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The Mean Labeling of Some Crowns

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ABSTRACT

Mean labelings are a type of additive vertex labeling. This labeling assigns non-negative integers to the vertices of a graph in such a way that all edge-weights are different, where the weight of an edge is defined as the mean of the end-vertex labels rounded up to the nearest integer. In this paper we focus on mean labelings of some graphs that are the result of the corona operation. In particular we prove the existence of mean labelings for graphs of the form $G \odot mK_1$ in the cases where G is an even cycle or G is an α-mean graph of odd size and the cardinalities of its stable sets differ by at most one unit. Under these conditions, we prove that $G \odot P_2$ and $G \odot P_3$ are also mean graphs, and that the class of α-graphs is equivalent to the class of α-mean graphs.

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1 Introduction

The concept of mean labeling was introduced in 2003 by Somasundaram and Ponraj [7]. A graph G of order m and size n is said to be a *mean graph* if there exists an injective function $f : V(G) \to \{0, 1, 2, \dots, n\}$ such that when each edge uv has assigned the *weight* $\lceil \frac{f(u)+f(v)}{2} \rceil$, the resulting weights are distinct. Only a few families of trees are known to be mean graphs. The above authors proved the following are mean graphs: the path P_n for every n, the star $S_n = K_{1,n}$ if and only if $n \leq 3$, bistars $B_{m,n}$ where $m > n$ if and only if $m < n + 2$, the subdivision graph of the star $K_{1,n}$ if and only if $n < 4$.
Recently Barrientos and Krop [1] proved that mean graphs must satisfy the following necessary conditions:

Proposition 1. *If G is a mean graph of order m and size n, then* (a) $n + 1 \geq m$, (b) $\Delta(G) \leq \frac{n+3}{2}$, *when n is odd and $\Delta(G) \leq \frac{n+2}{2}$ when n is even.*

We refer to part (b) as the *degree condition.* We have conjectured that all trees that satisfy the degree condition are mean graphs. Throughout the rest of this paper, we consider only graphs that satisfy this degree condition.
A more restrictive variation of mean labeling was also introduced in [1]. Let G be a mean bipartite graph. A mean labeling f of G is called an *α-mean labeling* if for every $uv \in E(G)$, $f(u)$ and $f(v)$ have different parity. We proved that the complemetary labeling, $\bar{f}$, of an α-mean labeling f, is an α-mean labeling. This is not the case with regular mean labelings.
In this paper we study mean labelings of graphs of the form $G \odot H$, where G is an α-graph of size n and order $n + 1$ such that the absolute difference of the cardinalities of its bipartite sets is at most 1, and $\odot$ denotes the corona product. The corona of two graphs was introduced in 1970 by Frucht and Harary [4]. Given two graphs G and H, their *corona product* is the graph with vertex set

$$V(G \odot H) = V(G) \cup \left(\bigcup_{i \in V(G)} V(H_i) \right)$$

and edge set

$$E(G \odot H) = E(G) \cup \left(\bigcup_{i \in V(G)} E(H_i) \right) \cup \{(i, u_i) : i \in V(G), u_i \in V(H_i)\}$$

In Section 2 we analyze the relationship between α-labelings and α-mean labelings. In Section 3 we present mean labelings of graphs obtained using the corona product. In

Section 4 we study coronas of the form $C_n \odot mK_1$, where n is an even number. In Section 5 we consider pseudo-mean labelings of some of the graphs considered in the previous sections. We present there a pseudo-mean labeling of the complete graph that uses Fibonacci numbers. Finally we close the paper with some questions that arose during this project.
For more information about graph labelings, the reader is referred to Gallian's survey [5]. Graphs considered in this paper are finite, with no loops or multiple edges. We follow the notation used in [3] and [5].

2 Alpha Versus Alpha-Mean

Recall that a graph G of order m and size n is a *graceful* graph if there exists an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that when the edge uv is assigned the *weight* $|f(u) - f(v)|$, the set of all weights is $\{1, 2, \ldots, n\}$. In addition, if there is an integer λ such that $f(u) \le \lambda < f(v)$ or $f(v) \le \lambda < f(u)$ for every edge uv in G then f is said to be an *α-labeling* and G is an *α-graph*. The number λ is named the *boundary value* of f.
It is well-known that if G is an α-graph, then G is bipartite. If G is a bipartite graph and $\{A, B\}$ is the bipartition of $V(G)$, we assume that with out loss of generality $|A| \ge |B|$. We refer to A and B as the *stable sets* of G.
Let $\mathcal{M}_n$ denote the family of all mean graphs of order $n+1$ and size n, and $\mathcal{A}_n \subset \mathcal{M}_n$ be the subfamily of all α-mean graphs such that the cardinalities of their stable sets differ by at most one unit. Notice that mean trees are members of $\mathcal{M}_n$. In [1] it was proved that if T is an α-tree of size n such that the cardinalities of its stable sets differ by at most one unit, then $T \in \mathcal{A}_n$. Now we generalize this result to include all α-graphs of order $n+1$ and size n whose stable sets satisfy the same condition.

Theorem 2. *$G \in \mathcal{A}_n$ if and only if G is an α-graph.*

Proof. Let G be an α-graph. Let f be an α-labeling of G with boundary value λ such that $f(v) = \lambda$ for some $v \in A$. Let g be a new labeling of G defined as:

$$g(v) = \begin{cases} 2f(v), & \text{if } v \in A; \\ (2n+1) - 2f(v) + 1, & \text{if } v \in B. \end{cases}$$

Hence, the labels assigned by g on the vertices of A are $0, 2, \ldots, 2\lambda$, and on the vertices of B are $1, 3, \ldots, 2(n-\lambda) - 1$. When n is even, $\lambda = \frac{n}{2}$; thus the labels assigned on A and B are $0, 2, \ldots, n$ and $1, 3, \ldots, n-1$ respectively. When n is odd, $\lambda = \frac{n-1}{2}$; then the labels are $0, 2, \ldots, n-1$ and $1, 3, \ldots, n$, respectively. Let $e = uv \in E(G)$ such that $u \in A$ and $v \in B$. Thus the weight of uv is given by

$$\begin{aligned}\left\lceil \frac{g(u)+g(v)}{2} \right\rceil &= \left\lceil \frac{2f(u)+(2n+1)-2f(v)+1}{2} \right\rceil \\ &= \left\lceil \frac{2n-2(f(v)-f(u))+2}{2} \right\rceil \\ &= n+1-(f(v)-f(u)).\end{aligned}$$

Since f is an α-labeling of G, $\{f(v)-f(u) : uv \in E(G)\} = \{1, 2, \ldots, n\}$, therefore the set of weights induced by g on the edges of G is $\{1, 2, \ldots, n\}$ and G is a mean graph. Given that g assigns even numbers to the elements of A and odd numbers to the elements of B, g is an α-mean labeling of G.

Suppose now that $G \in \mathcal{A}_n$. Let g be an α-mean labeling of G, then $g(w)$ is even when $w \in A$ and is odd when $w \in B$. Let f be a new labeling of G defined as:

$$f(w) = \begin{cases} g(w)/2, & \text{if } w \in A; \\ (2n+1-g(w))/2, & \text{if } w \in B. \end{cases}$$

We claim that f is an α-labeling of G. In fact, the labels assigned by f on the vertices of A are $0, 1, \ldots, \lceil \frac{n-1}{2} \rceil$, and on the vertices of B are $\lceil \frac{n+1}{2} \rceil, \lceil \frac{n+3}{2} \rceil, \ldots, n$. So, f assigns, injectively, the labels $0, 1, \ldots, n$.

For any $k \in \{1, 2, \ldots, n\}$, there exists an edge $e = uv \in G$, with $u \in A$ and $v \in B$, such that $\lceil \frac{g(u)+g(v)}{2} \rceil = k$. If $g(u) = 2a$ and $g(v) = 2b+1$, for certain non-negative integers a and b, then $a+b+1 = k$. Thus,

$$|f(u)-f(v)| = |a+b-n| = |a+b+1-(n+1)| = n+1-k.$$

Since $1 \le k \le n$, we have $1 \le n+1-k \le n$. Therefore f is an α-labeling of G, whose boundary value is $\lambda = \lceil \frac{n-1}{2} \rceil$. □

3 Mean Coronas of α-mean Graphs

In [2], Bailey and Barrientos proved that if G is a graph such that the difference of the cardinalities of its stable sets is at most one, then $G \odot mK_1$ is a mean graph when G is a mean graph and α-mean when G is an α-graph. Here we extend this result proving that for $m \in \{2, 3, 4\}$, the corona $G \odot mK_1$ is a mean graph when G is an α-mean graph of size n and order $n+1$ such that the cardinalities of its stable sets differs by at most one unit. Recall that when G has size n, $G \odot mK_1$ has size $m(n+1)+n$.

Proposition 3. *If $G \in \mathcal{A}_n$, then $G \odot mK_1$ is a mean graph for every $m \in \{2, 3, 4\}$.*

Proof. Let $G \in \mathcal{A}_n$. We split the proof into cases depending of the value of m.
Case I: When $m = 2$. Let f be an α-mean labeling of G. Consider now the following labeling of the vertices of G:

$$g(v) = \begin{cases} 3f(v) + 2, & \text{if } f(v) \text{ is even;} \\ 3f(v) + 1, & \text{if } f(v) \text{ is odd.} \end{cases}$$

This labeling assigns all even numbers in the interval $[0, 3n+2]$ that are not congruent to 0 (mod 6). That is, if $f(v)$ is even, $g(v) = 6k+2$ for some $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$; if $f(v)$ is odd, $g(v) = 6k+4$ for some $k \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. The weights induced by g on the edges of G are of the form $6k+3$, when the weight of the edge under f is odd, and f is of the form $6k$, when the weight of the edge under f is even, for some $k \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ or $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, respectively.
Now we attach to each vertex of G two pendant vertices to obtain $G \odot 2K_1$. If $f(v)$ is even, the pendant vertices adjacent to v have labels $g(v) - 2 = 6k$ and $g(v) - 1 = 6k + 1$, respectively. In this way the weights of these edges are $6k+1$ and $6k+2$, for $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. If $f(v)$ is odd, the pendant vertices adjacent to v have labels $g(v) - 1 = 6k + 3$ and $g(v) + 1 = 6k + 5$, respectively. So the weights of these edges are $6k + 4$ and $6k + 5$ for $k \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.
Therefore, g is a mean labeling of $G \odot 2K_1$.
Case II: When $m = 3$. In this case, $g(v) = 4f(v) + 2$. The three pendant vertices, attached to every vertex v of G, have labels $g(v) - 2, g(v) - 1$, and $g(v) + 1$.
Case III: When $m = 4$. Now, $g(v) = 5f(v) + 2$ if $f(v) \neq n$, and $g(v) = 5f(v) + 3$ if $f(v) = n$. The four pendant vertices, attached to every vertex v of G, have labels $g(v) - 3, g(v) - 1, g(v) + 1$, and $g(v) + 3$ when $f(v) \neq 0, n$. If $f(v) = 0$, the labels are $g(v) - 2, g(v) - 1, g(v) + 1$, and $g(v) + 3$. If $f(v) = n$, the labels are $g(v) - 4, g(v) - 2, g(v) - 1$, and $g(v) + 1$. □

In Figure 1 we show an example of this labeling corresponding to Case I.
Recall that all α-mean trees belong to $\mathcal{A}_n$. In the next theorem we extend the previous result in the case where G has odd size.

Theorem 4. *Let n be a positive odd integer. If $G \in \mathcal{A}_n$, then the corona $G \odot mK_1$ is an α-mean graph for all $m \geq 1$.*

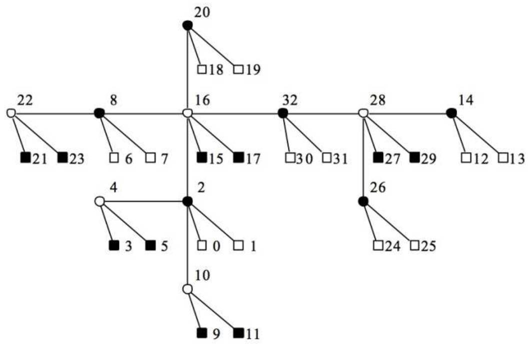

Figure 1: Mean labeling of $G \odot 2K_1$

Proof. Let $V(G) = \{v_0, v_1, \ldots, v_n\}$ where n is a positive odd number and f be an α-mean labeling of G. We assume that $f(v_i) = i$ for all $i \in \{0, 1, \ldots, n\}$. The m pendant vertices adjacent to v_i are denoted by $v_{i,j}$ for $1 \leq j \leq m$. Consider the following labeling of $G \odot mK_1$:

$$g(v_i) = \begin{cases} i(m+1), & \text{if } i \text{ is even;} \\ i(m+1) + m, & \text{if } i \text{ is odd;} \end{cases}$$

and

$$g(v_{i,j}) = \begin{cases} i(m+1) + 2j - 1, & \text{if } i \text{ is even;} \\ (i-1)(m+1) + 2j, & \text{if } i \text{ is odd.} \end{cases}$$

Clearly g assigns the labels $0, 1, \ldots, n(m+1) + m$. Over the vertices of G, g is just an amplification of f, thus the weights induced on G are $m+1, 2(m+1), \ldots, n(m+1)$. The edges of the form $v_i v_{i,j}$ have weights $i(m+1)+j$ for all $0 \leq i \leq n$. Since for every edge of $G \odot mK_1$ the labels of the end vertices have different parity, the labeling g is an α-mean labeling of $G \odot mK_1$. $\square$

The case where n is a positive even number is discussed in the last section.
In the rest of this section we study coronas of the form $G \odot P_m$, where $m \in \{2, 3\}$ and G is an α-graph such that the absolute difference between the cardinalities of its stable sets is at most one.

Proposition 5. *Let G be an α-graph such that the cardinalities of its stable sets differ by at most one. If G has size n and order $n+1$, then $G \odot P_2$ is a mean graph.*

Proof. Let G be an α-graph of size n and order $n+1$ such that $|A| - |B| \leq 1$, where $\{A, B\}$ is the bipartition of $V(G)$ and $|A| \geq |B|$. Suppose that f is an α-labeling of G with boundary value λ such that λ is assigned by f to a vertex of A.

First we label the vertices of G and later we label the vertices of each copy of P_2. Let $g : V(G \odot P_2) \to \{0, 1, \ldots, 4n + 3\}$ be defined on the vertices of G by

$$g(v) = \begin{cases} 8f(v) + 3, & \text{if } v \in A; \\ 8(n - f(v)) + 4, & \text{if } v \in B. \end{cases}$$

If $uv \in E(G)$, where $v \in A$ and $u \in B$, its weight under g is:

$$\begin{aligned} \left\lceil \frac{g(v) + g(u)}{2} \right\rceil &= \left\lceil \frac{8f(v) + 3 + 8(n - f(u)) + 4}{2} \right\rceil \\ &= \left\lceil \frac{8n + 7 + 8(f(v) - f(u))}{2} \right\rceil \\ &= \left\lceil \frac{8n + 7 - 8(f(u) - f(v))}{2} \right\rceil \\ &= 4(n + 1 - (f(u) - f(v))). \end{aligned}$$

Since $\{f(u) - f(v) : uv \in E(G)\} = \{1, 2, \ldots, n\}$, the weights induced by g on the edges of G form the set $W_G = \{4, 8, \ldots, 4n\}$.
Now we extend g to the vertices of the $n + 1$ copies of P_2. Let x and y be the vertices of P_2 adjacent to $v, v \in V(G)$.
If $v \in A$, $g(x) = g(v) - 3 = 8f(v)$ and $g(y) = g(v) - 1 = 8f(v) + 2$. In this way the weights of the edges vx, vy, and xy are $8f(v) + 2$, $8f(v) + 3$, and $8f(v) + 1$, respectively. Since $\{f(v) : v \in A\} = \{0, 1, \ldots, \lambda\}$, the set $W_A = \{8f(v) + 1, 8f(v) + 2, 8f(v) + 3 : v \in A\} = \{1, 2, 3, 9, 10, 11, \ldots, 8\lambda + 1, 8\lambda + 2, 8\lambda + 3\}$. If $v \in B$, $g(x) = g(v) + 2 = 8(n - f(v)) + 6, g(y) = g(v) + 3 = 8(n - f(v)) + 7$. Thus the weights of the edges vx, vy, and xy are $8(n - f(v)) + 5, 8(n - f(v)) + 6$, and $8(n - f(v)) + 7$, respectively. In this case $\{f(v) : v \in B\} = \{\lambda + 1, \lambda + 2, \ldots, n\}$, therefore $W_B = \{8(n - f(v)) + 5, 8(n - f(v)) + 6, 8(n - f(v)) + 7 : v \in B\} = \{5, 6, 7, 13, 14, 15, \ldots, 8(n - \lambda) - 3, 8(n - \lambda) - 2, 8(n - \lambda) - 1\}$.
Suppose first that n is even. In this case $|A| = |B| + 1$ and $\lambda = \frac{n}{2}$. So $W_A = \{1, 2, 3, 9, 10, 11, \ldots, 4n + 1, 4n + 2, 4n + 3\}$ and $W_B = \{5, 6, 7, 13, 14, 15, \ldots, 4n - 3, 4n - 2, 4n - 1\}$. Suppose now that n is odd. In this case $|A| = |B|$ and $\lambda = \frac{n-1}{2}$. Hence $W_A = \{1, 2, 3, 9, 10, 11, \ldots, 4n - 3, 4n - 2, 4n - 1\}$ and $W_B = \{5, 6, 7, 13, 14, 15, \ldots, 4n + 1, 4n + 2, 4n + 3\}$. In either case $W_G \cup W_A \cup W_B = \{1, 2, \ldots, 4n + 3\}$.
Therefore, g is a mean labeling of $G \odot P_2$. □

Similarly, we can prove that P_2 can be replaced by P_3 and the graph $G \odot P_3$ is a mean graph provided that G satisfies the conditions of the previous proposition. In this case

$g : V(G \odot P_3) \to \{0, 1, \ldots, 6n+5\}$. When v is a vertex of G, we define:

$$g(v) = \begin{cases} 6f(v)+5, & \text{if } v \in A; \\ 6f(v), & \text{if } v \in B. \end{cases}$$

If x, y, and z are the consecutive vertices of P_3 adjacent to $v, v \in V(G)$, the labeling g is defined on them as: $g(x) = g(v) - 3, g(y) = g(v) - 5, g(z) = g(v) - 1$ when $v \in A$; and $g(x) = g(v) + 2, g(y) = g(v) + 5, g(z) = g(v) + 4$ when $v \in B$.

Proposition 6. *Let G be an α-graph such that the cardinalities of its bipartite sets differ by at most one. If G has size n and order $n+1$, then $G \odot P_3$ is a mean graph.*

In Figure 2 we show an example of this labeling where $G = C_6 \cup K_2$.

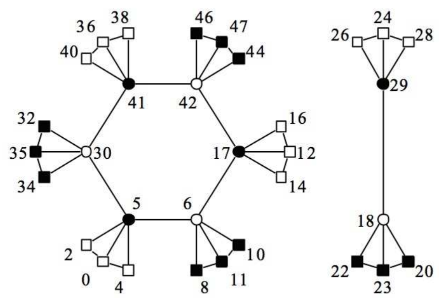

Figure 2: Mean labeling of $(C_6 \cup K_2) \odot P_3$

4 Mean Coronas of Even Cycles

In this section we study mean labelings of the coronas $C_n \odot mK_1$ where n is even; here $V(C_n \odot mK_1) = \{v_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_n \odot mK_1) = \{v_i v_{i+1} : 1 \leq i \leq n \text{ and } n+1 = 0\} \cup \{v_i u_{i,j} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. Thus, $C_n \odot mK_1$ is a graph where both, order and size, are equal to $n(m+1)$.

Proposition 7. *The corona $C_n \odot mK_1$ is a mean graph for every even value of $n \geq 4$ and $m \geq 1$.*

Proof. We analyze two cases depending of the congruence of n modulo 4. Let $f : V(C_n \odot mK_1) \to \{0, 1, \ldots, n(m+1)\}$.

Case I: When $n \equiv 0 (\mathrm{mod}\ 4)$.

$$f(v_i) = \begin{cases} (m+1)(i-1)+\delta_o, & \text{if } i \text{ is odd;} \\ (m+1)i+\delta_e, & \text{if } i \text{ is even.} \end{cases}$$

where

$$\delta_o = \begin{cases} 0, & \text{if } i \le \frac{n}{2}-1, \\ 1, & \text{if } i = \frac{n}{2}+1, \\ 2, & \text{if } i \ge \frac{n}{2}+3 \end{cases} \qquad \text{and} \qquad \delta_e = \begin{cases} 1, & \text{if } i \le \frac{n}{2}, \\ 0, & \text{if } i \ge \frac{n}{2}+2. \end{cases}$$

The pendant vertices are labeled as follows:

$$f(u_{i,j}) = \begin{cases} f(v_i)+2j-1, & \text{if } i \text{ odd}, i \ne \frac{n}{2}+1, 1 \le j \le m; \\ f(v_i)-1, & \text{if } i = \frac{n}{2}+1, j=1; \\ f(v_i)+2j-2, & \text{if } i = \frac{n}{2}+1, 2 \le j \le m; \\ f(v_i)+2j+1, & \text{if } i \text{ even}, 1 \le j \le m. \end{cases}$$

For $1 \le i \le \frac{n}{2}-1$, v_iv_{i+1} has weight $(m+1)i$. For $\frac{n}{2} \le i \le n-i$, v_iv_{i+1} has weight $(m+1)i+1$ and v_1v_n has weight $\frac{n}{2}(m+1)$. The pendant edges $v_iu_{i,j}$ have weights $(m+1)(i-1)+j$ when $1 \le i \le \frac{n}{2}$ and $(m+1)(i-1)+j+1$ when $\frac{n}{2}+1 \le i \le n$. Therefore, the set of induced weights is $\{1, 2, \ldots, n(m+1)\}$.
Notice that the smallest label assigned is 0 on v_1 and the largest is $n(m+1)$ on v_n. Moreover, when $i \le \frac{n}{2}-1$ is odd, $f(v_i) = (m+1)(i-1)$, $f(v_{i-1}) = (m+1)(i-1)-1$ and $f(v_{i+1}) = (m+1)i+(m-1)$; $f(u_{i-1,1}) = (m+1)(i-1)-2$ and $f(u_{i+1,m}) = (m+1)(i-1)+2$. When $i \ge \frac{n}{2}+2$ is even (except when $i = \frac{n}{2}+2$, here $f(v_{i-1}) = (m+1)i-2m$), $f(v_i) = (m+1)i$, $f(v_{i-1}) = (m+1)i-(2m+1)$ and $f(v_{i+1}) = (m+1)i+1$; $f(u_{i-1,m}) = (m+1)i-2$ and $f(u_{i+1,1}) = (m+1)i+2$.
Therefore, f is a mean labeling of $C_n \odot mK_1$ when $n \equiv 0 (\mathrm{mod}\ 4)$.
Case II: When $n \equiv 2 (\mathrm{mod}\ 4)$.
Given the similarity with the previous case, we omit the details and just present the labeling.

$$f(v_i) = \begin{cases} (m+1)(i-1)+\delta_o, & \text{if } i \text{ is odd ;} \\ (m+1)i+\delta_e, & \text{if } i \text{ is even .} \end{cases}$$

where

$$\delta_o = \begin{cases} 0, & \text{if } i \leq \frac{n-4}{2}, \\ 1, & \text{if } i \geq \frac{n}{2} \end{cases} \qquad \text{and} \qquad \delta_e = \begin{cases} 1, & \text{if } i \leq \frac{n-2}{2}, \\ 0, & \text{if } i \geq \frac{n+2}{2}. \end{cases}$$

In this case the pendant vertices are labeled as follows:

$$f(u_{i,j}) = \begin{cases} f(v_i) + 2j - 1, & \text{if } i \text{ odd}, \\ f(v_i) - 2j + 1, & \text{if } i \text{ even}. \end{cases}$$

□

5 Pseudo-Mean Labelings

A *pseudo-mean labeling* of a graph G is size n is an injective function $f : V(G) \to \{0, 1, \ldots, s\}$ where $s > n$, such that all the induced weights are distinct. This definition was intoduced in [2]. We can use this definition in a different manner. For instance, if G is not a mean graph, it would be useful to know how to close it is to actually being one. We just need to determine the smallest value of s that allows us to produce the weights $1, 2, \ldots, n$ under a pseudo-mean labeling of G.
Let $G \in \mathcal{A}_n$ where n is even. If we label the vertices of $G \odot mK_1$, $m \geq 2$, using the labeling g described in Theorem 4, the result is a pseudo-mean labeling of $G \odot mK_1$, whose largest label is $m-1$ units larger than the size of $G \odot mK_1$.
In fact, let $n = 2k$. Thus, an α-mean labeling of G has $k+1$ even-labeled vertices and k odd-labeled vertices. Therefore, when $G \odot mK_1$ is labeled using g in Theorem 4, we obtain $m(k+1)+k$ odd-labeled vertices. Since the size of $G \odot mK_1$ is $(n+1)m+n$ and the largest odd label used is $2(m(k+1)+k)-1 = (n+1)m+n+m-1$, we have a pseudo-mean labeling of $G \odot mK_1$ whose largest label is $m-1$ units larger than the size of $G \odot mK_1$.

Proposition 8. *Let $G \in \mathcal{A}_n$. For every $m \geq 2$ and n even, there exists a pseudo-mean labeling of $G \odot mK_1$ where the largest label used is $m-1$ units larger than the size of $G \odot mK_1$.*

Consider now the complete graph K_n, it is well known that when the integers $2^0, \ldots, 2^{n-1}$ are assigned to its vertices all the induced weights are distinct. Thus, subtracting 1 from each power of 2 we have a pseudo-mean labeling of K_n with $s = 2^{n-1}-1$. Now we present another labeling of K_n that reduces the value of s considerably.

Let F_n be the n^{th} Fibonacci number, with $F_0 = F_1 = 1$. We know that $2^{n-1} > F_{n+2}$ for large values of n. Hence, by assigning $F_3, F_4, \ldots, F_{n+2}$ to the vertices of K_n we also obtain a pseudo-mean labeling but now the largest label used is significantly smaller.

Proposition 9. *A pseudo-mean labeling of K_n exists where the largest label used is $F_{n+2} - 3$.*

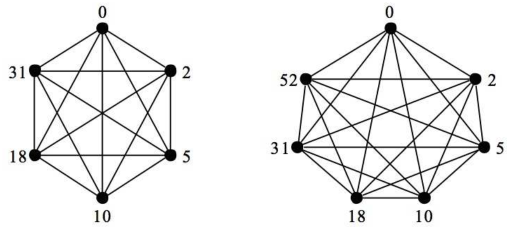

Figure 3: Pseudo-mean labelings of K_6 and K_7

6 Open Problems

1. Are all trees that satisfy the degree condition mean trees?
2. For all $m \geq 1$, find a mean labeling of $C_n \odot mK_1$ where n is odd.
3. For all $m \geq 2$, find a mean labeling of $G \odot mK_1$ where $G \in \mathcal{A}_n$ and n is even.

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