Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

A Survey On the Vulnerability Parameters of Networks

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ABSTRACT

The analysis of vulnerability in networks generally involves some questions about how the underlying graph is connected. One is naturally interested in studying the types of disruption in the network that maybe caused by failures of certain links or nodes. In terms of a graph, the concept of connectedness is used in different forms to study many of the measures of vulnerability. When certain vertices or edges of a connected graph are deleted, one wants to know whether the remaining graph is still connected, and if so, what its vertex - or edge - connectivity is. If on the other hand, the graph is disconnected, the determination of the number of its components or their orders is useful. Our purpose here is to describe and analyze the current status of the vulnerability measures, identify its more interesting variants, and suggest a most suitable measure of vulnerability.

Keyword: vulnerability measures, connectivity, binding number, toughness, integrity, tenacity

ARTICLE INFO

Article history:
Received 6, March 2016
Received in revised form 20, November 2016
Accepted 3 December 2016
Available online 25, December 2016

AMS subject Classification: 05C78.

1 PRELIMINARY

Many graph theoretical parameters have been used to describe the vulnerability of communication networks, including connectivity, toughness, binding number, integrity and

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tenacity. Before we start to describe and analyze the current status of the vulnerability measures, we will give some basic definitions and notations.
We shall for the most part, use the terminology and notation of Bondy and Murty [3]; so a graph G has vertex set V(G), edge set E(G), $\nu(G) = n$ vertices, $\epsilon(G) = m$ edges. We use $\alpha(G)$ to denote the independence number of G. Let A be a subset of V(G). The neighborhood of A, N(A), consists of all vertices of G adjacent to at least one vertex of A. We define G-A to be the graph induced by the vertices of V-A. Also, for any graph G, $\tau(G)$ is the number of vertices in a largest component of G and $\omega(G)$ is the number of components of G. A cutset of a connected graph G is a collection of vertices whose removal results in a disconnected graph.

2 VULNERABILITY PARAMETERS

1. CONNECTIVITY:
The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. There is a rich body of theorems concerning connectivity. Many of these are variations of a classical result of a Meneger, which involves the number of disjoint paths joining a given pair of vertices in a graph.

2. BINDING NUMBER:
In 1973 D. R. Woodall [27] introduced the concept of the binding number of a graph and studied some properties of binding number. The binding number of a graph G, denoted by bind(G), is defined to be $min\{\frac{|N(A)|}{|A|}; A \in F\}$, where $F = \{A \mid \phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)\}$.
Proposition 2.1: $bind(K_n) = n - 1 \qquad (n \geq 1)$
Proposition 2.2: $bind(K_{a,b}) = min(\frac{a}{b}, \frac{b}{a}) \qquad (a \geq 1, b \geq 1)$
Proposition 2.3: If $n \geq 3$, then $bind(C_n) = \begin{cases} 1 & \text{if } n \ is \ even, \\ \frac{n-1}{n-2} & \text{if } n \ is \ odd \end{cases}$

Proposition 2.4: (Fundemental Lemma). bind(G) is the largest number c such that $\mid N(A) \mid \geq \mid G \mid \frac{c-1}{c} + \frac{|A|}{c}$ for every $A \subseteq V(G), A \neq \phi$.
Proposition 2.5: If $\mid G \mid = n$ (≥ 1), and the connectivity of G is k (≥ 0) (so that G is k-connected but not (k+1)-connected), then $bind(G) \leq \frac{n+k}{n-k}$.

On the basis of these results in [27], Woodall gave a sufficient condition for the existence of a Hamiltonian circuit.

Theorem 2.1: Let G be a graph on n vertices such that $bind(G) \geq c$
a) If $c \geq \frac{3}{2}$, then G has a Hamiltonian circuit.
b) If $1 < c \leq \frac{3}{2}$, then G contains a circuit of length at least $\frac{3(n-1)(c-1)}{c}$, unless G consists either of two copies of K_4 with exactly one edge joining them, in which case the formula gives $4\frac{1}{2}$ and $4\frac{1}{5}$ respectively, and the longest circuit has length four.

Theorem 2.1 suggest the following conjectures:

Conjecture 2.1: If G is a graph on n vertices such that $bind(G) \geq c$, $(1 \leq c \leq \frac{3}{2})$, where n and c are sufficiently large (the precise conditions to be determined), and if G contains a circuit of length $m < \frac{3(n-1)(c-1)}{c}$, then G contains a circuit of length m+1.
Conjecture 2.2. If $bind(G) \geq \frac{3}{2}$, then G contains a triangle.
Conjecture 2.3: If $bind(G) \geq \frac{3}{2}$, then G is pancyclic (i.e., contains a circuit of every length m, $3 \leq m \leq | G |$).

The figure $\frac{3}{2}$ in Conjectures 2.2 and 2.3 is the least possible, in view of graphs of the following form: the vertices are spaced regularly round the circumference of a circle, and each vertex v is joined to all the vertices strictly within the arc of length $\frac{2}{3}\pi$ whose mid-point is diametrically opposite v. The conclusion of Conjecture 2 certainly follows if $bind(G) \geq \frac{1}{2}(1+\sqrt{5})$.

3. TOUGHNESS:
In 1972, Chvátal [6] introduced the concept of the toughness of a graph. It measures in a simple way how tightly various pieces of a graph hold together; therefore he called it toughness. Let G be a graph and t a real number such that the implication $\omega(G-A) > 1 \Rightarrow | A | \geq t \cdot \omega(G-A)$ holds for each set A of vertices of G. Then G will be said to be t-tough.

Proposition 3.1: $G \subset H \Rightarrow t(G) \leq t(H)$.

Thus toughness is a nondecreasing invariant whose values range from zero to infinity. A graph G is disconnected if and only if $t(G) = 0$; G is complete if and only if $t(G) = +\infty$.

Proposition 3.2: $t \geq \frac{\kappa}{\alpha}$.
Proposition 3.3: If G is not complete, then $t \leq \frac{1}{2}\kappa$.
Proposition 3.4: If G is not complete, then $t \leq \frac{n-\alpha}{\alpha}$.
Proposition 3.5: $m \leq n \Rightarrow t(K_{m,n}) = \frac{m}{n}$.
Theorem 3.1: $t(K_m \times K_n) = \frac{1}{2}(m+n) - 1, (m, n \geq 2)$.

Proposition 3.2, 3.3 indicate a relationship between toughness and connectivity. Another indication of this relationship is given by:

Theorem 3.2: $t(G^2) \geq \kappa(G)$.
Corollary 3.1: If m is a positive integer and $n = 2^m$, then $t(G^n) \geq \frac{1}{2}n\kappa(G)$.
Proposition 3.6: Every hamiltonian graph is 1-tough.

Unfortunately, the converse of Proposition 3.6 holds for graphs with at most six vertices only. Eventhough its converse in general does not hold, one may wonder what additional conditions placed upon a 1-tough graph G would imply the existence of hamiltonian cycle in G. As in next conjecture, such conditions may have the flavor of Ramsey's theorem.

Conjecture 3.1: If G is 1-tough, then either G is hamiltonian or its complement $\overline{G}$ contains the graph F.
Conjecture 3.2: Every t-tough with $t > \frac{3}{2}$ is hamiltonian.

The toughness has been studied extensively; see for example [11,12,13,20,21]. Woodall in[27] proved the following proposition:

Proposition 3.7. $bind(G) \leq t(G) + 1$.

4. INTEGRITY:
The integrity of a graph G was introduced by Barefoot, Entringer and Swart in [2] as a useful measure of the vulnerability of G. The integrity of a graph G is given by $I(G) = min(\mid S \mid + \tau(G - S)$, where the minimum is taken over all vertex cutsets A of G, $\tau(G - S)$ is the maximum number of vertices in a component of G-S. Integrity has been studied in numerous papers including [1,8].

5.TENACITY:
The tenacity is a new invariant for graphs. It is another vulnerability measure, incorporating ideas of both toughness and integrity. The tenacity of a graph G, T(G) is defined by $T(G) = min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$, where the minimum is taken over all vertex cutset A of G, G-A is the graph induced by the vertices of V-A, $\tau(G - A)$ is the number of vertices in the largest component of the graph induced by G-A and $\omega(G - A)$ is the number of components of G-A. A connected graph G is called T-tenacious if $\mid A \mid + \tau(G - A) \geq T\omega(G - A)$ holds for any subset A of vertices of G with $\omega(G - A) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $A \subset V(G)$ is said to be a T-set of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$.

Without attempting to obtain the best possible result, we can prove quite easily the following relation between T(G) and t(G). This result gives us a number of corollaries.

Theorem 5.1: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.
Proof: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} + \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.
Proposition 5.1: If G is Hamiltonian-connected and $n \geq 3$, then $T(G) > 1$.
Proof: By Proposition 3.6, every Hamiltonian graph is 1-tough. Hence $t(G) \geq 1$. But if G is Hamiltonian-connected and $p \geq 3$ then G is Hamiltonian. Therefore $T(G) > 1$.

The following theorem, proved by Chvátal and Erdös [7], enable us to relate Proposition 5.1 to the connectivity and the independence number of a graph.

Theorem 5.2: (Chvátal and Erdös). If G is k-connected and $k > \alpha$, then G is Hamiltonian-connected.

Thus from Proposition 5.1 we have three possibilities for a graph G:

1) $1 < \frac{\kappa(G)}{\alpha(G)} < \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$
2) $\frac{\kappa(G)+1}{\alpha(G)} \leq 1 \leq T(G)$
3) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1$

By Proposition 5.1, graphs satisfying the third inequality are not Hamiltonian-connected. By Theorem 5.2, graphs satisfying the first inequality are Hamiltonian-connected. The cycle C_p, $p \geq 6$, satisfies the second inequality but is not Hamiltonian-connected while the graph C_p^2, $p \geq 10$, satisfies the second inequality and is Hamiltonian-connected.

In [5] Chartrand, Kapoor and Lick considered some conditions necessary for a graph to be n-Hamiltonian. Let graph G be m-connected. By definition every Hamiltonian graph is 2-connected. Since the removal of any n vertices from an n-Hamiltonian graph G results in a Hamiltonian graph, it follows that G is (n+2)-connected.

Theorem 5.3: If G is n-Hamiltonian then $T(G) \geq 1 + \frac{n+1}{\alpha(G)}$.

To relate Theorem 5.2 to the connectivity of G, we use a generalization of the following theorem of Chvátal and Erdös [7].

Theorem 5.4: (Chvátal and Erdös). If G is k-connected and $k \geq \alpha$, then G is Hamiltonian.
Theorem 5.5: (Molluzzo [19]). If G is k-connected and for any integer n, $k - n \geq \alpha$, then G is n-Hamiltonian.

For such k and n, we have the following three possibilities for a graph G:

1) $1 + \frac{n+1}{\alpha(G)} \leq \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$
2) $\frac{\kappa(G)+1}{\alpha(G)} \leq 1 + \frac{n+1}{\alpha(G)} \leq T(G)$
3) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1 + \frac{n+1}{\alpha(G)}$

If G satisfies the third inequality it is not n-Hamiltonian by Theorem 5.3. If G satisfies the first inequality then G is n-Hamiltonian by Theorem 5.5. Define the graph C_p^k for any positive k as follows: $V(C_p^k) = V(C_p) = \{0, 1, 2, \cdots, p-1\}$ and two vertices i and j are adjacent if and only if $\mid i-j \mid\leq k$. The graph C_p^{n+2}, for p sufficiently large, satisfies the second inequality and is n-Hamiltonian while the graph $G_{p,2}$, defined below, for p sufficiently large, satisfies the second inequality and is not n-Hamiltonian.

The graph $G_{p,m}$, with $1 \leq m \leq \frac{p-1}{2}$, has p vertices and vertex v which is adjacent to all vertices of the two complete subgraphs, copies of K_m and K_{p-m-1}, in other words we have $G_{p,m} \cong K_1 + (K_m \cup K_{p-m-1})$.

Now we can discuss about tenacity and its operation on graphs. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower by too much. In [9] we proved the following two theorems and corollaries:

Theorem 5.6: For any nontrivial, noncomplete graph G with n vertics and any vertex v, $T(G-v) \geq T(G) - \frac{1}{2}$.

The following theorem allow us to find the tenacity of several important classes of graphs.

Theorem 5.7: If G is a bipartite, r-regular, r-connected graph on n vertices, then $T(G) = \frac{n+2}{n}$.

This result gives several interesting corollaries.

Corollary 5.1: If G_1 is a bipartite, d-regular, d-connected graph with n_1 vertices and G_2 is a bipartite, q-regular, q-connected graph with n_2 vertices, then $T(G_1 \times G_2) = \frac{n_1 n_2+2}{n_1 n_2}$.
Corollary 5.2: For any integer n, $T(Q_n) = \frac{2^n+2}{2^n}$.
Corollary 5.3: For any integers n and m, $T(C_n \times C_m) = \frac{nm+2}{nm}$.
Corollary 5.4: For any even integer n, $T(C_n \times K_2) = \frac{n+1}{n}$.

We next obtain some bounds on the tenacity of products of graphs. Note that the first inequality in the following theorem is a corollary to Theorem 5.1

Theorem 5.8: If $n \geq m$, then $\frac{m^2+mn-2m+2}{2m} \leq T(K_m \times K_n) \leq \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
Corollary 5.5: For any integer n, $T(K_n \times K_n) = n - 1 + \frac{1}{n}$.
Conjecture 5.1: If $n \geq m \geq 2$ then $T(K_m \times K_n) = \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.

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