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Vertex Equitable Labelings of Transformed Trees

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ABSTRACT

Let G be a graph with p vertices and q edges and let $A = \left\{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\right\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is vertex equitable if there exists a vertex labeling f such that for all a and b in $A, |v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. In this paper, we prove that $T\hat{O}P_n, T\hat{O}2P_n, T\hat{O}C_n, T\tilde{O}C_n$ are vertex equitable graphs.

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1 Introduction

All graphs considered here are simple, finite, connected and undirected. For the basic notations and terminology, we follow [3]. The symbols $V(G)$ and $E(G)$ denote the vertex set and the

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edge set of a graph G respectively. Let $G = (p, q)$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A labeling f of a graph G is a mapping that assigns elements of a graph to the set of numbers (usually to positive or non-negative integers). If the domain of the mapping is the set of vertices (respectively, the set of edges) then we call the labeling *vertex labeling* (respectively, *edge labeling*). The labels of the vertices induce the labels of the edges. There are several types of labeling and a detailed survey on graph labeling can be found in [2]. A vertex labeling f is said to be a difference labeling if it induces the label $|f(x) - f(y)|$ for each edge xy which is called the weight of an edge xy. A brief summary of the definitions and known results are given below.

The total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and the edges of G and the two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G. For each vertex v of a graph G, take a new vertex v' and join v' to the vertices of G which are adjacent to v. The graph thus obtained is called the splitting graph of G and is denoted by $S'(G)$.

Let G be a graph with p vertices and q edges. A graph H is said to be a super subdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some $m_i, 1 \leq i \leq q$ in such a way that ends of e_i are merged with the two vertices of the 2-vertices part of K_{2,m_i}, after removing the edge e_i from G. A super subdivision H of G is said to be an arbitrary super subdivision of G if every edge of G is replaced by $K_{2,m}$ (m vary for each edge arbitrarily). Fusion of two cycles C_m and C_n is a graph $C(m, n)$ obtained by identifying an edge of a cycle C_m with an edge of a cycle C_n.

The concept of equitable labeling of graphs was due to Bloom and Ruiz [1]. A function $f : V(G) \to \{0, 1, \ldots, k-1\}$ is called k-equitable labeling if the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_{\overline{f}}(i) - e_{\overline{f}}(j)| \leq 1$ for $i \neq j, i, j = 0, 1, 2, \ldots, k-1$ are satisfied, where $\overline{f}$ is the induced edge labeling given by $\overline{f}(uv) = |f(u) - f(v)|$ and $v_f(i)$ and $e_{\overline{f}}(i), i \in \{0, 1, \ldots, k-1\}$ are the number of vertices and edges of G respectively with label i.

A. Lourdusamy and M. Seenivasan introduced the concept of vertex equitable labeling in [7]. Let G be a graph with p vertices and q edges and let $A = \left\{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\right\}$. A vertex labeling $f : V(G) \to A$ induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv. For $a \in A$, let $v_f(a)$ be the number of vertices v with $f(v) = a$. A graph G is said to be vertex equitable if there exists a vertex labeling f such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. They proved that the graphs like path, bistar $B(n, n)$, comb graph, cycle C_n if $n \equiv 0$ or $3 (mod\ 4)$, $K_{2,n}, C_3^{(t)}$ for $t \geq 2$, quadrilateral snake, $K_2 + mK_1$, $K_{1,n} \cup K_{1,n+k}$ if and only if $1 \leq k \leq 3$, ladder, arbitrary super division of any path and cycle C_n with $n \equiv 0$ or $3 (mod\ 4)$ are vertex equitable. Also they proved that the graphs $K_{1,n}$ if $n \geq 4$, any Eulerian graph with n edges where $n \equiv 1$ or $2 (mod\ 4)$, the wheel W_n, the complete graph K_n if $n > 3$ and triangular cactus with $q \equiv 0$ or 6 or $9 (mod\ 12)$ are not vertex equitable. In addition, they proved that if G is a graph with p vertices and q edges, q is even and $p < \frac{q}{2} + 2$ then G is not vertex equitable.

P. Jeyanthi and A. Maheswari [5, 6] proved that T_p-trees, $T \odot \overline{K_n}$ where T is a T_p-tree with an

even number of vertices, the bistar $B(n, n+1)$, the caterpillar $S(x_1, x_2, \dots, x_n)$, $C_n \odot \overline{K_1}, P_n^2$, tadpoles, $C_m \oplus C_n$, armed crowns, $[Pm; C_n^2]$, $\langle P_m \hat{o} K_{1,n} \rangle$, the graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle C_n, total graph of P_n, splitting graph of P_n and $C(m, n)$ are vertex equitable graphs.
In this paper, we prove that $T\hat{O}P_n, T\hat{O}2P_n, T\hat{O}C_n, T\tilde{O}C_n$ are vertex equitable graphs. We use the following definitions.

Definition 1.1. [4] *Let T be a tree and u_0 and v_0 be two adjacent vertices in T. Let u and v be two pendant vertices of T such that the length of the path u_0-u is equal to the length of the path v_0-v. If the edge u_0v_0 is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (ept) and the edge u_0v_0 is called transformable edge.*

If by the sequence of ept's, T can be reduced to a path, then T is called a T_p-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P is denoted as $P(T)$. A T_P-tree and a sequence of two ept's reducing it to a path are illustrated in Figure-1.

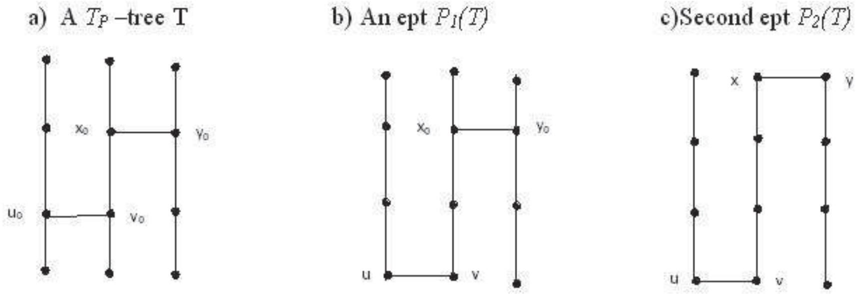

Figure 1: A T_P-tree and a sequence of two ept's reducing it to a path

Definition 1.2. *Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1 \hat{o} G_2$ is obtained from G_1 and p copies of G_2 by identifying one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1.*

Definition 1.3. *Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1 \hat{o} G_2$ is obtained from G_1 and p copies of G_2 by joining one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1 by an edge.*

2 Main Result

In this paper, we prove that $T\hat{O}P_n, T\hat{O}2P_n, T\hat{O}C_n, T\tilde{O}C_n$ are vertex equitable graphs.

Theorem 2.1. *Let T be a T_p-tree on m vertices. Then the graph $T\hat{o}P_n$ is a vertex equitable graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T)E_d)\Box E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges.
Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u_1^j, u_2^j, \ldots, u_n^j (1 \le j \le m)$ be the vertices of j^{th} copy of P_n. Then $V(T\hat{o}P_n) = \{u_i^j : 1 \le i \le n, 1 \le j \le m \text{ with } u_n^j = v_j\}$. The graph $T\hat{o}P_n$ has mn vertices and $mn-1$ edges. Let $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{mn-1}{2} \right\rceil \right\}$.
We define a vertex labeling $f : V(T\hat{o}P_n) \to A$ as follows:
For $1 \le i \le n$, let $f(u_i^j) = \begin{cases} \frac{nj}{2} - \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is even, } 1 \le j \le m \\ \frac{n(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is odd, } 1 \le j \le m. \end{cases}$
Let v_iv_j be a transformed edge in T for some indices $i, j, 1 \le i \le j \le m$ and let P_1 be the ept that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.
Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity, that is, i is odd and j is even or vice-versa.
The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= f(v_i) + f(v_{i+2t+1}) \\ &= \begin{cases} \frac{n(i-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor + \frac{n(i+2t+1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor & \text{if } i \text{ is odd} \\ \frac{ni}{2} - \left\lfloor \frac{n}{2} \right\rfloor + \frac{n(i+2t+1-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } i \text{ is even.} \end{cases} \\ &= n(i+t) \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= f(v_{i+t}) + f(v_iv_{i+t+1}) \\ &= n(i+t). \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
It can be verified that the induced edge labels of $T\hat{o}P_n$ are $1, 2, 3, \ldots, mn-1$ and for $|v_f(a) - v_f(b)| \le 1$ for all $a, b \in A$. Hence, $T\hat{o}P_n$ is a vertex equitable graph. □

The vertex equitable labeling of $T\hat{o}P_5$ where T is a T_p-tree with 13 vertices is given in Figure 2.

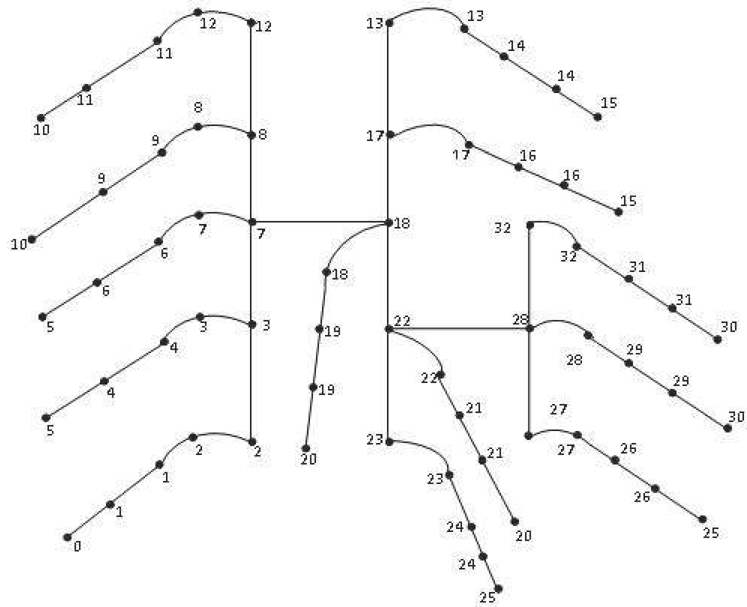

Figure 2

Theorem 2.2. *Let T be a T_p-tree on m vertices. Then the graph $T\hat{o}2P_n$ is a vertex equitable graph.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d)\square E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \dots, P_k)$ of the epts P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges.
Now denote the vertices of $P(T)$ successively by $v_1, v_2, \dots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u^j_{1,1}, u^j_{1,2}, \dots, u^j_{1,n}$ and $u^j_{2,1}, u^j_{2,2}, \dots, u^j_{2,n}$ $(1 \leq j \leq m)$ be the vertices of the two vertex disjoint paths joined by the j^{th} vertex of T such that $v_j = u^j_{1,n} = u^j_{2,n}$. Then $V(T\hat{o}2P_n) = \{v_j, u^j_{1,i}, u^j_{2,i} : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u^j_{1,n} = u^j_{2,n} = v_j\}$.
Let $A = \left\{0, 1, 2, \dots, \left\lceil \frac{m(2n-1)-1}{2} \right\rceil \right\}$.
Define a vertex labeling $f : V(T\hat{o}2P_n) \to A$ as follows:

$$\text{For } 1 \leq i \leq n, \text{ let } f(u^j_{1,i}) = \begin{cases} \frac{(2n-1)j}{2} - n + \left\lceil \frac{i}{2} \right\rceil & \text{if } j \text{ is even, } 1 \leq j \leq m \\ \frac{(2n-1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is odd, } 1 \leq j \leq m, \end{cases}$$

$$f(u^j_{2,i}) = \begin{cases} \frac{(2n-1)j}{2} - \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is even, } 1 \leq j \leq m \\ \frac{(2n-1)(j-1)}{2} + n - \left\lceil \frac{i}{2} \right\rceil & \text{if } j \text{ is odd, } 1 \leq j \leq m. \end{cases}$$

Let $v_i v_j$ be the transformed edge in T for some indices $i, j, 1 \leq i \leq j \leq m$ and let P_1 be the ept that deletes the edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} as

also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.
Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i+2t+1$. Therefore i and j are of opposite parity, that is, i is odd and j is even or vice-versa.
The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\ &= f(v_i) + f(v_{i+2t+1}) \\ &= (2n-1)(i+t) \ \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\ &= f(v_{i+t}) + f(v_iv_{i+t+1}) \\ &= (2n-1)(i+t). \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
It can be verified that the induced edge labels of $T\hat{o}2P_n$ are $1, 2, 3, \ldots, m(2n-1)-1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence, $T\hat{o}2P_n$ is a vertex equitable graph. □

The vertex equitable labeling of $T\hat{o}2P_4$ where T is a T_p-tree with 10 vertices is given in Figure 3.

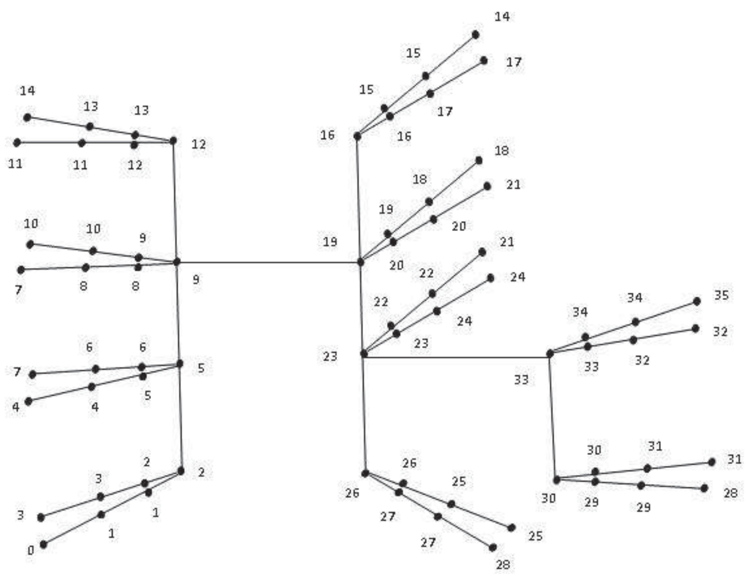

Figure 3

Theorem 2.3. *Let T be a T_p-tree on m vertices. Then the graph $T\hat{o}C_n$ is a vertex equitable graph if $n \equiv 0, 3(mod\ 4)$.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d)\Box E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges.
Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u_1^j, u_2^j, \ldots, u_n^j$ $(1 \leq j \leq m)$ be the vertices of j^{th} copy of P_n. Then $V(T\hat{o}C_n) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u_1^j = v_j\}$. Let $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{m(n+1)-1}{2} \right\rceil \right\}$.
Define a vertex labeling $f : V(T\hat{o}C_n) \to A$ as follows:
Case (i) $n \equiv 3(mod\ 4)$.
For $1 \leq j \leq m$ and j is odd,

$$\text{let } f(u_1^j) = \left\lceil \frac{n}{2} \right\rceil j,$$

$$f(u_i^j) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor \text{ if } i \text{ is odd, } 2 \leq i \leq n,$$

$$f(u_i^j) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is odd, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil (j-1) + \frac{i}{2} & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

For $1 \leq j \leq m$ and j is even.

$$\text{let } f(u_1^j) = \left\lceil \frac{n}{2} \right\rceil (j-1),$$

$$f(u_i^j) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor \text{ if } i \text{ is even, } 2 \leq i \leq n,$$

$$f(u_i^j) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } i \text{ is odd, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lceil \frac{i}{2} \right\rceil & \text{if } i \text{ is odd, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

Let v_iv_j be a transformed edge in T for some indices $i, j, 1 \leq i \leq j \leq m$ and let P_1 be the ept that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.
Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity, that is , i is odd and j is even or vice-versa.
The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\ &= (n+1)(i+t) \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_iv_{i+t+1}) \\ &= (n+1)(i+t). \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
Case(ii) $n \equiv 0(mod\ 4)$.
For $1 \leq j \leq m$ and j is odd.

$$\text{let } f(u_1^j) = \left\lfloor \frac{(n+1)j}{2} \right\rfloor,$$
$$f(u_i^j) = \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \text{ if } i \text{ is odd, } 2 \leq i \leq n,$$
$$f(u_i^j) = \begin{cases} \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ \frac{(n+1)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

For $1 \leq j \leq m$ and j is even.

$$\text{let } f(u_1^j) = \left\lceil \frac{(n+1)(j-1)}{2} \right\rceil,$$
$$f(u_i^j) = \left\lfloor \frac{(n+1)(j-1)}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil \text{ if } i \text{ is odd, } 2 \leq i \leq n,$$
$$f(u_i^j) = \begin{cases} \left\lceil \frac{(n+1)(j-1)}{2} \right\rceil + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ \left\lceil \frac{(n+1)(j-1)}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. \end{cases}$$

Let v_iv_j be a transformed edge in T for some indices $i, j, 1 \leq i \leq j \leq m$ and let P_1 be the ept that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.
Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity, that is, i is odd and j is even or vice-versa.
The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\ &= (n+1)(i+t) \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_iv_{i+t+1}) \\ &= (n+1)(i+t). \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
It can be verified that the induced edge labels of $T\hat{o}C_n$ are $1, 2, 3, \ldots, m(n+1)-1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $T\hat{o}C_n$ is a vertex equitable graph. □

The vertex equitable labeling pattern of $T\hat{o}C_7$, where T is a T_p-tree with 8 vertices, is given in Figure 4.

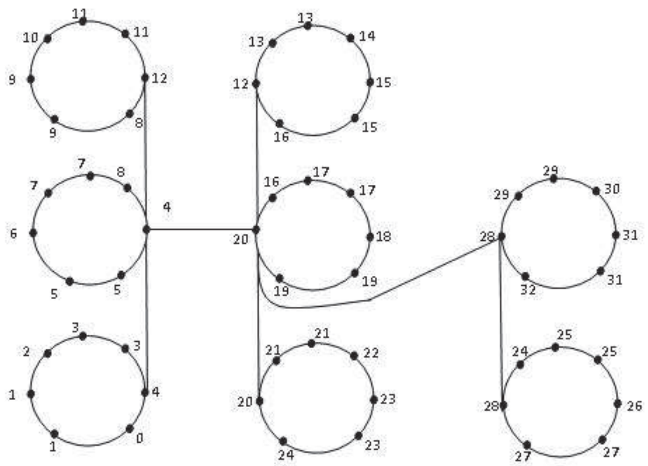

Figure 4

Theorem 2.4. *Let T be a T_p-tree on m vertices. Then the graph $T\tilde{o}C_n$ is a vertex equitable graph if $n \equiv 0, 3(mod\ 4)$.*

Proof. Let T be a T_p-tree with m vertices. By the definition of a transformed tree there exists a parallel transformation P of T such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d)\square E_p$ where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the ept's P used to arrive at the path $P(T)$. Clearly, E_d and E_p have the same number of edges.
Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u_1^j, u_2^j, \ldots, u_n^j$ $(1 \leq j \leq m)$ be the vertices of j^{th} copy of P_n then $V(T\tilde{o}C_n) = \{v_j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\tilde{o}C_n) = E(T) \cup E(C_n) \cup \{v_j u_1^j : 1 \leq j \leq m\}$. Let $A = \left\{0, 1, 2, \ldots, \left\lceil \frac{m(n+2)-1}{2} \right\rceil \right\}$. Define a vertex labeling $f : V(T\tilde{o}C_n) \to A$ as follows:

Case(i) $n \equiv 3(mod 4)$.

For $1 \leq j \leq m$, let $f(v_i) = \begin{cases} \frac{(i-1)(n+2)}{2} + \left\lceil \frac{n}{2} \right\rceil & \text{if } i \text{ is odd} \\ \frac{(i-2)(n+2)}{2} + 1 + \left\lceil \frac{n}{2} \right\rceil & \text{if } i \text{ is even} \end{cases}$ and $f(u_1^j) = f(v_i)$.

For $1 \leq j \leq m$ and j is odd,

$$\text{let } f(u_i^j) = \begin{cases} \frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ \frac{(n+2)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, \end{cases}$$

$$f(u_i^j) = \frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \quad \text{if } i \text{ is odd, } 2 \leq i \leq n.$$

For $1 \le j \le m$ and j is even,

$$\text{let } f(u_i^j) = \begin{cases} \frac{(n+2)(j-2)}{2} + 1 + \lceil \frac{n}{2} \rceil + \frac{i-1}{2} & \text{if } i \text{ is odd, } 2 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{(n+2)(j-2)}{2} + 1 + \lceil \frac{n}{2} \rceil + \lceil \frac{i}{2} \rceil & \text{if } i \text{ is odd, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, \end{cases}$$

$$f(u_i^j) = \frac{(n+2)(j-2)}{2} + 1 + \left\lceil \frac{n}{2} \right\rceil + \frac{i}{2} \text{ if } i \text{ is even, } 2 \le i \le n.$$

Let $v_i v_j$ be a transformed edge in T for some indices i and $j, 1 \le i \le j \le m$ and let P_1 be the ept that deletes this edge $v_i v_j$ and adds the edge $v_{i+t} v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.

Since $v_{i+t} v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, i and j are of opposite parity, that is, i is odd and j is even or vice-versa.

The induced label of the edge $v_i v_j$ is given by

$$\begin{aligned} f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\ &= (n+2)(i+t) \text{ and} \\ f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) = f(v_{i+t}) + f(v_i v_{i+t+1}) \\ &= (n+2)(i+t). \end{aligned}$$

Therefore, $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.

Case (ii) $n \equiv 0(mod\ 4)$.

For $1 \le i \le m$, let $f(v_i) = \begin{cases} \frac{i(n+2)}{2} & \text{if } i \text{ is odd} \\ \frac{(i-1)(n+2)}{2} & \text{if } i \text{ is even.} \end{cases}$

For $1 \le j \le m$ and j is odd,

$$\text{let } f(u_1^j) = \frac{j(n+2)}{2} - 1$$

$$f(u_i^j) = \begin{cases} \frac{(n+2)(j-1)}{2} + \lfloor \frac{i-1}{2} \rfloor & \text{if } i \text{ is even, } 2 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{(n+2)(j-1)}{2} + \lceil \frac{i}{2} \rceil & \text{if } i \text{ is even, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, \end{cases}$$

$$f(u_i^j) = \frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \text{ if } i \text{ is odd, } 2 \le i \le n.$$

For $1 \le j \le m$ and j is even,

$$\text{let } f(u_1^j) = \frac{(j-1)(n+2)}{2} + 1,$$

$$f(u_i^j) = \begin{cases} \frac{(n+2)(j-1)}{2} + \lceil \frac{i}{2} \rceil & \text{if } i \text{ is even, } 2 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{(n+2)(j-1)}{2} + \lceil \frac{i+1}{2} \rceil & \text{if } i \text{ is even, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, \end{cases}$$

$$f(u_i^j) = \frac{(n+2)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil \text{ if } i \text{ is odd, } 2 \le i \le n.$$

Let v_iv_j be a transformed edge in T for some indices $i, j, 1 \leq i \leq j \leq m$ and let P_1 be the ept that deletes the edge v_iv_j and adds the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} as also the distance of v_j from v_{j-t}. Let P be a parallel transformation of T that contains P_1 as one of the constituent epts.
Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore i and j are of opposite parity, that is, i is odd and j is even or vice-versa.
The induced label of the edge v_iv_j is given by

$$\begin{aligned} f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\ &= (n+2)(i+t) \text{ and} \\ f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_iv_{i+t+1}) \\ &= (n+2)(i+t). \end{aligned}$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.
It can be verified that the induced edge labels of $T\tilde{o}C_n$ are $1, 2, 3, \ldots, m(n+1) - 1$ and for all $a, b \in A$. Hence $T\tilde{o}C_n$ is a vertex equitable graph. □

The vertex equitable labeling pattern of $T\tilde{o}C_n$, where T is a T_p-tree with 8 vertices, is given in Figure 5.

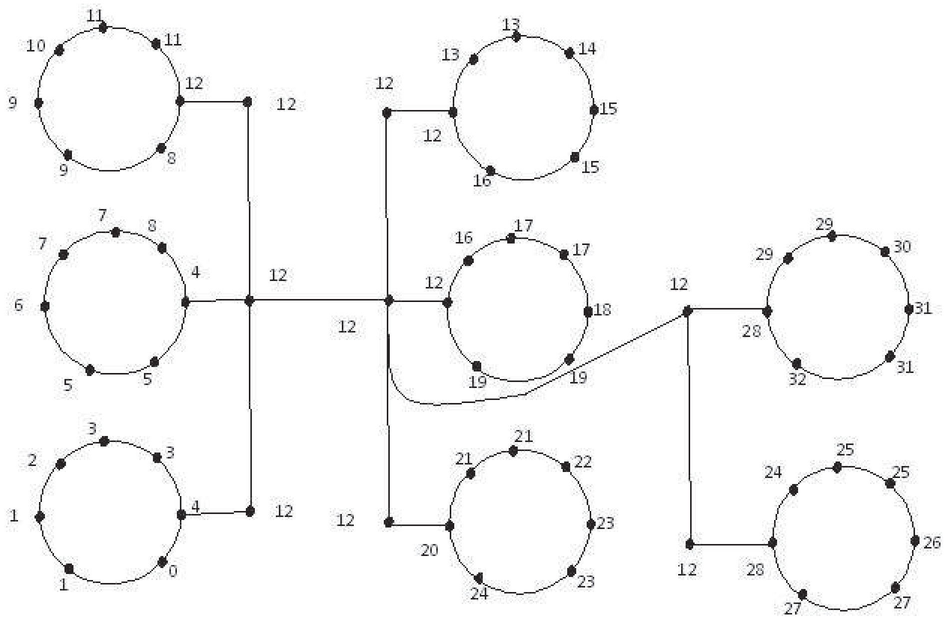

Figure 5

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