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Analytical Solutions for Entanglement A Superposition of Spin Coherent States with Non-Phase Coherence Parameters

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Abstract

In this work, the entanglement of a superposition of bipartite qubit coherent states with non-phased coherent parameters is studied. We use Generalized-concurrence as the measure to quantify the entanglement and drive analytical results in terms of the effective parameters involved. Analyzing the results, we conclude that such states may attain maximum entanglement or no entanglement at all, depending on the choice of the parameters involved.

Keywords: Entanglement; Generalized concurrence; Spin coherent states; Qubit.

Introduction

Entanglement is one of the most interesting and mysterious phenomena in the world of quantum mechanics. This phenomenon, which has become one of the most important and interesting topics in the quantum information theory in recent decades, was first introduced in 1935 by Schrödinger [1]. Entanglement has very important applications in the field of quantum information processing theory [2, 3], like quantum computing, quantum teleportation and quantum cryptography [4-8]. Recently, entangled coherent states have also found many applications in quantum information theory. In this paper, we study the entanglement of the superposition of the bipartite qubit spin coherent states with non-phased coherence parameters using the generalized concurrence as the entanglement measure. Numerical studies are the approach that is usually taken to calculate the entanglement of the quantum states. However, analytical computations and discussions of the results have always been of great importance in physics. Such analytical studies are means of understanding of the concepts related to various physics problems; this is the approach we take in this work.

Materials and Methods

Generalized concurrence measure

Among the most widely used measures for analyzing the degree of entanglement of bipartite systems, are concurrence and the generalized concurrence [9, 10]; the latter is in fact a generalization of concurrence [11, 12] for systems with dimensions greater than two. It is defined for a bipartite system consisting of A and B parts as follows [9].

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$$IC(\left|\psi^{AB}\right\rangle) = \sqrt{2(1-tr\,\rho_A{}^2)} \quad (1)$$

In this relation, ρ_A, is the reduced density matrix of subsystem A and is given by

$$\rho_A = tr_B \left|\psi^{AB}\right\rangle\left\langle\psi^{AB}\right| \quad (2)$$

we may expand the state of such a system in terms of computational bases, in the related Hilbert space,

$$\left|\psi^{AB}\right\rangle = \sum_{i,j} c_{i,j}\left|i^A, j^B\right\rangle$$

Where $c_{i,j} =: \langle i, j|\psi\rangle$. Then, it can be shown that generalized concurrence is obtained in terms of the expansion of coefficients in relation (2) as follows [13].

$$IC(\left|\psi^{AB}\right\rangle) = 2\sqrt{\sum_{i'>i,j'>j}\left|c_{i,j}c_{i',j'} - c_{i',j}c_{i,j'}\right|^2} \quad (3)$$

The generalized concurrence defined in relations (1) and (3) can be used as a measure of entanglement for bipartite systems of arbitrary dimensions. The minimum of its value for a separable state is equal to zero, and the maximum for entangled bipartite systems with arbitrary dimensions d is equal to $\sqrt{2(d-1)/d}$. For example, the maximum amount of this measure for entangled qubits and qutrits is 1 and $\sqrt{4/3}$, respectively

Density matrix of superposed coherent states and generalized concurrence

The general form of the spin coherent states, also known as Radcliffe states, is expressed as follows [13]

$$|\alpha, j\rangle = \frac{1}{(1+|\alpha|^2)^j}\sum_{m=-j}^{j}\binom{2j}{m+j}^{\frac{1}{2}}\alpha^{j+m}\,|j,m\rangle \quad (4)$$

where $|j,m\rangle$ are the eigenstates of angular momentum operator. For $j=\frac{1}{2}$

$$|\alpha\rangle = \frac{1}{\left(1+|\alpha|^2\right)^{\frac{1}{2}}}\left[|0\rangle + \alpha|1\rangle\right] \quad (5)$$

we use the following definitions

$$|0\rangle := \left|\frac{1}{2},-\frac{1}{2}\right\rangle,\ |1\rangle := \left|\frac{1}{2},\frac{1}{2}\right\rangle,\ \left|\alpha,\frac{1}{2}\right\rangle = |\alpha\rangle \quad (6)$$

To study the entanglement of coherent states, we must consider a superposition of these states. One of the purely entangled states of the two qubits, which consists of the superposition of separable coherent states, can be written as follows

$$|\chi\rangle = \frac{1}{\sqrt{N}}[Cos\theta(|\alpha\rangle\otimes|\beta\rangle) + e^{i\phi}Sin\theta(|\alpha'\rangle\otimes|\beta'\rangle)] \quad (7)$$

Using relation (5), we calculate $|\alpha\rangle$, $|\beta\rangle$, $|\alpha'\rangle$, $|\beta'\rangle$ and put them in relation (7)

$$|\chi\rangle = M_1|00\rangle + M_2|01\rangle + M_3|10\rangle + M_4|11\rangle \quad (8)$$

In which we use the following definitions:

$$M_1 = \frac{1}{\sqrt{N}}[\eta+\kappa];\ M_2 = \frac{1}{\sqrt{N}}[\beta\eta+\beta'\kappa]$$
$$M_3 = \frac{1}{\sqrt{N}}[\alpha\eta+\alpha'\kappa];\ M_4 = \frac{1}{\sqrt{N}}[\alpha\beta\eta+\alpha'\beta'\kappa] \quad (9\text{-a})$$

$$\eta = \frac{\cos\theta}{\left(1+|\alpha|^2\right)^{1/2}\left(1+|\beta|^2\right)^{1/2}}$$
$$\kappa = \frac{e^{i\phi}\sin\theta}{\left(1+|\alpha'|^2\right)^{1/2}\left(1+|\beta'|^2\right)^{1/2}} \quad (9\text{-b})$$

In order to calculate generalized concurrence for the state introduced in relation (8), we must obtain a reduced density matrix of ρ_A based on relation (1). To calculate the reduced density matrix, we first compute the density matrix of the state introduced in relation (8) as follows

$$\begin{aligned}\rho = |\chi\rangle\langle\chi| = &|M_1|^2|00\rangle\langle00| + M_1M_2^*|00\rangle\langle01| + M_1M_3^*|00\rangle\langle10| + M_1M_4^*|00\rangle\langle11| \\ &+ M_2M_1^*|01\rangle\langle00| + |M_2|^2|01\rangle\langle01| + M_2M_3^*|01\rangle\langle10| + M_2M_4^*|01\rangle\langle11| \\ &+ M_3M_1^*|10\rangle\langle00| + M_3M_2^*|10\rangle\langle01| + |M_3|^2|10\rangle\langle10| + M_3M_4^*|10\rangle\langle11| \\ &+ M_4M_1^*|11\rangle\langle00| + M_4M_2^*|11\rangle\langle01| + M_4M_3^*|11\rangle\langle10| + |M_4|^2|11\rangle\langle11|\end{aligned}$$

(10)

In which the Hilbert space bases for this two-qubit system are defined as follows

$$|00\rangle \equiv \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ |01\rangle \equiv \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ |10\rangle \equiv \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ |11\rangle \equiv \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad (11)$$

By inserting the computational bases of relation (11) in relation (10) we have a density matrix

$$\rho \equiv \begin{pmatrix} |M_1|^2 & M_1M_2^* & M_1M_3^* & M_1M_4^* \\ M_2M_1^* & |M_2|^2 & M_2M_3^* & M_2M_4^* \\ M_3M_1^* & M_3M_2^* & |M_3|^2 & M_3M_4^* \\ M_4M_1^* & M_4M_2^* & M_4M_3^* & |M_4|^2 \end{pmatrix} \quad (12)$$

Finally, after calculating the reduced density matrix, ρ_A , generalized concurrence is obtained using relation (1) for the state introduced in relation (8) as follows

$$IC = \left(2\left(1 - \left[\left(|M_1|^2 + |M_2|^2 \right)^2 + \left(M_1M_3^* + M_2M_4^* \right)^2 + \left(M_3M_1^* + M_4M_2^* \right)^2 + \left(|M_3|^2 + |M_4|^2 \right)^2 \right] \right) \right)^{1/2} \quad (13)$$

Now, using the relation (13), we can investigate the effect of different coherence and superposition parameters on the entanglement of superposed coherent states.

Results and Discussion

Analytical solutions for entanglement of superposed coherent states

To consider a combination of real and imaginary parameters, we insert the values of $\alpha' = -i\,\alpha$ and $\beta' = -i\,\beta$ in relation (7), we will examine the generalized concurrence obtained in relation (13) for different values of α , β , θ and ϕ . We consider several cases as follows:

Case 1: For $\phi = 0$ and $\theta = \frac{\pi}{4}$, These are among the values obtained by maximizing the relation (13) in terms of superposition parameters, By analytical calculations for the generalized concurrence in relation (13), we have,

$$IC = \frac{2|\alpha\beta|}{\alpha^2 + \beta^2 + 2} \quad (14)$$

Where, its variations and its contour are displayed in Figures 1 and 2 respectively. According to our expectations, expect for α and β or both being zero, the state introduced in relation (8) is entangled.

Case 2: For $\beta = \alpha$, generalized concurrence in relation (13) in terms of α , θ and ϕ leads to the following analytical solution,

$$IC = \frac{2\alpha^2 |\sin 2\theta|}{(\alpha^4 + 2\alpha^2 + 1) + \delta} \quad (15)$$

Where, δ is defined as follows

$$\delta = \mathrm{Sin}\,2\theta[2\alpha^2\,\mathrm{Sin}\,\varphi + \mathrm{Cos}\,\varphi(1-\alpha^4)]$$

Now, by fixing one of the parameters in relation (15), we can examine the variations of generalized concurrence for different values of the other two parameters. Assuming $\varphi = 0$ in relation (15), we plot the variations in generalized concurrence as a function of α and θ in Figure 3. For $\theta = (2n+1)\frac{\pi}{4}$ and $\alpha \neq 0$, we obtain the largest value, while, for $\theta = \frac{n\pi}{2}$ and any α the concurrence is zero. We note that for the latter choices the state introduced in relation (7) will turn to a separable state. We have displayed related graphs for $\varphi = 0$ in Figure 3.

By inserting $\theta = \frac{\pi}{4}$ in relation (15), we have plotted the variation of generalized concurrence as a function of α and φ in Figure 4. It is observed that the generalized concurrence as a functions of α and φ, varies between 0 and 1. We also note that, for $\alpha = 0$ and any φ, the generalized concurrence is 0. We have plotted the relevant graphs in Figure 4.

Finally, by inserting $\alpha = 1$ in relation (15), we

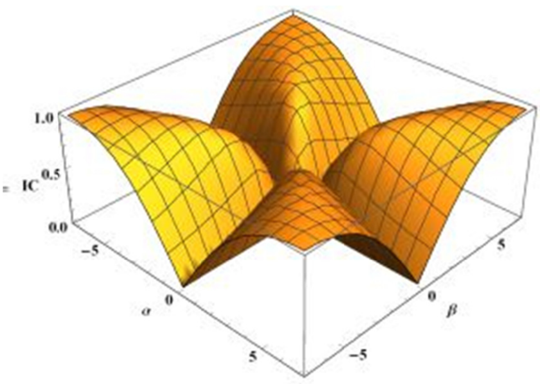

Figure 1. Generalized concurrence as a function of α, β for $\theta = \frac{\pi}{4}, \phi = 0$, and it's contour.

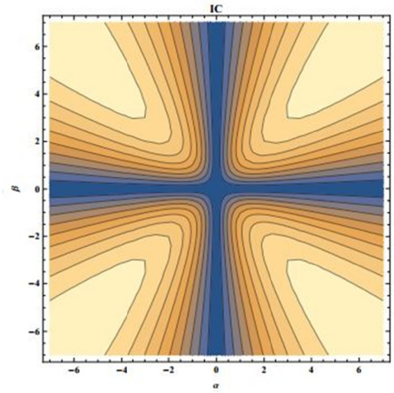

Figure 2. Contour of Generalized concurrence as a function of α, β for $\theta = \frac{\pi}{4}, \phi = 0$.

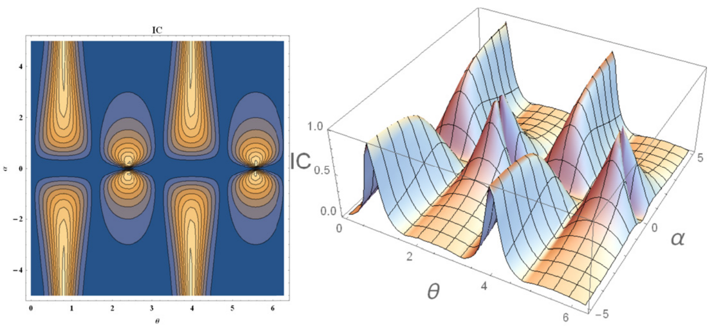

Figure 3. Generalized concurrence as a function of α, θ for $\phi = 0$, and it's contour.

study the variation of the generalized concurrence in terms of superposition parameters θ and φ. We have displayed the relevant graphs in Figure 5. For $\theta = (2n+1)\frac{\pi}{4}$, a relative maximum of generalized concurrence occurs, which depends on the value of φ.

Also, we obtain the minimum of generalized concurrence (zero) for all φs taking the value $\theta = \frac{n\pi}{2}$. For a better analysis of Figure 5, we have plotted the contour plot next to it.

We have studied, the entanglement of a superposition of bipartite spin coherent states with non-phased coherent parameters. Using generalized concurrence, analytical solutions were derived in terms of the parameters of the

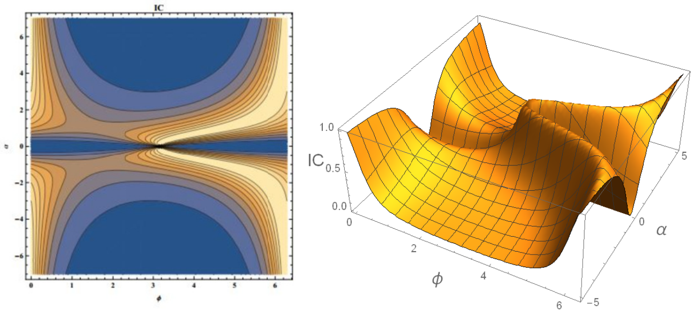

Figure 4. Generalized concurrence as a function of α, φ for $\theta = \frac{\pi}{4}$, and it's contour.

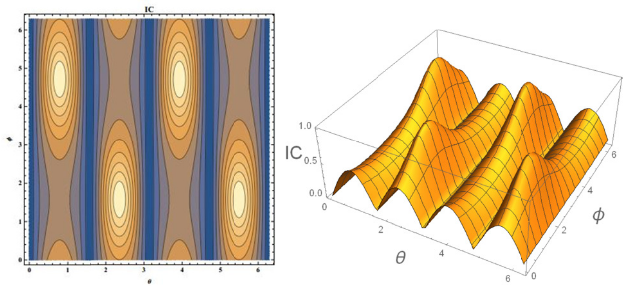

Figure 5. Generalized concurrence as a function of θ, φ for $\alpha = 1$, and it's contour.

superposition and the states. We observed that one may adjust the degree of the entanglement in these states, which can vary between 1 and zero, by choosing the required parameters. These results are consistent with the findings of Borata ,et al [14], who have reported a variation range of zero to the maximum for the superposition of spin coherent states with real coherent parameters [15]. We also note that by choosing the right values of the parameters, we may turn the state (8) into a bell-type state with the maximum value of generalized concurrence (IC=1); as we found for some specific values of the parameters.

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