E(K_{1,m}) = \{u_0u_i : 1 \le i \le m\}$. Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of $K_{1,n}$ and $E(K_{1,n}) = \{v_0v_i : 1 \le i \le n\}$. Without loss of generality assume that $m < n$.
Define $f : V(G) \cup E(G) \to \{\pm 1, \pm 2, \ldots, \pm(m+n+1)\}$ as follows:

$$\begin{aligned} f(u_0) &= -(m+n+1), \\ f(u_i) &= m+n+1-2i, 1 \le i \le m, \\ f(v_0) &= m+n+1, \\ f(v_i) &= \begin{cases} 1-2i, & 1 \le i \le m \\ -m-i, & m+1 \le i \le n, \end{cases} \\ f(u_0u_i) &= -2i, 1 \le i \le m, \\ f(v_0v_i) &= \begin{cases} m+n-2(i-1), & 1 \le i \le m \\ n+1-i, & m+1 \le i \le n. \end{cases} \end{aligned}$$

Thus, $f$ is a super pair sum labeling and hence $K_{1,m} \cup K_{1,n}$ is a super pair sum graph. □

For example, a super pair sum labeling of $K_{1,4} \cup K_{1,7}$ is shown in Figure 8.

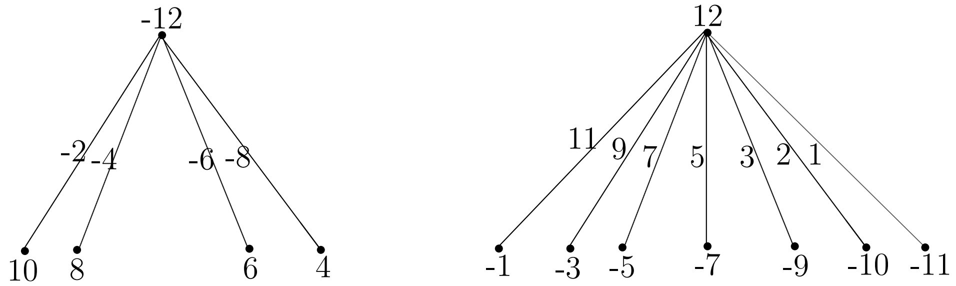


Figure 8: A super pair sum labeling of $K_{1,4} \cup K_{1,7}$

**Theorem 2.8.** *The caterpillar $S(X_1, X_2, \ldots, X_n)$ where $X_1 = m, X_2 = X_3 = \cdots = X_n = 0$ is a super pair sum graph.*

*Proof.* Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$. The vertex $u_1$ is attached to $X_1 = m$ number of leaves $b_{1_j}(1 \leq j \leq m)$.
Define $f : V(G) \cup E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm(m+n)\}$ as follows:

$$f(u_i) = \begin{cases} -\left\lfloor \frac{2n-i}{2} \right\rfloor - m & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ \left\lfloor \frac{n+i-1}{2} \right\rfloor + m & \text{if } i \text{ is even and } 2 \leq i \leq n, \end{cases}$$

$$f(b_{1_j}) = \left\lfloor \frac{n-1}{2} \right\rfloor + j, 1 \leq j \leq m,$$

$$f(u_1 b_{1_j}) = -m - \left\lfloor \frac{n}{2} \right\rfloor + j, 1 \leq j \leq m \text{ and}$$

$$f(u_i u_{i+1}) = i - \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq i \leq n-1.$$

Then, $f$ is a super pair sum labeling and hence $S(m, 0, 0, \ldots, 0)$ is a super pair sum graph. □

For example, a super pair sum labeling of $S(6, 0, 0, 0, 0, 0, 0, 0)$ is shown in Figure 9.

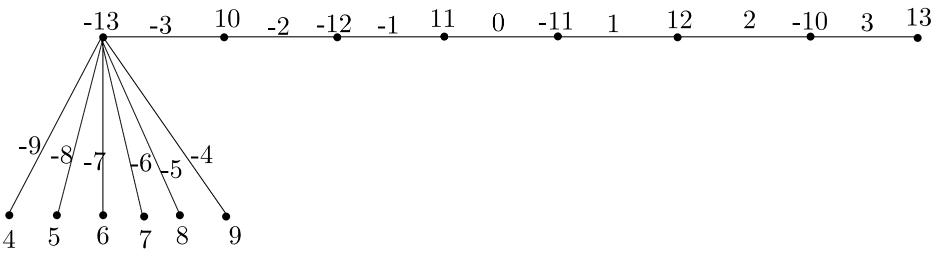


Figure 9: A super pair sum labeling of $S(6, 0, 0, 0, 0, 0, 0, 0)$

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