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All Ramsey $(2K_2, C_4)-$Minimal Graphs

Kristiana Wijaya[*1], Lyra Yulianti [†2], Edy Tri Baskoro [‡1], Hilda Assiyatun [§1] and Djoko Suprijanto [¶1]

[1]Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Jalan Ganesa 10 Bandung 40132 Indonesia

[2]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Andalas University, Kampus UNAND Limau Manis Padang 25136 Indonesia

ABSTRACT

Let F, G and H be non-empty graphs. The notation $F \rightarrow (G, H)$ means that if any edge of F is colored by red or blue, then either the red subgraph of F contains a graph G or the blue subgraph of F contains a graph H. A graph F (without isolated vertices) is called a Ramsey $(G, H)-$minimal if $F \rightarrow (G, H)$ and for every $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. The set of all Ramsey $(G, H)-$minimal graphs is denoted by $\mathfrak{R}(G, H)$. In this paper, we characterize all graphs which are in $\mathfrak{R}(2K_2, C_4)$.

Keyword: Ramsey minimal graph; edge coloring; graph $2K_2$; cycle graph.

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[*]Corresponding author. E-mail: *kristiana.w@students.itb.ac.id*
[†]Email:*lyra@fmipa.unand.ac.id*
[‡]Email:*ebaskoro@math.itb.ac.id*
[§]Email:*hilda@math.itb.ac.id*
[¶]Email:*djoko@math.itb.ac.id*

1 Introduction

Let F be a graph without isolated vertices. Let G and H be non-empty graphs. We write $F \to (G, H)$ to mean that any red-blue coloring on the edges of F contains a red copy of G or a blue copy of H. Any red-blue coloring on the edges of F is called a $(G, H)-$*coloring* if neither a red G nor a blue H occurs. If a graph F has a $(G, H)-$coloring, then we write $F \nrightarrow (G, H)$. A graph F is called a *Ramsey* $(G, H)-$*minimal* if $F \to (G, H)$, but for every $e \in E(F)$, $(F - e) \nrightarrow (G, H)$. The set of all Ramsey $(G, H)-$minimal graphs is denoted by $\mathfrak{R}(G, H)$. The pair (G, H) is called *Ramsey-finite* if $\mathfrak{R}(G, H)$ is finite and *Ramsey-infinite* if otherwise.

The characterization of all graphs in $\mathfrak{R}(G, H)$ for any given graphs G and H is an interesting problem. However, it is a difficult even for small graphs G and H. There are many papers dealing with this characterization of the members of $\mathfrak{R}(G, H)$. Nešetřil and Rödl (1978) gave some properties of G and H such that (G, H) is Ramsey-infinite. Some researchers have characterized some infinite families of Ramsey $(K_{1,2}, H)-$minimal graphs (see [1, 4, 5, 6, 11, 15, 16]). Burr *et al.* [10] showed that the pair (G, H) is Ramsey infinite whenever both G and H are forests, with at least one of G or H having a non-star component. Moreover, Borowiecka-Olszewska and Hałuszczak [7] gave a method for constructing infinitely many graphs which belong to $\mathfrak{R}(K_{1,m}, \mathcal{G})$, where $m \geq 2$ and $\mathcal{G}$ is a family of 2–connected graphs.

Burr *et al.* [8] proved that if G is a matching, then $\mathfrak{R}(G, H)$ is Ramsey-finite for all graphs H. In the same paper, Burr *et al.* proved that for any graph H, $\mathfrak{R}(K_2, H) = \{H\}$ and they gave some examples of the set $\mathfrak{R}(2K_2, H)$ where $H = 2K_2$ and $H = C_3$, that is $\mathfrak{R}(2K_2, 2K_2) = \{C_5, 3K_2\}$ and $\mathfrak{R}(2K_2, C_3) = \{K_5, 2C_3, G_1\}$ (see Figure 1). Moreover Burr *et al.* [9] investigated $\mathfrak{R}(G, H)$ in the special case where G is a 2–matching and H is a t–matching. They also proved that $\mathfrak{R}(2K_2, 3K_2) = \{C_7, 4K_2, G_2\}, \mathfrak{R}(2K_2, 4K_2) = \{5K_2, 2C_5, C_5 \cdot C_5, C_9, G_3, G_4\}$ (see Figure 1), $\mathfrak{R}(2K_2, K_{1,2}) = \{C_4, C_5, 2K_{1,2}\}$.

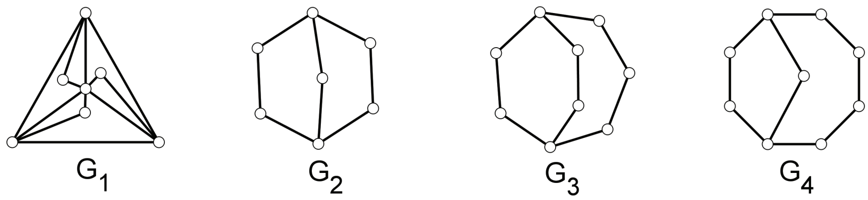

Figure 1: Graphs G_1, G_2, G_3 and G_4.

Mengersen and Oeckermann [12] characterized graphs which belong to $\mathfrak{R}(2K_2, K_{1,n})$ for $n \geq 3$ and determined explicitly all graphs in $\mathfrak{R}(2K_2, K_{1,n})$ for $n \leq 3$. Baskoro and Yulianti [3] determined all graphs in $\mathfrak{R}(2K_2, P_n)$ for $n = 4, 5$. Moreover, Tatanto and Baskoro [14] determined all graphs in $\mathfrak{R}(2K_2, 2P_3)$. Mushi and Baskoro [13] derived the properties of graphs belonging to the class $\mathfrak{R}(3K_2, P_3)$ and obtained all graphs in this set, which can be also found in [9] without proof, except one graph. Recently, Baskoro and Wijaya [2] gave the necessary and sufficient conditions of graphs in $\mathfrak{R}(2K_2, H)$ for any connected H.

Theorem 1.1. [2] *Let H be any connected graph. $F \in \mathfrak{R}(2K_2, H)$ if and only if the following conditions are satisfied:*

(i) *for every $v \in V(F), F - v \supseteq H$,*

(ii) *for every K_3 in $F, F - E(K_3) \supseteq H$,*

(iii) *for every $e \in E(F)$, there exists $v \in V(F)$ or K_3 in F such that $(F - e) - v \not\supseteq H$ or $(F - e) - E(K_3) \not\supseteq H$.*

They determined all graphs in $\mathfrak{R}(2K_2, K_4)$ with at most 8 vertices.
In this paper, we give some properties of graphs belonging to $\mathfrak{R}(2K_2, C_n)$ for $n \geq 3$. Furthermore, we characterize all graphs in the set $\mathfrak{R}(2K_2, C_4)$.

2 Main Results

As usual, V and E are used to denote the vertex set and the edge set of a graph G. If a vertex u is adjacent to v in G, then this edge is denoted by uv. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident to a vertex v. If G has n vertices, then the *degree sequence* of G is $(d_1, d_2, \ldots, d_n)$, where d_i is the degree of vertex i for every $i \in [1, n]$ and $d_1 \geq d_2 \geq \ldots \geq d_n$.
Let G be a graph with n vertices and m edges. For $v \in V(G)$, define $G - v$ as a subgraph of G obtained by removing the vertex v and all edges incident with v. Similarly, for $e \in E(G)$, define $G - e$ as a subgraph of G obtained by deleting the edge e but leaving two vertices incident to e. A complete graph and cycle with n vertices is denoted by K_n and C_n, respectively. mK_2 is a graph consisting of m disjoint copies of K_2. In this paper, we use the notation $(uvwx)$ to describe a cycle C_4 with the vertex set $\{u, v, w, x\}$ and the edge set $\{uv, vw, wx, ux\}$. So, $(uvwx)$ and $(uwxv)$ denote two different cycles. Similarly, the notation (uvw) describes a triangle K_3 with the edge set $\{uv, vw, uw\}$.
We will determine explicitly all graphs in $\mathfrak{R}(2K_2, C_4)$ by using Theorem 1.1. In general, the case (ii) of Theorem 1.1 can be replaced by (ii') for every induced subgraph S of order 3 in F, $F - E(S) \supseteq H$.
The following result gives some properties of graphs in $\mathfrak{R}(2K_2, C_n)$, for any $n \geq 3$.

Lemma 2.1. *Let $F \in \mathfrak{R}(2K_2, C_n)$. Then,*

(i) $|V(F)| \geq n + 1$,

(ii) *$d(v) \geq 2$ for every $v \in V(F)$,*

(iii) *every vertex v is contained in C_n, and*

(iv) *every edge e is contained in C_n.*

Proof.

(i) If $|V(F)| = n$, then $F - v$ has $n - 1$ vertices. So, F does not contain a cycle C_n, a contradiction to Theorem 1.1(i).

(ii) Suppose that there exists a vertex $v \in V(F)$ having $d(v) = 1$. Then v is incident to exactly an edge e. Thus, there exists a $(2K_2, C_n)-$coloring ϕ' on the edges of $F - e$. Next, we define a red-blue coloring ϕ on the edges of F such that $\phi(x) = \phi'(x)$ for $x \in E(F - e)$ and $\phi(e) =$ blue. It is easy to verify that ϕ is a $(2K_2, C_n)-$coloring of F, a contradiction.

(iii) Suppose that there exists a vertex $v \in V(F)$ not contained in any C_n. By the minimality of F, we have a $(2K_2, C_n)-$coloring ϕ' of $F - v$. Next, we define a red-blue coloring ϕ on the edges of F such that $\phi(x) = \phi'(x)$ for $x \in E(F - v)$ and $\phi(x) =$ blue, for all edges x incident to v. Then, ϕ is a $(2K_2, C_n)-$coloring of F, a contradiction.

(iv) Suppose that there exists an edge $e \in E(F)$ not contained in any C_n. By the minimality of F, we have a $(2K_2, C_n)-$coloring ϕ' of $F - e$. Next, we define a red-blue coloring ϕ on the edges of F such that $\phi(x) = \phi'(x)$ for all $x \in E(F - e)$ and $\phi(e) =$ blue. Then, we obtain ϕ as a $(2K_2, C_n)-$coloring of F, a contradiction. □

The next result gives all disconnected graphs which belong to $\mathfrak{R}(2K_2, C_n)$.

Theorem 2.2. *Let $n \geq 3$. The only disconnected graph in $\mathfrak{R}(2K_2, C_n)$ is $2C_n$.*

Proof. It is easy to see that $2C_n \in \mathfrak{R}(2K_2, C_n)$. Let $F \in \mathfrak{R}(2K_2, C_n)$ be a disconnected graph. So, F contains more than one component. By Theorem 1.1(i), each component contains a cycle C_n. Then, the minimality of F implies that $F = 2C_n$. □

By Theorem 2.2 and the minimality of graphs in $\mathfrak{R}(2K_2, C_n)$, we can conclude that any connected graph in $\mathfrak{R}(2K_2, C_n)$ must satisfy the following lemma.

Lemma 2.3. *Let F be a connected graph in $\mathfrak{R}(2K_2, C_n)$ for $n \geq 3$. Then,*

(i) *every two cycles of length n in F intersects in at least one vertex,*

(ii) *F contains at least three cycles of length n.*

Proof. Let F be a connected graph in $\mathfrak{R}(2K_2, C_n)$.

(i) Assume that we have two disjoint cycles of length n in F. Then, $F \supseteq 2C_n$. This contradicts to the minimality of F.

(ii) Let v be an intersection vertex of two cycles C_n in F. By Theorem 1.1(i), we have $F - v \supseteq C_n$. Of course, the last cycle will be different from the previous two cycles. Therefore, F must contain at least 3 cycles of length n. □

From now on, we will determine all graphs in $\mathfrak{R}(2K_2, C_4)$. By Theorem 2.2, a graph $2C_4$ is the only disconnected graph in $\mathfrak{R}(2K_2, C_4)$. So, to complete our task, we must determine all connected graphs F in $\mathfrak{R}(2K_2, C_4)$. The minimality of F implies that $F \not\supseteq 2C_4$. To determine all connected graphs $F \in \mathfrak{R}(2K_2, C_4)$, we will find such graphs of certain order n. Trivially, $n \geq 5$. We will show that these graphs exist if $n \leq 10$. The following theorem shows that $K_5 - e$ is the only connected graph of order 5 in $\mathfrak{R}(2K_2, C_4)$.

Theorem 2.4. *The only connected graph of order 5 in $\mathfrak{R}(2K_2, C_4)$ is $K_5 - e$.*

Proof. First, we show that $K_5 - e \in \mathfrak{R}(2K_2, C_4)$. For every $v \in V(K_5 - e)$, $(K_5 - e) - v$ can be either K_4 or $K_4 - e$. Obviously, $(K_5 - e) - v \supseteq C_4$. Moreover, for every triangle K_3 in $K_5 - e$, we obtain a graph of order 5, $K_5 - e - E(K_3)$ with a degree sequence $(4, 3, 2, 2, 1)$ or $(3, 3, 2, 2, 2)$. Certainly, $K_5 - e - E(K_3)$ contains a C_4. So, $K_5 - e \rightarrow (2K_2, C_4)$. Next, if an edge of $K_5 - e$ is deleted, then we obtain two non-isomorphic graphs $K_5 - 2e$, say F_a with a degree sequence $(4, 4, 3, 3, 2)$ or F_b with a degree sequence $(4, 3, 3, 3, 3)$. For $v \in V(F_a)$ with $d(v) = 4$, we obtain $F_a - v$ which does not contain a C_4. For the graph F_b, we choose a triangle K_3 where two edges of K_3 incident to a vertex of degree 4 in F_b. We obtain $F_b - E(K_3)$ which does not contain a C_4. So, $K_5 - 2e \nrightarrow (2K_2, C_4)$. Hence, $K_5 - e \in \mathfrak{R}(2K_2, C_4)$.
Since $K_5 \supseteq K_5 - e$, $K_5 \notin \mathfrak{R}(2K_2, C_4)$. Beside that, other non complete graphs of order 5 are subgraphs of $K_5 - e$. So, $K_5 - e$ is the only graph of order 5 in $\mathfrak{R}(2K_2, C_4)$. □
We need the following lemma to find all connected Ramsey $(2K_2, C_4)-$minimal graphs of order greater than 5.

Lemma 2.5. *Let A_1 and A_2 be graphs of orders 7 and 6 containing two cycles of length 4, respectively, with $E(A_1) = E(\mathcal{C}_1) \cup E(\mathcal{C}_2)$ and $E(A_2) = E(\mathcal{C}_1) \cup E(\mathcal{C}_3)$, where $\mathcal{C}_1 = (v_1v_2v_3v_4)$, $\mathcal{C}_2 = (v_4v_5v_6v_7)$ and $\mathcal{C}_3 = (v_3v_4v_5v_6)$. Let F be a connected graph and $F \in \mathfrak{R}(2K_2, C_4)$. If F has order at least 6, then F must contain A_1 or A_2. Precisely, we have*

(i) *If F has order 6, then F contains A_2,*

(ii) *If F has order 7, then F contains A_1 or A_5, where A_5 is a graph having the vertex set $V(A_5) = V(A_2) \cup \{v_7\}$ and the edge set $E(A_5) = E(A_2) \cup \{v_2v_4, v_2v_7, v_4v_6\}$,*

(iii) *If F has order at least 8, then F contains A_1.*

Proof. Let $F \in \mathfrak{R}(2K_2, C_4)$. Then, F must contain a cycle of length 4, say $\mathcal{C}_1 = (v_1v_2v_3v_4)$. Since there is no triangle in $\mathcal{C}_1$, we take an induced subgraph S of order 3 to apply Theorem 1.1(ii'). So, by Theorem 1.1(ii'), for $S = \{v_1, v_2, v_4\}$, there must be a C_4 in $F - E(S)$. This C_4 is formed by involving a vertex of $\mathcal{C}_1$ and three other vertices v_5, v_6, v_7 in F, say $C_4 = (v_4v_5v_6v_7)$ (the graph A_1); or involving two vertices of $\mathcal{C}_1$ and two other vertices v_5, v_6 in F, say $C_4 = (v_3v_4v_5v_6)$ (the graph A_2) or $C_4 = (v_3v_5v_4v_6)$ (the graph A_3); or involving three vertices of $\mathcal{C}_1$ and a vertex v_5 in F, say $C_4 = (v_2v_3v_4v_5)$ (the graph A_4), as depicted in Figure 2. Let F be of order at least 6. We now consider that

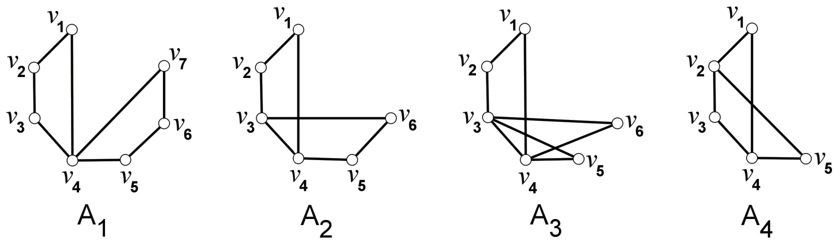

Figure 2: The possibilities of forming C_4 in $F - E(S)$ for some $V(S) = \{v_1, v_2, v_4\}$ when $F \supseteq C_4$.

F contains either A_3 or A_4. If $F \supseteq A_3$, then by Theorem 1.1(i), $F - v_3$ must contain a C_4. As a consequence, F will contain either A_1 or A_2. If $F \supseteq A_4$, then by Theorem 1.1(i), $F - v_2$ must contain a C_4. Since $|V(F)| \geq 6$, there exists at least a vertex in F other than $V(\mathcal{C}_1) \cup \{v_5\}$. But every C_4 in $F - v_2$ will yield that F contains either A_1 or A_2 or A_3. Thus, the claim follows immediately.

(i) Let F be of order 6. Clearly F does not contain A_1.

(ii) Let F be of order 7. Then, F contains either A_1 or A_2. Suppose that F contains A_2. Then, there must be a C_4 in $F - v_3$, by Theorem 1.1(i). This cycle is formed by involving three vertices in A_2 and a vertex v_7 in F. There are five possibilities (up to isomorphism), say $C_4 = (v_2v_4v_6v_7)$, $C_4 = (v_1v_4v_5v_7)$, $C_4 = (v_1v_7v_4v_5)$, $C_4 = (v_1v_4v_6v_7)$ or $C_4 = (v_2v_6v_4v_7)$, (see the graph A_5, A_6, A_7, A_8 or A_9, respectively, as depicted in Figure 3). From these possibilities, it is enough to consider $F \supseteq A_5$, since $F - v_4$ contains a C_4.

(iii) Let F be of order at least 8. Then, F contains either A_1 or A_2. If F contains A_2, then there must be a C_4 in $F - v_3$, by Theorem 1.1(i). This cycle can be formed by involving two vertices in A_2 and two vertices v_7 and v_8 in F; or involving three vertices in A_2 and a vertex v_7 in F. But, if this C_4 contains v_7 and v_8, then F will contain either A_1 or $2C_4$. So, this C_4 is formed by involving v_7 in F. By case (ii),

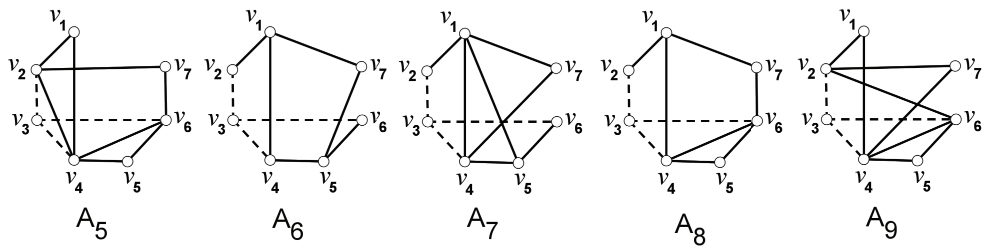

Figure 3: The possibilities of forming C_4 in $F - v_3$ when $F \supseteq A_2$.

F contains A_5. Next, by Theorem 1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_3v_4v_6)$. This cycle is formed by involving three vertices of A_5 and a vertex v_8 in F. So, the edge v_iv_8 must be in F for some $i \in [1, 7]$. But, this cycle cannot involve the edge v_iv_8 for every $i \in [1, 7]$, since it causes F containing A_1. $\square$

In the next result, we consider all graphs in Figure 4. First, we consider the graph F_1 which has the vertex set $V(F_1) = \{v_1, v_2, \ldots, v_6\}$ and the edge set $E(F_1) = \{v_iv_{i+1} \mid i = 1, 2, \ldots, 5\} \cup \{v_1v_4, v_1v_6, v_3v_6\}$. We prove that the graph F_1 is the only graph of order 6 in $\mathfrak{R}(2K_2, C_4)$ by the following theorem.

Theorem 2.6. *The only connected graph of order 6 in $\mathfrak{R}(2K_2, C_4)$ is F_1.*

Proof. First, we prove that $F_1 \in \mathfrak{R}(2K_2, C_4)$. We can verify easily that for every $i \in [1, 6]$, $F_1 - v_i$ contains a C_4. Next, consider two cycles of length 4, namely $\mathcal{C}_1 = (v_1v_2v_3v_4)$, $\mathcal{C}_3 = (v_3v_4v_5v_6)$ and an edge v_1v_6 in F_1. If an edge e of the cycle $\mathcal{C}_1$ is deleted from F, then we obtain $(F_1 - e) - v_6 \not\supseteq C_4$. Moreover, $(F_1 - e) - v_1 \not\supseteq C_4$ if an edge e on the cycle $\mathcal{C}_3$ is deleted from F. Finally, $(F_1 - v_1v_6) - v_3 \not\supseteq C_4$. So, $F_1 \in \mathfrak{R}(2K_2, C_4)$.
Now, we prove that the connected graph of order 6 in $\mathfrak{R}(2K_2, C_4)$ is F_1. Let F be a connected graph having the vertex set $V(F) = \{v_1, v_2, \ldots, v_6\}$. Suppose that $F \in \mathfrak{R}(2K_2, C_4)$ but $F \neq F_1$. By Lemma 2.5(i), F contains A_2. By the minimality of F, $F \not\supseteq 2C_4$ and $F \not\supseteq K_5 - e$. By Theorem 1.1(i), $F - v_3$ must contain a C_4. Since $F \neq F_1$, the edges $v_1v_6, v_2v_5 \notin E(F)$. So, there must be the edges v_2v_4 and v_1v_5 in F (the graph A_{21}) or edges v_2v_6 and v_4v_6 in F (the graph A_{22}) (see Figure 5). Next, if F contains either A_{21} or A_{22}, then $F - v_4$ must contain a C_4, by Theorem 1.1(i). This cycle is formed by involving an edge v_3v_5 in F when $F \supseteq A_{21}$ (see the graph A_{2a}) and an edge v_1v_3 in F when $F \supseteq A_{22}$ (see the graph A_{2b}). Otherwise F contains F_1.
Furthermore, by Theorem 1.1(ii), we must form a C_4 in $F - E(K_3)$ for some $K_3 = (v_1v_4v_5)$ when F contains A_{2a} or $K_3 = (v_3v_4v_6)$ when F contains A_{2b}. As a consequence, F will contain F_1, a contradiction. $\square$
Next, we will find all Ramsey $(2K_2, C_4)-$minimal graphs of order 7. We consider the graphs F_2, F_3, F_4 and F_5 as depicted in Figure 4. We will prove that F_2, F_3, F_4 and F_5

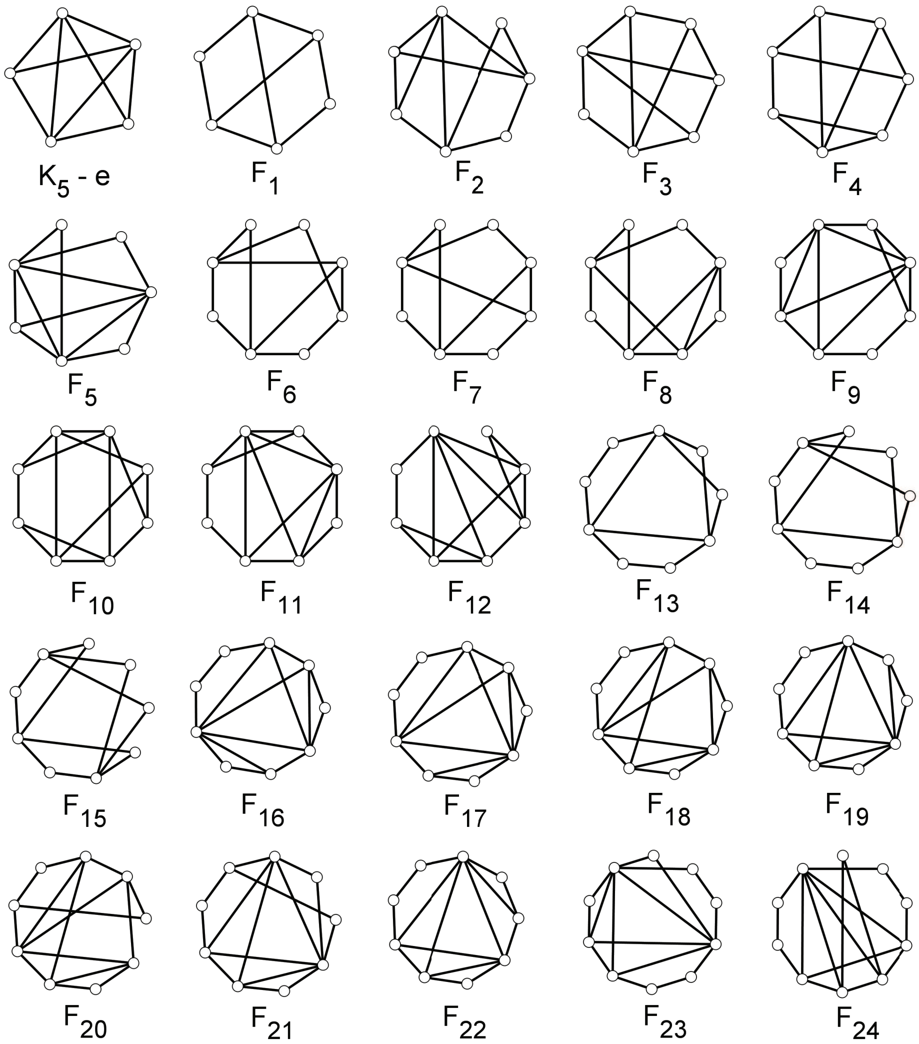

Figure 4: All connected Ramsey $(2K_2, C_4)$−minimal graphs.

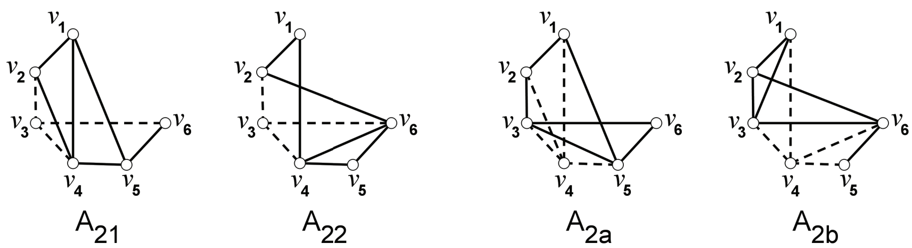

Figure 5: The process of forming a C_4 when F contains A_2.

are the only Ramsey $(2K_2, C_4)-$minimal graphs of order 7 as follows.

Theorem 2.7. *The only connected graphs of order 7 in* $\mathfrak{R}(2K_2, C_4)$ *are* F_2, F_3, F_4 *and* F_5.

Proof. We can show easily that F_2, F_3, F_4, F_5 satisfy Theorem 1.1 (i) and (ii). To show the minimality of F_2, consider the first line of Figure 6. Let e be any edge in F_2. Then, e must be in a (bold) C_4. Now, consider $F - e$. Then, color all dash-line edges by red and the remaining edges by blue. Thus, this coloring is a $(2K_2, C_4)-$coloring of $F_2 - e$. Therefore, F_2 is a Ramsey $(2K_2, C_4)-$minimal graph. Similarly, by considering Figure 6, we can show the minimality of F_3, F_4 and F_5.

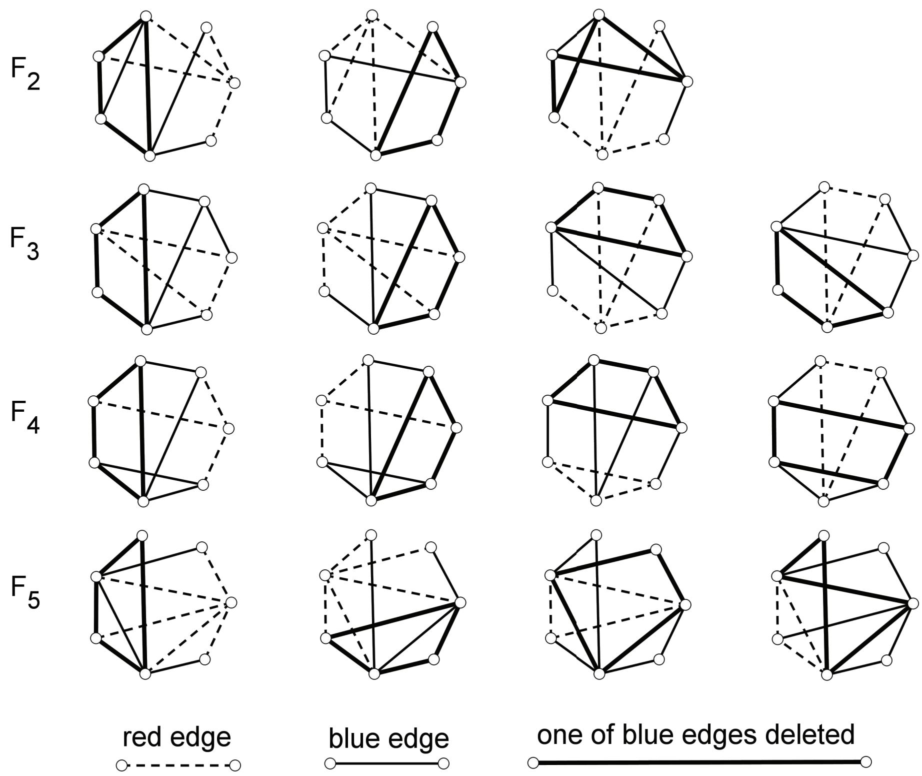

Figure 6: Some red-blue coloring of F_2, F_3, F_4 and F_5 contain a red K_2 and a blue C_4.

Now, we prove that the connected graphs of order 7 in $\mathfrak{R}(2K_2, C_4)$ are F_2, F_3, F_4 and F_5. Let $F \in \mathfrak{R}(2K_2, C_4)$ be a connected graph having the vertex set $V(F) = \{v_1, v_2, \ldots, v_7\}$. By the minimality, F does not contain $K_5 - e$ or F_1. By Lemma 2.5(ii), F contains A_1 or A_5.

First, we observe that F contains A_1.There must be a C_4 in $F - v_4$ by Theorem 1.1(i). Up to isomorphism, there are five possibilities to form this C_4 by involving some edges other than $e \in E(A_1)$, that is $C_4 = (v_1v_3v_2v_6)$ (the graph B_1), $C_4 = (v_1v_2v_3v_7)$ (the graph B_2), $C_4 = (v_1v_3v_2v_7)$ (the graph B_3), $C_4 = (v_1v_2v_6v_7)$ (the graph B_4) or $C_4 = (v_1v_3v_5v_7)$ (the graph B_5), as depicted in Figure 7.

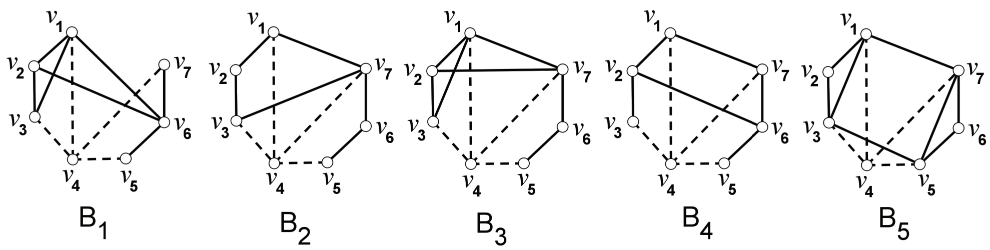

Figure 7: The possibilities of forming C_4 in $F - v_4$ when $F \supseteq A_1$.

Now, consider that F contains B_i for every $i \in [1, 5]$. If $F \supseteq B_1$, then $F = B_1$ since B_1 is isomorphic to F_2. Consider a triangle $K_3 = (v_1v_4v_7)$. If $F \supseteq B_i$ for $i \in [2, 5]$, then there must be a C_4 in $F - E(K_3)$, by Theorem 1.1(ii). But this C_4 makes F not minimal, when $F \supseteq B_i$ for $i = 2, 3, 5$. Next, if $F \supseteq B_4$, this cycle is formed by involving an edge in F, namely v_2v_5 or v_3v_5. We obtain the graph F with either the edge set $E(F) = E(B_4) \cup \{v_2v_5\}$ or $E(F) = E(B_4) \cup \{v_3v_5\}$ which is isomorphic to either F_3 or F_4, respectively. So, F_2, F_3 and F_4 are all Ramsey $(2K_2, C_4)-$minimal graphs of order 7 containing A_1.

Now, we will find all Ramsey $(2K_2, C_4)-$minimal graphs of order 7 containing A_5. By Theorem 1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_3v_4v_6)$. Since F does not contain A_1, one of the edges $v_1v_5, v_1v_6, v_1v_7, v_2v_5, v_4v_7, v_5v_7$ does not involve in F. By the minimality, this cycle is only formed by involving an edge v_2v_6 in F. We obtain the graph F with $E(F) = E(A_5) \cup \{v_2v_6\}$ which is isomorphic to F_5. □

Next, we will determine all graphs of order 8 in $\mathfrak{R}(2K_2, C_4)$. We consider the graphs F_6, F_7, F_8, F_9, F_{10}, F_{11} and F_{12} as depicted in Figure 4. The following theorem prove that F_6, F_7, F_8, F_9, F_{10}, F_{11} and F_{12} are the only Ramsey $(2K_2, C_4)-$minimal graphs of order 8.

Theorem 2.8. *The only connected graphs of order 8 in* $\mathfrak{R}(2K_2, C_4)$ *are* F_6, F_7, F_8, F_9, F_{10}, F_{11} *and* F_{12}.

Proof. We can easily notice that for $i \in [6, 12]$, the graph F_i satisfy Theorem 1.1 (i) and (ii). The proof of the minimality of $F \in \{F_6, F_7, \ldots, F_{12}\}$ is done in the same fashion as in Theorem 2.7. In Figure 8, for any edge $e \in E(F)$, we construct a red-blue coloring such that there exist a red K_2 and exactly a blue C_4. Thus for every edge $e \in E(F)$, we obtain a $(2K_2, C_4)-$coloring of $F - e$.

Now, we prove that the connected graphs of order 8 in $\mathfrak{R}(2K_2, C_4)$ are F_6, F_7, F_8, F_9, F_{10}, F_{11} and F_{12}. Let $F \in \mathfrak{R}(2K_2, C_4)$ be a connected graph with the vertex set $V(F) =$

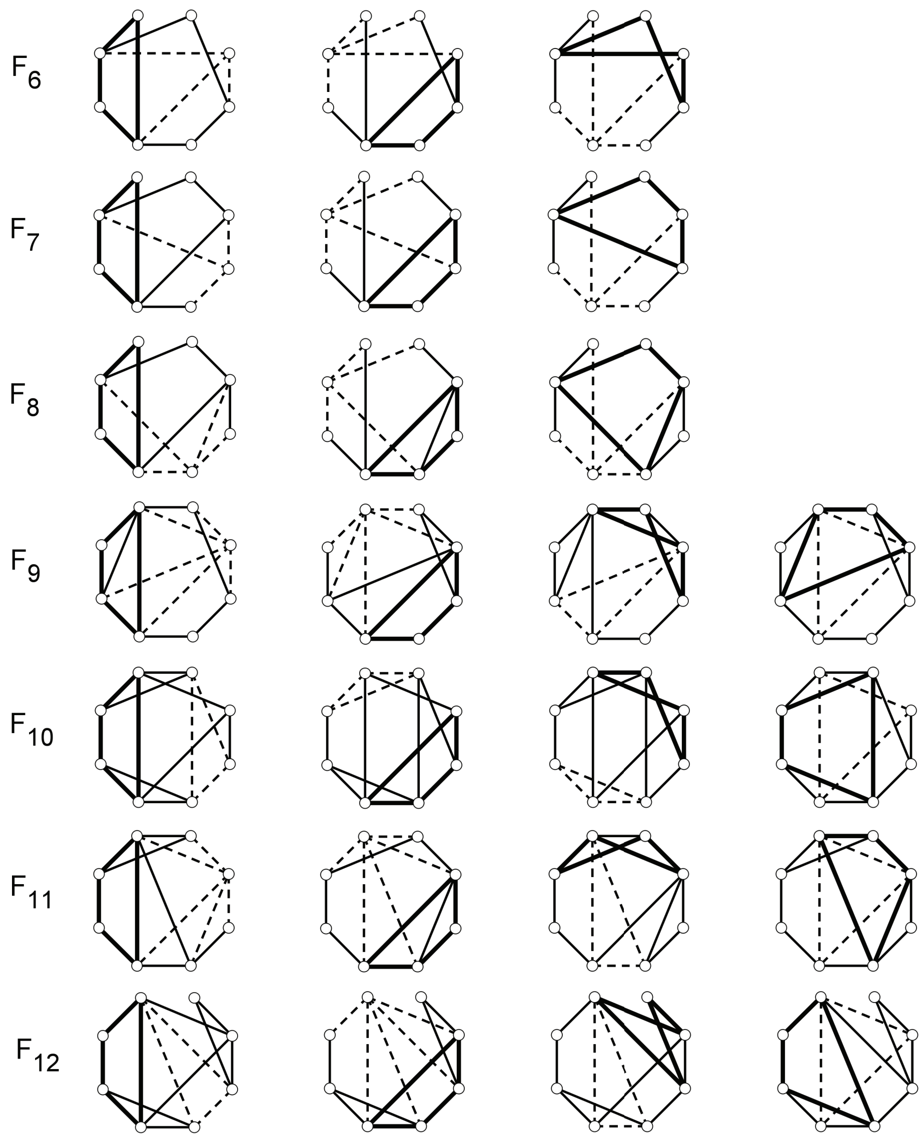

Figure 8: Some red-blue coloring of F of order 8 contain a red K_2 and a blue C_4.

$\{v_1, v_2, \dots, v_8\}$. By the minimality, F does not contain $2C_4$, $K_5 - e$, F_1, F_2, F_3, F_4 or F_5. By Lemma 2.5(iii), F contains A_1. There must be a C_4 in $F - v_4$ by Theorem 1.1(i). This cycle is formed by involving three vertices in A_1 and a vertex, say v_8. Up to isomorphism, there are six possibilities, say $C_4 = (v_1v_2v_8v_6)$ (the graph D_1), $C_4 = (v_1v_2v_6v_8)$ (the graph D_2), $C_4 = (v_1v_3v_8v_6)$ (the graph D_3), $C_4 = (v_1v_2v_8v_7)$ (the graph D_4), $C_4 = (v_1v_7v_2v_8)$ (the graph D_5) or $C_4 = (v_1v_3v_7v_8)$ (the graph D_6), as illustrated in Figure 2.

Now, we consider that F contains D_i for every $i \in [1, 6]$. The graphs D_1, D_2 and D_3 are isomorphic to F_6, F_7 and F_8, respectively. Furthermore, if F contains either D_4 or

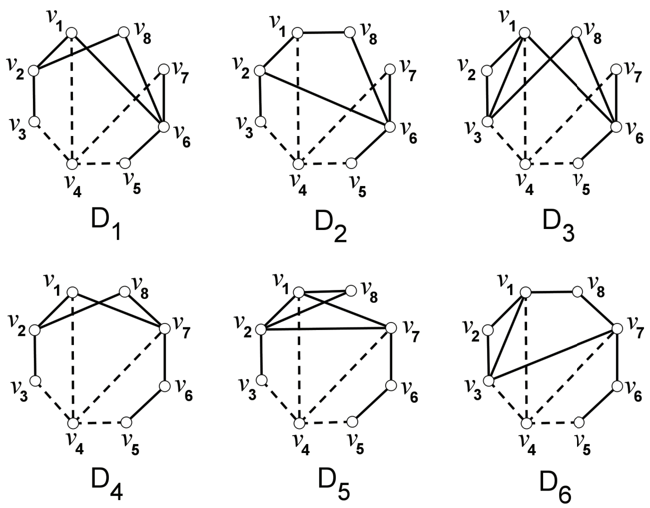

Figure 9: The possibilities of forming C_4 in $F - v_4$ when $F \supseteq A_1$ by involving v_8 in F.

D_5, then there exists a triangle $K_3 = (v_1v_4v_7)$ which yields that $F - E(K_3)$ does not contain a C_4. If $F \supseteq D_4$, then this cycle is formed by involving three edges in F, say either $C_4 = (v_1v_5v_7v_8)$ or $C_4 = (v_3v_5v_6v_8)$. We obtain the graph F with either the edge set $E(F) = E(D_4) \cup \{v_1v_5, v_1v_8, v_5v_7\}$ or $E(F) = E(D_4) \cup \{v_3v_5, v_3v_8, v_6v_8\}$ which is isomorphic to either F_9 or F_{10}, respectively. Next, if $F \supseteq D_5$, then this cycle is formed by involving two edges in F, say either $v_2v_4, v_3v_5 \in E(F)$ or $v_3v_5, v_3v_7 \in E(F)$. We obtain the graph F with either the edge set $E(F) = E(D_5) \cup \{v_2v_4, v_3v_5\}$ or $E(F) = E(D_5) \cup \{v_3v_5, v_3v_7\}$ which is isomorphic to either F_9 or F_{12}, respectively. Moreover, we consider F containing D_6. There exists a triangle $K_3 = (v_3v_4v_7)$ such that $F - E(K_3)$ does not contain a C_4. By the minimality, this C_4 is formed by involving two edges in F, say $v_1v_7, v_6v_8 \in E(F)$. We obtain the graph F with the edge set $E(F) = E(D_6) \cup \{v_1v_7, v_6v_8\}$ which is isomorphic to F_{11}. $\square$

In the next result, we will find all graphs of order 9 which belong to $\mathfrak{R}(2K_2, C_4)$. We consider the graph of order 9 in Figure 4, namely $F_{13}, F_{14}, \ldots, F_{22}$.

Theorem 2.9. *The only connected graphs of order 9 in* $\mathfrak{R}(2K_2, C_4)$ *are* $F_{13}, F_{14}, \ldots, F_{22}$.

Proof. We can easily notice that $F_{13}, F_{14}, \ldots, F_{22}$ satisfy Theorem 1.1 (i) and (ii). The proof of the minimality is done in the same fashion as in Theorem 2.7. In Figure 10 and 11, for every edge e in $F \in \{F_{13}, F_{14}, \ldots, F_{22}\}$, we construct a red-blue coloring of F such that there exists a red K_2 and exactly a blue C_4. Thus, for every $e \in E(F)$ we obtain a $(2K_2, C_4)-$coloring of $F - e$.

Now, we prove that the connected graphs of order 9 in $\mathfrak{R}(2K_2, C_4)$ are $F_{13}, F_{14}, \ldots, F_{22}$. Let $F \in \mathfrak{R}(2K_2, C_4)$ be a connected graph having the vertex set $V(F) = \{v_1, v_2, \ldots, v_9\}$.

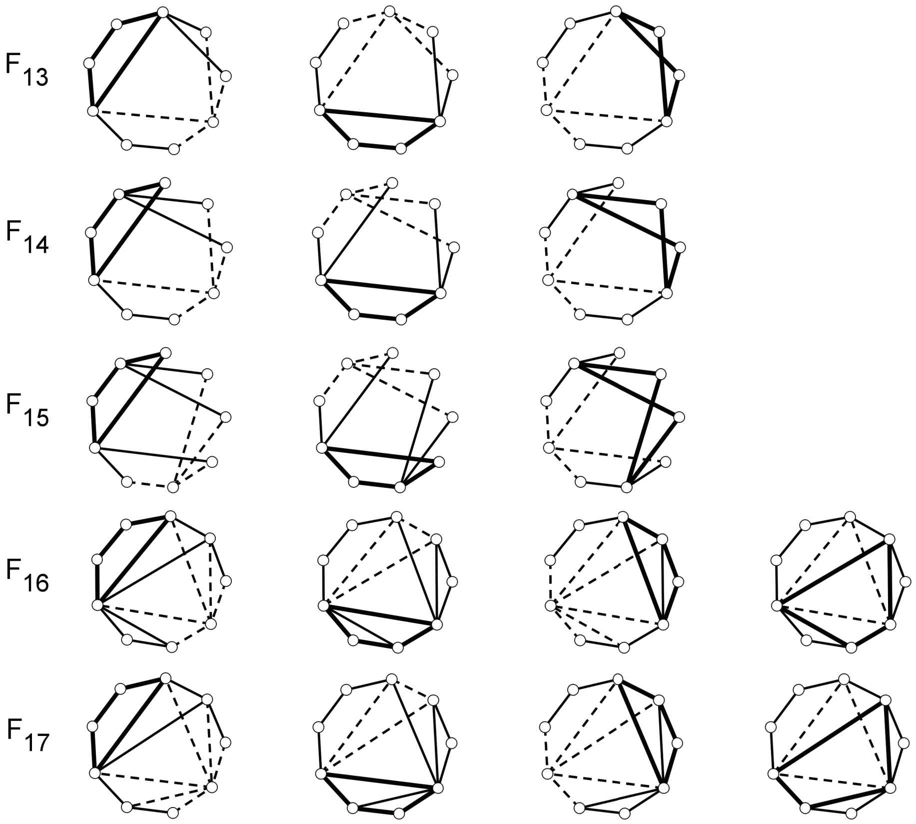

Figure 10: Some red-blue coloring of F_{13}, F_{14}, F_{15}, F_{16} and F_{17} contain a red K_2 and a blue C_4.

By Lemma 2.5(iii), F contains A_1. By Theorem 1.1(i), there must be a C_4 in $F - v_4$. Then, this C_4 must contain at least one vertex of v_8 and v_9. So, (up to isomorphism) there are seven possibilities to form this cycle, that is $C_4 = (v_1v_8v_7v_9)$ (the graph E_1), $C_4 = (v_1v_8v_6v_9)$ (the graph E_2), $C_4 = (v_2v_8v_6v_9)$ (the graph E_3), $C_4 = (v_1v_7v_8v_9)$ (the graph E_4), $C_4 = (v_1v_3v_7v_8)$ (the graph E_5), $C_4 = (v_1v_2v_8v_7)$ (the graph E_6) or $C_4 = (v_1v_7v_2v_8)$ (the graph E_7), as illustrated in Figure 12.
Now, consider that F contains E_i for every $i \in [1, 7]$. The graphs E_1, E_2 and E_3 are isomorphic to F_{13}, F_{14} and F_{15}, respectively. If $F \supseteq E_4$, then by Theorem 1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_1v_4v_7)$. By the minimality property and up to isomorphism, there are three possibilities to form this cycle, that is $C_4 = (v_1v_2v_4v_8)$, $C_4 = (v_1v_3v_4v_8)$ or $C_4 = (v_1v_3v_7v_8)$. We obtain the graph F having the edge set $E(F) = E(E_4) \cup \{v_1v_8, v_2v_4, v_4v_8\}$, $E(F) = E(E_4) \cup \{v_1v_3, v_1v_8, v_4v_8\}$ or $E(F) = E(E_4) \cup \{v_1v_3, v_1v_8, v_3v_7\}$ which is isomorphic to F_{16}, F_{17} or F_{19}, respectively. If $F \supseteq E_5$, then by Theorem 1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_3v_4v_7)$. By the minimality property, there are five possibilities to form this C_4, that is $C_4 = (v_1v_4v_8v_9)$, $C_4 = (v_1v_7v_8v_9)$, $C_4 =$

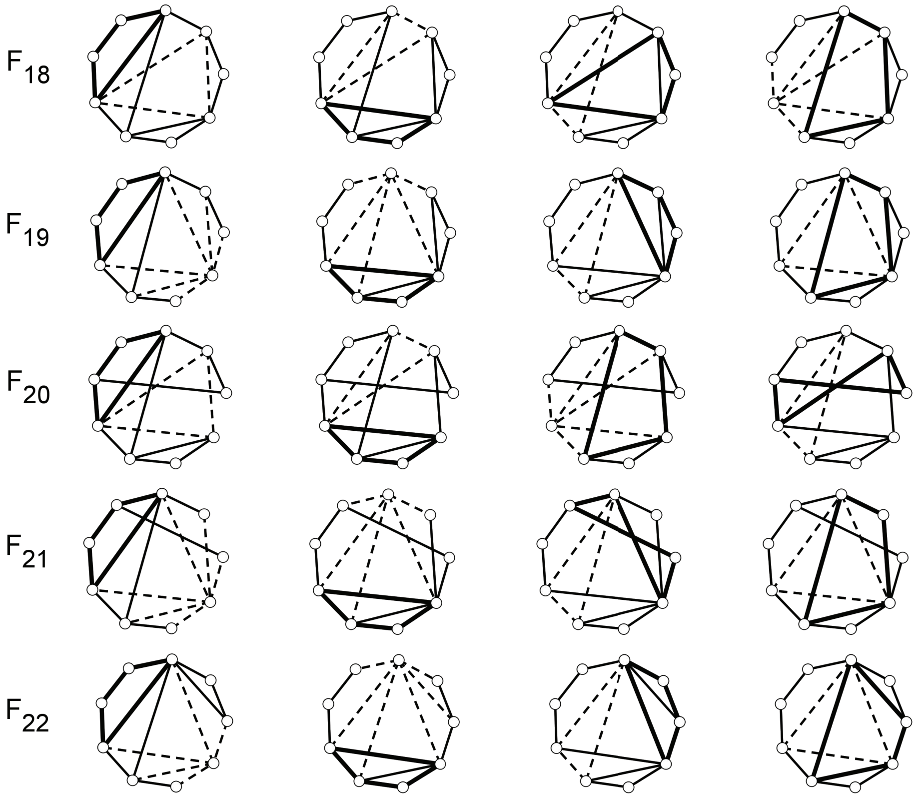

Figure 11: Some red-blue coloring of F_{18}, F_{19}, F_{20}, F_{21} and F_{22} contain a red K_2 and a blue C_4.

$(v_1v_7v_9v_8)$, $C_4 = (v_4v_8v_9v_5)$ or $C_4 = (v_1v_7v_6v_9)$. We obtain the graph F having the edge set $E(F) = E(E_5) \cup \{v_1v_9, v_4v_8, v_8v_9\}$, $E(F) = E(E_5) \cup \{v_1v_7, v_1v_9, v_8v_9\}$, $E(F) = E(E_5) \cup \{v_1v_7, v_7v_9, v_8v_9\}$, $E(F) = E(E_5) \cup \{v_4v_8, v_5v_9, v_8v_9\}$ or $E(F) = E(E_5) \cup \{v_1v_7, v_1v_9, v_6v_9\}$ which is isomorphic to F_{18}, F_{19}, F_{22}, F_{20} or F_{21}, respectively. Next, if F contains either E_6 or E_7, then by Theorem 1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_1v_4v_7)$. If $F \supseteq E_6$, then this C_4 can be formed by involving four edges in F, that is either $C_4 = (v_1v_5v_9v_6)$ or $C_4 = (v_1v_5v_7v_9)$. We obtain the graph F having either the edge set $E(F) = E(E_6) \cup \{v_1v_5, v_1v_6, v_5v_9, v_6v_9\}$ or $E(F) = E(E_6) \cup \{v_1v_5, v_1v_9, v_5v_7, v_7v_9\}$ which is isomorphic to either F_{20} or F_{21}, respectively. If $F \supseteq E_7$, then this C_4 is formed by involving three edges in F, that is $C_4 = (v_3v_7v_6v_9)$. We obtain the graph F with the edge set $E(F) = E(E_7) \cup \{v_3v_7, v_3v_9, v_6v_9\}$ which is isomorphic to F_{20}. Hence, the connected graphs of order 9 in $\mathfrak{R}(2K_2, C_4)$ are $F_{13}, F_{14}, \ldots, F_{22}$. □

Finally, we will find all graphs of order 10 belonging to $\mathfrak{R}(2K_2, C_4)$. We consider the graphs F_{23} and F_{24} in Figure 4.

Theorem 2.10. *The only connected graphs of order 10 in $\mathfrak{R}(2K_2, C_4)$ are F_{23} and F_{24}.*

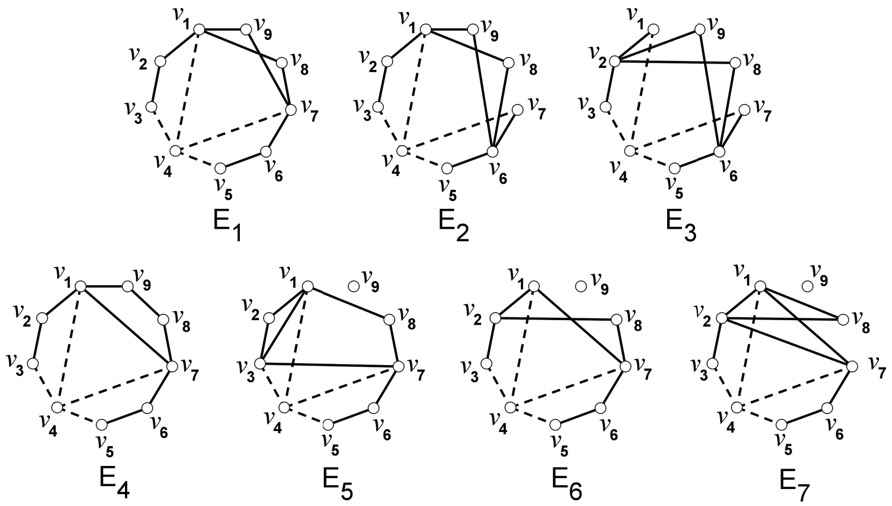

Figure 12: The possibilities of forming a C_4 in $F - v_4$ when $F \supseteq A_4$ by involving two vertices v_8 and v_9 in F.

Proof. We can notice easily that F_{23} and F_{24} satisfy Theorem 1.1 (i) and (ii). The proof of the minimality of F_{23} and F_{24} is done in the same fashion as in Theorem 2.7. In Figure 13, for every edge e in $F \in \{F_{23}, F_{24}\}$ we construct a red-blue coloring of F such that there exists a red K_2 and exactly a blue C_4. Thus, we obtain a $(2K_2, C_4)-$coloring of $F - e$.

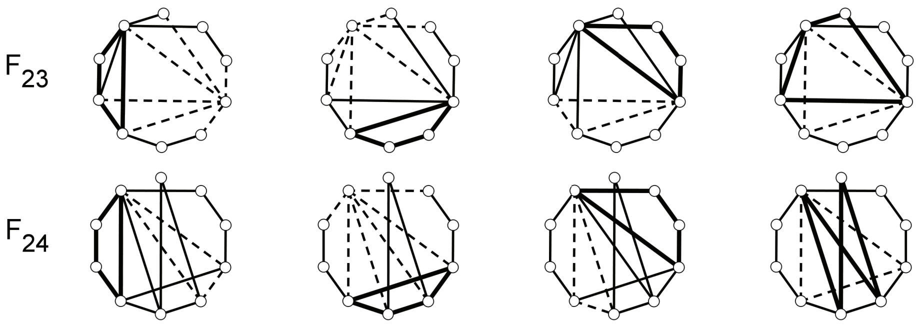

Figure 13: The red-blue coloring of F of order 10 contains a red K_2 and a blue C_4.

Now, we prove that the connected graphs of order 10 in $\mathfrak{R}(2K_2, C_4)$ are F_{23} and F_{24}. Let F be a connected graph in $\mathfrak{R}(2K_2, C_4)$ where $V(F) = \{v_1, v_2, \ldots, v_{10}\}$. By Lemma 2.5(iii), F contain A_1. By Theorem 1.1(i), F must contain a C_4 in $F - v_4$. Since F does not contain $2C_4$, this C_4 must contain a vertex in cycle $\mathcal{C}_1 = (v_1v_2v_3v_4)$ and a vertex in cycle $\mathcal{C}_2 =(v_4v_5v_6v_7)$. So, this cycle is formed by involving two vertices, say v_7 and v_8 in F, that is $C_4 = (v_1v_7v_8v_9)$ (the graph E_4 in Figure 12). Next, if F contains E_4, then by Theorem

1.1(ii), $F - E(K_3)$ must contain a C_4 for some $K_3 = (v_1v_4v_7)$. By the minimality and up to isomorphism, this C_4 is formed by involving three edges in E_4 and a vertex v_{10} in F, say either $C_4 = (v_1v_3v_7v_{10})$ or $C_4 = (v_1v_5v_{10}v_6)$. We obtain the graph F having either the edge set $E(F) = E(E_4) \cup \{v_1v_3, v_1v_{10}, v_3v_7, v_7v_{10}\}$ or $E(F) = E(E_4) \cup \{v_1v_5, v_1v_6, v_5v_{10}, v_6v_{10}\}$ which is isomorphic to either F_{23} and F_{24}, respectively. $\square$

Lemma 2.11. *The order of a connected graph F in $\mathfrak{R}(2K_2, C_4)$ is at most 10.*

Proof. Let F be a connected graph in $\mathfrak{R}(2K_2, C_4)$ and $|V(F)| = 11$ where $V(F) = \{v_1, v_2, \ldots, v_{11}\}$. By Lemma 2.5(iii), F contains A_1. By Theorem 1.1 and the minimality of F, there is only one possibility to form this C_4, say $(v_1v_7v_8v_9)$ (the graph E_4 in Figure 12). Next, by Theorem 1.1(iii), there must be a C_4 in $F - E(K_3)$ for some $K_3 = (v_1v_4v_7)$. This C_4 must contain both vertices v_{10} and v_{11}. Since F does not contain $2C_4$, at least another vertex must be contained in two different cycles of length 4 in F, that is v_1, v_4 or v_7. Without loss of generality, we may assume v_1 is contained in $C_4 \subseteq F - E(v_1v_4v_7)$. So, up to isomorphism, the other vertex is either v_4 or v_5. But the resulted graph contains either F_{13} or F_{14}. It implies that F is not minimal. $\square$

By Theorems 2.2, 2.4, 2.6, 2.7, 2.8, 2.9, 2.10 and Lemma 2.11, we have the following theorem.

Theorem 2.12. $\mathfrak{R}(2K_2, C_4) = \{2C_4, K_5 - e\} \cup \{F_i \mid i \in [1, 24]\}$. $\square$

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