Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

# On the tenacity of cycle permutation graph

D. Jelodar, *[1], D. Moazzami, †[2] and P. Nasehpour‡[3]

[1]University of Tehran, Department of Algorithms and Computation

[2,3]University of Tehran, College of Engineering, Department of Engineering Science.

**ABSTRACT**

A special class of cubic graphs are the cycle permutation graphs. A cycle permutation graph $P_n(\alpha)$ is defined by taking two vertex-disjoint cycles on $n$ vertices and adding a matching between the vertices of the two cycles.

In this paper we determine a good upper bound for tenacity of cycle permutation graphs.

**ARTICLE INFO**

*Article history:*

Received 20, February 2016

Received in revised form 27, October 2016

Accepted 29, November 2016

Available online 8, December 2016

*Keyword:* Tenacity, Tenacious, Cycle Permutation, Toughness, Integrity.

AMS subject Classification: Primary 05$C$78, 05$C$70

## 1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Our terminology will be standard except as indicated; a good reference for any undefined terms is [1]. Throughout the paper $G = (V, E)$ will denote a graph with vertex set $V(G)$, edge set $E(G)$. The minimum degree $\delta(G)$, the maximum degree $\Delta(G)$, connectivity $\kappa(G)$, the independence number $\beta(G)$, the number of components $\omega(G)$ and the toughness $\tau(G)$. If no ambiguity is possible, the symbols will be used without reference to $G$. A cut-set of $G$

*Email:*djelodar@alumni.ut.ac.ir*;

†Email:*dmoazzami@ut.ac.ir*;

‡E-mail: *nasehpour@gmail.com*

is a proper subset $S$ of $V(G)$ such that $\omega(G-S)>1$. The toughness of a graph $G$ was introduced by Chvátal, [3], where he observed the relationships between this parameter and the existence of Hamiltonian cycles in the given graph, and several results regarding this invariant were obtained. The original approach to toughness, $\tau(G)$, as follows. A connected graph G is called $\tau$-tough if $\tau\omega(G-A)\leq\mid A\mid$ for any subset $A$ of $V(G)$ with $\omega(G-A)>1$.

A set $S\subset V(G)$ is said to be a $\tau$-set of $G$ if $\tau(G)=\frac{|S|}{\omega(G-S)}$.

The concept of graph tenacity was introduced by Cozzens, Moazzami and Stueckle in [5, 6], as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens et al. introduced two measures of network vulnerability termed the tenacity, $T(G)$, and the Mix-tenacity, $T_m(G)$, of a graph.

The tenacity $T(G)$ of a graph $G$ is defined as

$$T(G)=\min_{S\subset V(G)}\left\{\frac{|S|+m(G-S)}{\omega(G-S)}\right\}$$

where $m(G-S)$ denotes the order (the number of vertices) of a largest component of $G-S$ and $\omega(G-S)$ is the number of components of $G-S$. A set $S\subset V(G)$ is said to be a $T$-set of $G$ if $T(G)=\frac{|S|+m(G-S)}{\omega(G-S)}$.
The Mix-tenacity $T_m$ of a graph $G$ is defined as

$$T_m=\min_{S\subset E(G)}\left\{\frac{|S|+m(G-S)}{\omega(G-S)}\right\}$$

where $m(G-S)$ denotes the order (the number of vertices) of a largest component of $G-S$ and $\omega(G-S)$ is the number of components of $G-S$. we called this parameter Mix-tenacity. $T(G)$ and $T_m(G)$ turn out to have interesting properties. After the pioneering work of Cozzens, Moazzami, and Stueckle several groups of researchers have investigated tenacity, and related problems.

Permutation graphs were introduced by Chartrand and Harary [4]. For a labeled graph, $G$ with $V(G)=\{1,2,\cdots,n\}$, and a permutation $\alpha$ in symmetric group $S_n$, the $\alpha$-permutation graph, $P_G(\alpha)$, is a graph with two disjoint copies of $G$, $G_a=(V_a,E_a)$ and $G_b=(V_b,E_b)$, with $V_k=\{k_1,k_2,\cdots,k_n\}$ for $k=\{a,b\}$, and with the edges $(a_i,b_{\alpha(i)})$, for

$1 \leq i \leq n$. If $G$ is an $n$-cycle, labeled consecutively around the cycle, then $P_n(\alpha) = P_{C_n}(\alpha)$ is a *cycle permutation graph*. The copy of $C_n$ labeled $a_1, a_2, \cdots, a_n$ will be called the *outer cycle*, the copy of $C_n$ labeled $b_1, b_2, \cdots, b_n$ will be called the *inner cycle*, and the edges of the form $(a_i, b_{\alpha(i)})$ will be called *permutation edges* (This is the same definition used in [10]). The permutation cycle graphs are a family of cubic graphs.

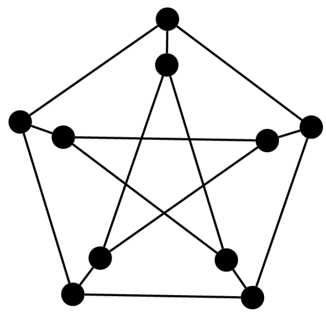


Figure 1: $P_5(1, 3, 5, 2, 4)$, Petersen Graph

For $\alpha = (1, 3, 5, 2, 4)$, Fig.[1] shows $P_5(\alpha)$ that is Petersen graph.

Now, we list some known results on toughness and tenacity.

**Proposition 1.** *[3, 8] For a non-complete graph $G$*

$$\frac{\kappa(G)}{\Delta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}.$$

**Proposition 2.** *[5] For a non-complete graph $G$*

$$\tau(G) < T(G).$$

Piazza, Ringeisen and Stueckle in [9] conjectured that a permutation cycle graph has $\tau \leq 4/3$. Goddard in [7] was very close to prove that this conjecture is true.

**Proposition 3.** *[7] Let $G$ be a cycle permutation graph on $2n$ vertices. Then*

$$\tau \begin{cases} \leq 4/3 & n \equiv 0, 1 \ mod\ 4 \\ < 4/3 & n \equiv 2 \ mod\ 4 \\ \leq 4/3 + 4/(9n - 3) & n \equiv 3 \ mod\ 4 \end{cases}$$

# 2 Upper Bound

In this section, we use a special vertex coloring for separate vertices into some independent sets. Consider the outer cycle $C_O$ is $a_1 a_2 \cdots a_n a_1$ in clockwise and the inner cycle $C_I$ is

$b_1 b_2 \cdots b_n b_1$ in clockwise.
For the outer cycle, assume $\{a_1, a_3, \cdots\}$ are colored with black and $\{a_2, a_4, \cdots\}$ are colored with white, now if $n$ is an even, $a_n$ will be white otherwise $a_n$ is colored with gray. For the inner cycle, suppose $\alpha(n) = m$, so $\{b_{m+1}, b_{m+3}, b_{m+5}, \cdots\}$ are colored with black and $\{b_{m+2}, b_{m+4}, b_{m+6}, \cdots\}$ are colored with white and if $n$ is an odd so $b_m$ is colored with gray otherwise $b_m$ is white.

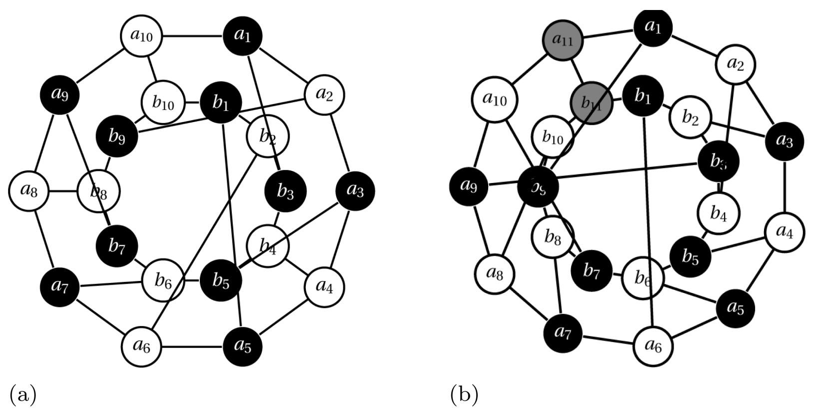
Now we can separate all vertices of $P_n(\alpha)$ into nine subsets that eight of them are independent sets. $\mathcal{A}_{bb}$ is included all **black** vertices of **outer** cycle that they are connected to **black** vertices of **inner** cycle, $\mathcal{A}_{bw}$ is a subset of vertices of **outer** cycle that they are **black** themselves and connected to **white** vertices of **inner** cycle. $\mathcal{B}_{bw}$ is included all **white** vertices of **inner** cycle that they are connected to **black** vertices of **outer** cycle. We can use these definitions same as above to declare $\mathcal{A}_{wb}$, $\mathcal{A}_{ww}$, $\mathcal{B}_{wb}$, $\mathcal{B}_{ww}$ and $\mathcal{B}_{bb}$ and if $n$ is an odd then $\mathcal{D} = \{a_n, b_{\alpha(n)}\}$ otherwise $\mathcal{D} = \emptyset$.

**Example 4.** For $n = 10$ and $\alpha = (3, 9, 5, 4, 1, 2, 6, 8, 7, 10)$, in $P_{10}(\alpha)$ (Fig.2a) we have:

$$\mathcal{A}_{bb} = \{a_1, a_3, a_5, a_9\}, \quad \mathcal{A}_{ww} = \{a_4, a_6, a_8, a_{10}\}, \mathcal{A}_{bw} = \{a_7\}, \quad \mathcal{A}_{wb} = \{a_2\}$$
$$\mathcal{B}_{bb} = \{b_1, b_3, b_5, b_7\}, \quad \mathcal{B}_{ww} = \{b_2, b_4, b_8, b_{10}\}, \mathcal{B}_{bw} = \{b_6\}, \quad \mathcal{B}_{wb} = \{b_9\}.$$
$$\mathcal{D} = \emptyset.$$

For $n = 11$ and $\alpha = (9, 4, 2, 5, 6, 1, 8, 10, 3, 7, 11)$, in $P_{11}(\alpha)$ (Fig. 2b) we have:

$$\mathcal{A}_{bb} = \{a_1, a_9\}, \quad \mathcal{A}_{ww} = \{a_2, a_8\}, \mathcal{A}_{bw} = \{a_3, a_5, a_7\}, \quad \mathcal{A}_{wb} = \{a_4, a_6, a_{10}\}$$
$$\mathcal{B}_{bb} = \{b_3, b_9\}, \quad \mathcal{B}_{ww} = \{b_4, b_{10}\}, \mathcal{B}_{bw} = \{b_2, b_6, b_8\}, \quad \mathcal{B}_{wb} = \{b_1, b_5, b_7\}$$
$$\mathcal{D} = \{a_{11}, b_{11}\}.$$




(a) $P_{10}(3, 9, 5, 4, 1, 2, 6, 8, 7, 10)$ (b) $P_{11}(9, 4, 2, 5, 6, 1, 8, 10, 3, 7, 11)$

Figure 2: Permutation cycle graphs

**Lemma 5.** *Let $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices, then:*

$$|\mathcal{A}_{bb}| = |\mathcal{A}_{ww}| = |\mathcal{B}_{bb}| = |\mathcal{B}_{ww}|$$

$$|\mathcal{A}_{bw}| = |\mathcal{A}_{wb}| = |\mathcal{B}_{bw}| = |\mathcal{B}_{wb}|.$$

*Proof.* Consider, $|\mathcal{A}_{bb}| = a_{bb}$, $|\mathcal{A}_{bw}| = a_{bw}$, $|\mathcal{A}_{wb}| = a_{wb}$, $|\mathcal{A}_{ww}| = a_{ww}$, $|\mathcal{B}_{bb}| = b_{bb}$, $|\mathcal{B}_{bw}| = b_{bw}$, $|\mathcal{B}_{wb}| = b_{wb}$, $|\mathcal{B}_{ww}| = b_{ww}$ and $|\mathcal{D}| = d$.
It is clear that $a_{bb} = b_{bb}$, $a_{ww} = b_{ww}$, $a_{bw} = b_{bw}$ and $a_{wb} = b_{wb}$.
Assume to the contrary and $a_{bb} \neq a_{ww}$. Without loss of generality, suppose that $a_{bb} \leq a_{ww} - 1$. $k = \lfloor n/2 \rfloor$ so:

$$\begin{aligned}
&a_{bb} \leq a_{ww} - 1 \\
\Longrightarrow &a_{bw} \geq k - a_{ww} + 1 \qquad (\text{ by } a_{bb} + a_{bw} = k) \\
\Longrightarrow &b_{bw} \geq k - a_{ww} + 1 \qquad (\text{ by } a_{bw} = b_{bw}) \\
\Longrightarrow &b_{ww} \leq a_{ww} - 1 \qquad (\text{ by } a_{bw} + b_{ww} = k) \\
\Longrightarrow &a_{ww} \leq a_{ww} - 1 \qquad (\text{ by } b_{ww} = a_{ww})
\end{aligned}$$

that is inconsistency, So the result is desired. □

Consider $S_1 = \mathcal{A}_{bb} \bigcup \mathcal{A}_{bw} \bigcup \mathcal{B}_{ww}$, $S_2 = \mathcal{A}_{ww} \bigcup \mathcal{A}_{wb} \bigcup \mathcal{B}_{bb}$, $S_3 = \mathcal{B}_{bw}$ and $S_4 = \mathcal{B}_{wb}$. Notice, if $b_{bw} \geq b_{bb}$ we exchange $\mathcal{B}_{bb}$ with $\mathcal{B}_{bw}$ in $S_1$ and $S_3$, $\mathcal{B}_{ww}$ with $\mathcal{B}_{wb}$ in $S_2$ and $S_4$. We know that $S_3$ and $S_4$ are independent sets. For Example 4 we have:

a) $S_1 = \mathcal{A}_{bb} \bigcup \mathcal{A}_{bw} \bigcup \mathcal{B}_{ww} = \{a_1, a_3, a_5, a_9, a_7, b_2, b_4, b_8, b_{10}\}$
$S_2 = \mathcal{A}_{ww} \bigcup \mathcal{A}_{wb} \bigcup \mathcal{B}_{bb} = \{a_4, a_6, a_8, a_{10}, a_2, b_1, b_3, b_5, b_7\}$
$S_3 = \mathcal{B}_{bw} = \{b_6\}, S_4 = \mathcal{B}_{wb} = \{b_9\}, S_5 = \emptyset.$

b) $S_1 = \mathcal{A}_{bb} \bigcup \mathcal{A}_{bw} \bigcup \mathcal{B}_{wb} = \{a_1, a_9, a_3, a_5, a_7, b_1, b_5, b_7\}$
$S_2 = \mathcal{A}_{ww} \bigcup \mathcal{A}_{wb} \bigcup \mathcal{B}_{bw} = \{a_2, a_8, a_4, a_6, a_{10}, b_2, b_6, b_8\}$
$S_3 = \mathcal{B}_{bb} = \{b_3, b_9\}, S_4 = \mathcal{B}_{ww} = \{b_4, b_{10}\}, S_5 = \{a_{11}, b_{11}\}.$

**Lemma 6.** *The set $S_1$ and $S_2$ are independent sets.*

*Proof.* It is clear that there are no edges between vertices of $\mathcal{A}_{bb}$ and $\mathcal{A}_{bw}$. Furthermore vertices of $B_{ww}$ are connected to white vertices of outer cycle that means $\mathcal{A}_{ww}$ and $\mathcal{A}_{ww} \not\subseteq S_1$, so, we have the desired result. For $S_2$, proof is the same. □

**Lemma 7.** *Every vertex of $S_3$ has exactly one neighbor in $S_1$.*

*Proof.* At first, every vertex of $S_3 = \mathcal{B}_{bw}$ has at least one neighbor on $S_1$, because each vertex of $S_3$ must be connected to a black vertex of outer cycle and all black vertices of outer cycle are in $S_1$ . Now we prove that it is only one and no more. According to the definition, each vertex of $S_3$ has two black neighbors in inner cycle that they are not in $S_1$, so the lemma is proved. □

This proof can be extended to $S_2$ and $S_4$. By lemma above, for the cut set of $S = S_2 \cup S_4 \cup S_5$, the largest component will be $m(P_n(\alpha)) \leq 2$ and $\omega(P_n(\alpha) - S) = |S_1|$.

**Theorem 8.** *Let $P_n(\alpha)$ be a cycle permutation with $2n$ vertices and $b = \min(b_{bb}, b_{bw})$, then:*

$$T(P_n(\alpha)) \leq \begin{cases} \frac{n+2}{n-b} & \text{if } n \text{ is even,} \\ \frac{n+3}{n-b-1} & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* Now, to prove the theorem we start with even $n$, let the cut set is $S = S_2 \cup S_4$, then:

$$\begin{aligned} T(P_n(\alpha) &\leq \frac{|S| + m(P_n(\alpha) - S)}{\omega(P_n(\alpha) - S)} \\ &\leq \frac{a_{bb} + a_{bw} + b_{ww} + b_{wb} + 2}{a_{ww} + a_{wb} + b_{bb}} \quad \text{by } m(P_n(\alpha) - S) \leq 2 \\ &\leq \frac{n+2}{n-b} \end{aligned}$$

For odd $n$, let the cut set is $S = S_2 \cup S_4 \cup S_5$, so:

$$\begin{aligned} T(P_n(\alpha) &\leq \frac{|S| + m(P_n(\alpha) - S)}{\omega(P_n(\alpha) - S)} \\ &\leq \frac{a_{bb} + a_{bw} + b_{ww} + b_{wb} + |S_5| + 2}{a_{ww} + a_{wb} + b_{bb}} \quad \text{by } m(P_n(\alpha) - S) \leq 2 \\ &\leq \frac{n+3}{n-b-1} \end{aligned}$$

so the theorem is proved. □

For example 4, $T(P_{10}(\alpha)) \leq \frac{12}{9}$ and $T(P_{11}(\alpha)) \leq \frac{14}{8}$.

When the number of vertices goes to infinity, if $b$ is small enough, so the tenacity value close to $1^+$. We next investigate the sharpness of the bounds provided above if $b$ is less than or equal 1.

**Theorem 9.** *The above theorem provide the close answer.*

*Proof.* The graph $G = P_n(\alpha)$ is defined as follows. Let the cycle of $C_1$ with $n$ vertices is $a_1a_2 \cdots a_na_1$ in clockwise as outer cycle and $C_2$ with $n$ vertices is $b_1b_2 \cdots b_nb_1$ in clockwise as inner cycle. At first, suppose $n$ be an even, let the vertices $\{a_1, a_3, \cdots, a_{n-1}, b_1, b_3, \cdots, b_{n-1}\}$

are colored black and the other vertices are colored with white. For permutation edges, suppose each vertex of outer cycle except two optional vertices with opposite color is connected to just one of the vertices of inner cycle of opposite color, and the exceptional outer cycle vertices are matched to the identical color vertices of the inner cycle. By theorem [8], $|S| = n$, $m(G - S) = 2$ and $\omega(G - S) = n - 1$ therefore $T \leq \frac{n+2}{n-1}$.
For odd number $n$, suppose the vertices $\{a_1, a_3, \cdots, a_{n-2}, b_1, b_3, \cdots, b_{n-2}\}$ are colored with black and the vertices $\{a_2, a_4, \cdots, a_{n-1}, b_2, b_4, \cdots, b_{n-1}\}$ are colored with white, $a_n$ and $b_n$ are gray. For permutation edges, let each vertex of outer cycle is connected to just one of the vertices of inner cycle of opposite color, and at last $a_n$ is connected to $b_n$. By theorem [8], $|S| = n + 1$, $m(G - S) = 2$ and $\omega(G - S) = n - 1$ therefore $T \leq \frac{n+3}{n-1}$.
The second result has obtained is $\beta(G) \geq n - 1$. On the other hand, by preposition [1] for each cut set $S$, $|S| \geq \omega(G - S)$ so $\beta \leq n$. The independent number $\beta \neq n$, since the graph $G$ is not $3 - regular$ bipartite graph. So let $S$ is an $T$-set of $G$, $\omega(G - S) = \omega$ and $m(G - S) = m$:

$$\begin{aligned} T = \frac{|S| + m}{\omega} &\geq \frac{\omega + m}{\omega} \\ &\geq 1 + \frac{2}{\omega} \\ &\geq 1 + \frac{2}{\beta} \\ &= \frac{n+1}{n-1} \end{aligned}$$

for $n$ is an odd number, proof is the same, so:

$$\frac{n+1}{n-1} \leq T \leq \begin{cases} \dfrac{n+2}{n-1} & n \text{ is even} \\ \dfrac{n+3}{n-1} & n \text{ is odd} \end{cases}$$

□

**Acknowledgement**
This work supported by University of Tehran. Our special thanks go to University of Tehran, College of Engineering and Department of Engineering Science for providing all the necessary facilities available to me for successfully conducting this research.

# References

[1] J.A. Bundy, U.S.R. Murty, Graph Theory with Applications, The Macmillan Press Ltd, 1976.

[2] Chong-Yun Chao; Shaocen Han, On the toughness of cycle permutation graphs, Czechoslovak Mathematical Journal, Vol. 51, (2001), No. 2, 239-260.

[3] Chvatal V.: Tough graphs and hamiltonian circuits. Discrete Math. 5, 215228.

[4] G.Chartrand, F.Harary: Planar permutation graphs. Ann. Inst. Henri. Poincar, Nouv. Sr., Sect. B 3 (1967), 433438.

[5] M.B. Cozzens, D. Moazzami, S. Stueckle, The tenacity of the Harary graphs, J. Combin. Math. Combin. Comput, 16, 1994

[6] M.B. Cozzens, D. Moazzami, S. Stueckle, The tenacity of a graph, in: Yousef Alavi, Allen Schwenk (Eds.), Graph Theory, Combinatorics, and Algorithms, Wiley, New York, 1995, pp. 1111-1112

[7] W. Goddard, The toughness of cubic graphs, Graphs and Combin. 12 (1996) 1722.

[8] Goddard, W., Swart, H.C.: On the toughness of a graph. Quaestiones Math. 13, 217232 (1990).

[9] B. Piazza, R. Ringeisen, and S. Stueckle, On the vulnerability of cycle permutation graphs, Ars Combinatoria 29 (1990) 289296.

[10] B. Piazza and S. Stueckle, On the cut frequency vector of permutation graphs, Congr. Numer. 49 (1985), 143-145 (1973).