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On computing total double Roman domination number of trees in linear time

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ABSTRACT

Let $G = (V, E)$ be a graph. A double Roman dominating function (DRDF) on G is a function $f : V \to \{0, 1, 2, 3\}$ such that for every vertex $v \in V$ if $f(v) = 0$, then either there is a vertex u adjacent to v with $f(u) = 3$ or there are vertices x and y adjacent to v with $f(x) = f(y) = 2$ and if $f(v) = 1$, then there is a vertex u adjacent to v with $f(u) \geq 2$. A DRDF f on G is a total DRDF (TDRDF) if for any $v \in V$ with $f(v) > 0$ there is a vertex u adjacent to v with $f(u) > 0$. The weight of f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a TDRDF on G is the total double Roman domination number of G. In this paper, we give a linear algorithm to compute the total double Roman domination number of a given tree.

Keyword: Total double Roman dominating function, linear algorithm, Dynamic programming, Combinatorial optimization, Tree.

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1 Introduction

Let $G = (V, E)$ be a graph. A *double Roman dominating function* (DRDF) $f : V \to \{0, 1, 2, 3\}$ of G has the property that for every vertex $v \in V$ with $f(v) = 0$ either there

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is a vertex $u \in V$ adjacent to v with $f(u) = 3$ or there are vertices $x, y \in V$ adjacent to v with $f(x) = f(y) = 2$ and for every vertex $v \in V$ with $f(v) = 1$ there is a vertex $u \in V$ adjacent to v with $f(u) \geq 2$. Beeler et al. [2] introduced the concept of double Roman dominating function. The concept of double Roman domination was further studied, see for example [1, 5, 7, 8].
Shao et al. [6] introduced a new variant of double Roman dominating functions. A *total* DRDF (TDRDF) is a DRDF f on G with an additional property that for every vertex $v \in V$ with $f(v) > 0$ there is a vertex $u \in V$ adjacent to v with $f(u) > 0$.
The weight of a TDRDF f on G is the sum $f(V) = \sum_{v \in V} f(v)$, denoted by $w(f)$, and the minimum weight of a TDRDF f is the *total double Roman domination number* of G, denoted by $\gamma_{tdR}(G)$. They showed that the decision problem for the total double Roman domination is NP-hard even when restricted to chordal and bipartite graphs. There are many works that compute a variant of domination for a given tree, see for example, [3, 4, 8]. In This paper we give a dynamic programming algorithm that computes the total double Roman domination of a given tree in linear time.

2 Total double Roman domination of trees

In this section, we give a linear algorithm (Algorithm 2.1) that computes the total double Roman domination number of a given tree. Let $G = (V, E)$ be a graph with $v \in V$, let a vertex $w \notin V$ and let $a \in \{0, 1, 2, 3\}$ and $b, c \in \{1, 2, 3\}$. We define the following.

- $\gamma_{tdR}(G, v = a) = \min\{w(f) : f \text{ is a TDRDF on } G \text{ with } f(v) = a\}$,
- $\gamma'_{tdR}(G, v = 0, w = 2) = \min\{w(f) : f$ is a DRDF on $G+vw$ such that the restriction of f to $G - v$ is a TDRDF on $G - v$, $f(v) = 0$ and $f(w) = 2\}$,
- $\gamma_{tdR}(G, v = b, w = c) = \min\{w(f) : f$ is a TDRDF on $G + vw$ with $f(v) = b$ and $f(w) = c\}$.

A $\gamma_{tdR}(G, v = a)$-function is a minimum TDRDF f on G with $f(v) = a$, a $\gamma'_{tdR}(G, v = 0, w = 2)$-function is a minimum DRDF on $G + vw$ such that the restriction of f to $G - v$ is a TDRDF on $G - v$, $f(v) = 0$ and $f(w) = 2$ and a $\gamma_{tdR}(G, v = b, w = c)$-function is a minimum TDRDF on $G + vw$ with $f(v) = b$ and $f(w) = c$.
Lemma 1:
Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$, $v \in V_1$ and $u \in V_2$, let w be a vertex not in $V_1 \cup V_2$, let $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv\})$ and let $a \in \{1, 2, 3\}$ and $b \in \{2, 3\}$. Then,

(i) $\gamma_{tdR}(G, v = 0) = \min\{\gamma_{tdR}(H_1, v = 0) + \gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_1, v = 0) + \gamma_{tdR}(H_2, u = 1), \gamma'_{tdR}(H_1, v = 0, w = 2) + \gamma_{tdR}(H_2, u = 2) - 2, \gamma_{tdR}(H_1 - v) + \gamma_{tdR}(H_2, u = 3)\}$,

(ii) $\gamma_{tdR}(G, v = 1) = \min\{\gamma_{tdR}(H_1, v = 1) + \gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_1, v = 1) + \gamma_{tdR}(H_2, u = 1), \gamma_{tdR}(H_1, v = 1, w = 2) + \gamma_{tdR}(H_2, u = 2, w = 1) - 3, \gamma_{tdR}(H_1, v = 1, w = 3) + \gamma_{tdR}(H_2, u = 3, w = 1) - 4\}$,

(iii) $\gamma_{tdR}(G, v = 2) = \min\{\gamma_{tdR}(H_1, v = 2) + \gamma'_{tdR}(H_2, u = 0, w = 2) - 2, \gamma_{tdR}(H_1, v = 2, w = 1) + \gamma_{tdR}(H_2, u = 1, w = 2) - 3, \gamma_{tdR}(H_1, v = 2, w = 2) + \gamma_{tdR}(H_2, u = 2, w = 2) - 4, \gamma_{tdR}(H_1, v = 2, w = 3) + \gamma_{tdR}(H_2, u = 3, w = 2) - 5\}$,

(iv) $\gamma_{tdR}(G, v = 3) = \min\{\gamma_{tdR}(H_1, v = 3) + \gamma_{tdR}(H_2 - u), \gamma_{tdR}(H_1, v = 3, w = 1) + \gamma_{tdR}(H_2, u = 1, w = 3) - 4, \gamma_{tdR}(H_1, v = 3, w = 2) + \gamma_{tdR}(H_2, u = 2, w = 3) - 5, \gamma_{tdR}(H_1, v = 3, w = 3) + \gamma_{tdR}(H_2, u = 3, w = 3) - 6\}$,

(v) $\gamma'_{tdR}(G, v = 0, w = 2) = \min\{\gamma'_{tdR}(H_1, v = 0, w = 2) + \gamma_{tdR}(H_2, u = 0), \gamma'_{tdR}(H_1, v = 0, w = 2) + \gamma_{tdR}(H_2, u = 1), \gamma_{tdR}(H_1 - v) + \gamma_{tdR}(H_2, u = 2) + 2, \gamma_{tdR}(H_1 - v) + \gamma_{tdR}(H_2, u = 3) + 2\}$,

(vi) $\gamma_{tdR}(G, v = 1, w = b) = \min\{\gamma_{tdR}(H_1, v = 1, w = b) + \gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_1, v = 1, w = b) + \gamma_{tdR}(H_2, u = 1), \gamma_{tdR}(H_1, v = 1, w = b) + \gamma_{tdR}(H_2, u = 2, w = 1) - 1, \gamma_{tdR}(H_1, v = 1, w = b) + \gamma_{tdR}(H_2, u = 3, w = 1) - 1\}$,

(vii) $\gamma_{tdR}(G, v = 2, w = a) = \min\{\gamma_{tdR}(H_1, v = 2, w = a) + \gamma'_{tdR}(H_2, u = 0, w = 2) - 2, \gamma_{tdR}(H_1, v = 2, w = a) + \gamma_{tdR}(H_2, u = 1, w = 2) - 2, \gamma_{tdR}(H_1, v = 2, w = a) + \gamma_{tdR}(H_2, u = 2, w = 2) - 2, \gamma_{tdR}(H_1, v = 2, w = a) + \gamma_{tdR}(H_2, u = 3, w = 2) - 2\}$,

(viii) $\gamma_{tdR}(G, v = 3, w = a) = \min\{\gamma_{tdR}(H_1, v = 3, w = a) + \gamma_{tdR}(H_2 - u), \gamma_{tdR}(H_1, v = 3, w = a) + \gamma_{tdR}(H_2, u = 1, w = 3) - 3, \gamma_{tdR}(H_1, v = 3, w = a) + \gamma_{tdR}(H_2, u = 2, w = 3) - 3, \gamma_{tdR}(H_1, v = 3, w = a) + \gamma_{tdR}(H_2, u = 3, w = 3) - 3\}$,

(ix) $\gamma_{tdR}(G - v) = \gamma_{tdR}(H_1 - v) + \min\{\gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_2, u = 1), \gamma_{tdR}(H_2, u = 2), \gamma_{tdR}(H_2, u = 3)\}$.

Proof: Let f be a $\gamma_{tdR}(G)$-function and let $x \in \{0, 1, 2, 3\}$ and $c \in \{1, 2, 3\}$. Clearly, $f(v) = x$ if and only if both $f(v) = x$ and $f(u) = 0$, both $f(v) = x$ and $f(u) = 1$, both $f(v) = x$ and $f(u) = 2$ or both $f(v) = x$ and $f(u) = 3$. Let f_1, f_2, f_1^{-v} and f_2^{-u} be restrictions of f to H_1, H_2, $H_1 - v$ and $H_2 - u$, respectively. Let $g_1^{=x}$, $g_2^{=x}$, $g_1^{=b,=c}$, $g_2^{=b,=c}$, g_1^{-v}, g_2^{-u}, $h_1^{=0,=2}$ and $h_2^{=0,=2}$ be a $\gamma_{tdR}(H_1, v = x)$-function, $\gamma_{tdR}(H_2, u = x)$-function, $\gamma_{tdR}(H_1, v = b, w = c)$-function, $\gamma_{tdR}(H_2, u = b, w = c)$-function, $\gamma_{tdR}(H_1 - v)$-function, $\gamma_{tdR}(H_2 - u)$-function, $\gamma'_{tdR}(H_1, v = 0, w = 2)$-function and $\gamma'_{tdR}(H_2, u = 0, w = 2)$-function, respectively, and let $0_y = \{(y, 0)\}$, $1_y = \{(y, 1)\}$, $2_y = \{(y, 2)\}$ and $3_y = \{(y, 3)\}$, where y is a vertex. Assume that for every $f, g \in \{g_1^{=b,=c}, g_2^{=b,=c}, h_1^{=0,=2}, h_2^{=0,=2} : b, c \in \{1, 2, 3\}\}$ we have $D_f \cap D_g = \emptyset$.

Let $f(v) = 0$ and $\gamma_{tdR} = \min\{\gamma_{tdR}(H_1, v = 0) + \gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_1, v = 0) + \gamma_{tdR}(H_2, u = 1), \gamma'_{tdR}(H_1, v = 0, w = 2) + \gamma_{tdR}(H_2, u = 2) - 2, \gamma_{tdR}(H_1 - v) + \gamma_{tdR}(H_2, u = 3)\}$. So, f_1 is a TDRDF on H_1 with $f_1(v) = 0$ and f_2 is a TDRDF on H_2 with $f_2(u) = 0$, function f_1 is a TDRDF on H_1 with $f_1(v) = 0$ and f_2 is a TDRDF on H_2 with $f_2(u) = 1$, function $h = f_1 \cup 2_w$ is a DRDF on $H_1 + vw$ such that the restriction of h to $H_1 - v$ is a TDRDF on $H_1 - v$, $h(v) = 0$ and $h(w) = 2$ and f_2 is a TDRDF on H_2 with $f_2(u) = 2$ or f_1^{-v} is a TDRDF on $H_1 - v$ and f_2 is a TDRDF on H_2 with $f_2(u) = 3$. Hence, $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 0)$. Conversely, $g_1 = g_1^{=0} \cup g_2^{=0}$ is a TDRDF on G with $g_1(v) = 0$, function $g_2 = g_1^{=0} \cup g_2^{=1}$ is a TDRDF on G with $g_2(v) = 0$, the restriction of $g_3 = h_1^{=0,=2} \cup g_2^{=2}$ to G is a TDRDF on G with $g_3(v) = 0$ and $g_4 = g_1^{-v} \cup g_2^{=3} \cup 0_v$ is a TDRDF on G with $g_4(v) = 0$. Hence, $\gamma_{tdR}(G, v = 0) \leq \gamma_{tdR}$. This, together with $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 0)$, completes the proof of part (i).

Let $f(v) = 1$, let $z \neq w$ be a vertex not in $V(G)$ and let $\gamma_{tdR} = \min\{\gamma_{tdR}(H_1, v = 1) + \gamma_{tdR}(H_2, u = 0), \gamma_{tdR}(H_1, v = 1) + \gamma_{tdR}(H_2, u = 1), \gamma_{tdR}(H_1, v = 1, w = 2) + \gamma_{tdR}(H_2, u = 2, w = 1) - 3, \gamma_{tdR}(H_1, v = 1, w = 3) + \gamma_{tdR}(H_2, u = 3, w = 1) - 4\}$. So, f_1 is a TDRDF on H_1 with $f_1(v) = 1$ and f_2 is a TDRDF on H_2 with $f_2(u) = 0$, function f_1 is a TDRDF on H_1 with $f_1(v) = 1$ and f_2 is a TDRDF on H_2 with $f_2(u) = 1$, function $h_1 = f_1 \cup 2_w$ is a TDRDF on $H_1 + vw$ with $h_1(v) = 1$ and $h_1(w) = 2$ and $h_2 = f_2 \cup 1_z$ is a TDRDF on $H_2 + uz$ with $h_2(u) = 2$ and $h_2(z) = 1$ or $h_3 = f_1 \cup 3_w$ is a TDRDF on $H_1 + vw$ with $h_3(v) = 1$ and $h_3(w) = 3$ and $h_4 = f_2 \cup 1_z$ is a TDRDF on $H_2 + uz$ with $h_4(u) = 3$ and $h_4(z) = 1$. Hence, $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 1)$. Conversely, $g_1 = g_1^{=1} \cup g_2^{=0}$ is a TDRDF on G with $g_1(v) = 1$, function $g_2 = g_1^{=1} \cup g_2^{=1}$ is a TDRDF on G with $g_1(v) = 1$, the restriction of $g_3 = g_1^{=1,=2} \cup g_2^{=2,=1}$ to G is a TDRDF on G with $g_3(v) = 1$ and the restriction of $g_4 = g_1^{=1,=3} \cup g_2^{=3,=1}$ is a TDRDF on G with $g_3(v) = 1$. Hence, $\gamma_{tdR}(G, v = 1) \leq \gamma_{tdR}$. This, together with $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 1)$, completes the proof of part (ii).

Let $f(v) = 2$, let $z \neq w$ be a vertex not in $V(G)$ and let $\gamma_{tdR} = \min\{\gamma_{tdR}(H_1, v = 2) + \gamma'_{tdR}(H_2, u = 0, w = 2) - 2, \gamma_{tdR}(H_1, v = 2, w = 1) + \gamma_{tdR}(H_2, u = 1, w = 2) - 3, \gamma_{tdR}(H_1, v = 2, w = 2) + \gamma_{tdR}(H_2, u = 2, w = 2) - 4, \gamma_{tdR}(H_1, v = 2, w = 3) + \gamma_{tdR}(H_2, u = 3, w = 2) - 5\}$. So, f_1 is a TDRDF on H_1 with $f_1(v) = 2$ and $h_1 = f_2 \cup 2_w$ is a DRDF on $H_2 + uw$ such that the restriction of h_1 to $H_2 - u$ is a TDRDF on $H_2 - u$, $h_1(w) = 2$ and $h_1(u) = 0$, function $h_2 = f_1 \cup 1_w$ is a TDRDF on $H_1 + vw$ with $h_2(v) = 2$ and $h_2(w) = 1$ and $h_3 = f_2 \cup 2_z$ is a TDRDF on $H_2 + uz$ with $h_3(u) = 1$ and $h_3(z) = 2$, function $h_4 = f_1 \cup 2_w$ is a TDRDF on $H_1 + vw$ with $h_4(v) = 2$ and $h_4(w) = 2$ and $h_5 = f_2 \cup 2_z$ is a TDRDF on $H_2 + uz$ with $h_5(u) = 2$ and $h_5(z) = 2$ or $h_6 = f_1 \cup 3_w$ is a TDRDF on $H_1 + vw$ with $h_6(v) = 2$ and $h_6(w) = 3$ and $h_7 = f_2 \cup 2_z$ is a TDRDF on $H_2 + uz$ with $h_7(u) = 3$ and $h_7(z) = 2$. Hence, $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 3)$. Conversely, the restriction of $g_1 = g_1^{=2} \cup h_2^{=0,=2}$ to G is a TDRDF on G with $g_1(v) = 2$, the restriction of $g_2 = g_1^{=2,=1} \cup g_2^{=1,=2}$ to G is a TDRDF on G with $g_2(v) = 2$, the restriction of $g_3 = g_1^{=2,=2} \cup g_2^{=2,=2}$ to G is a TDRDF on G with $g_3(v) = 2$ and the restriction of $g_4 = g_1^{=2,=3} \cup g_2^{=3,=2}$ to G is a TDRDF on G with $g_4(v) = 2$. Hence, $\gamma_{tdR}(G, v = 3) \leq \gamma_{tdR}$. This, together with $\gamma_{tdR} \leq \gamma_{tdR}(G, v = 2)$, completes the proof of part (iii).

Let $f(v)=3$, let $z\neq w$ be a vertex not in $V(G)$ and let $\gamma_{tdR}=\min\{\gamma_{tdR}(H_1,v=3)+\gamma_{tdR}(H_2-u),\gamma_{tdR}(H_1,v=3,w=1)+\gamma_{tdR}(H_2,u=1,w=3)-4,\gamma_{tdR}(H_1,v=3,w=2)+\gamma_{tdR}(H_2,u=2,w=3)-5,\gamma_{tdR}(H_1,v=3,w=3)+\gamma_{tdR}(H_2,u=3,w=3)-6\}$. So, f_1 is a TDRDF on H_1 with $f_1(v)=3$ and f_2^{-u} is a TDRDF on H_2-u, function $h_2=f_1\cup 1_w$ is a TDRDF on H_1+vw with $h_2(v)=3$ and $h_2(w)=1$ and $h_3=f_2\cup 3_z$ is a TDRDF on H_2+uz with $h_3(u)=1$ and $h_3(z)=3$, function $h_4=f_1\cup 2_w$ is a TDRDF on H_1+vw with $h_4(v)=3$ and $h_4(w)=2$ and $h_5=f_2\cup 3_z$ is a TDRDF on H_2+uz with $h_5(u)=2$ and $h_5(z)=3$ or $h_6=f_1\cup 3_w$ is a TDRDF on H_1+vw with $h_6(v)=3$ and $h_6(w)=3$ and $h_7=f_2\cup 3_z$ is a TDRDF on H_2+uz with $h_7(u)=3$ and $h_7(z)=3$. Hence, $\gamma_{tdR}\leq\gamma_{tdR}(G,v=3)$. Conversely, $g_1=g_1^{=3}\cup g_2^{-u}\cup 0_u$ is a TDRDF on G with $g_1(v)=3$, the restriction of $g_2=g_1^{=3,=1}\cup g_2^{=1,=3}$ to G is a TDRDF on G with $g_2(v)=3$, the restriction of $g_3=g_1^{=3,=2}\cup g_2^{=2,=3}$ to G is a TDRDF on G with $g_3(v)=3$ and the restriction of $g_4=g_1^{=3,=3}\cup g_2^{=3,=3}$ to G is a TDRDF on G with $g_4(v)=3$. Hence, $\gamma_{tdR}(G,v=3)\leq\gamma_{tdR}$. This, together with $\gamma_{tdR}\leq\gamma_{tdR}(G,v=3)$, completes the proof of part (iv).

Similarly, we can prove parts $(v)-(viii)$.

Since $G-v=(H_1-v)\cup H_2$ and graphs H_1-v and H_2 are disjoint, $\gamma_{tdR}(G-v)=\gamma_{tdR}(H_1-v)+\gamma_{tdR}(H_2)=\gamma_{tdR}(H_1-v)+\min\{\gamma_{tdR}(H_2,u=0),\gamma_{tdR}(H_2,u=1),\gamma_{tdR}(H_2,u=2),\gamma_{tdR}(H_2,u=3)\}$. This completes the proof of part (ix).

We say that a rooted tree T with the vertex set $V=\{v_1,v_2,\ldots,v_n\}$ has Property 1 if $j<i$, where $v_j\in V$ is the parent of $v_i\in V$.

Theorem 1. Let T be a tree. Algorithm **TDRDNT**(T) computes the total double Roman domination number of T in linear time.

Proof. Let f be a $\gamma_{tdR}(T)$-function and let $v\in V(T)$. Clearly, $f(v)\in\{0,1,2,3\}$. So, $\gamma_{tdR}(T)=\min\{\gamma_{tdR}(T,v=0),\gamma_{tdR}(T,v=1),\gamma_{tdR}(T,v=2),\gamma_{tdR}(T,v=3)\}$. We can compute a rooted tree T_v with the root v and Property 1 for T in linear time. Clearly, $\gamma_{tdR}(T)=\gamma_{tdR}(T_v)$. Let u be a child of v in T_v and let T_u be the subtree of T_v with the root u. Clearly, T_u is a rooted tree with Property 1. Since T_v has Property 1, Algorithm **TDRDNT**(T) considers T_u before T_v. If T_u is only a vertex, then in Lines 2-4 of Algorithm **TDRDNT**(T) computes values $(i)-(ix)$ of Lemma 1 correctly. So, by Lemma 1, Algorithm **TDRDNT**(T) computes values $(i)-(ix)$ of Lemma 1 for vertex v correctly. Since Algorithm **TDRDNT**(T) returns $\min\{\gamma_1^0,\gamma_1^1,\gamma_1^2,\gamma_1^3\}$, it returns $\gamma_{tdR}(T_v)$, that is, the total double Roman domination number of T.

Clearly, the running time of each iteration of the **for** loops of Algorithm **TDRDNT**(T) is $\mathcal{O}(1)$ and so the running time of Algorithm **TDRDNT**(T) is linear. This completes the proof.

Algorithm 2.1: TDRDNT(T)

Input: A tree T of order n.
Output: The total double Roman domination number of T.

Compute a rooted tree $T' = (V, E)$ with $V = \{v_1, \cdots, v_n\}$ and Property 1.
for $(i \in \{1, \ldots, n\}) \wedge (a \in \{0, 1, 2, 3\}) \wedge (b, c \in \{1, 2, 3\})$ **do**
  $\gamma_i^a = \gamma_i'(02) = \infty$;
  $\gamma_i^{bc} = b + c$;
  $\gamma(v_i) = 0$;
for $i = n$ **to** 2 **do**
  Let v_j be the parent of v_i;
  $\gamma_j^0 = \min\{\gamma_j^0 + \gamma_i^0, \gamma_j^0 + \gamma_i^1, \gamma_j'(02) + \gamma_i^2 - 2, \gamma(v_j) + \gamma_i^3\}$;
  $\gamma_j^1 = \min\{\gamma_j^1 + \gamma_i^0, \gamma_j^1 + \gamma_i^1, \gamma_j^{12} + \gamma_i^{21} - 3, \gamma_j^{13} + \gamma_i^{31} - 4\}$;
  $\gamma_j^2 = \min\{\gamma_j^2 + \gamma_i'(02) - 2, \gamma_j^{21} + \gamma_i^{12} - 3, \gamma_j^{22} + \gamma_i^{22} - 4, \gamma_j^{23} + \gamma_i^{32} - 5\}$;
  $\gamma_j^3 = \min\{\gamma_j^3 + \gamma(v_i), \gamma_j^{31} + \gamma_i^{13} - 4, \gamma_j^{32} + \gamma_i^{23} - 5, \gamma_j^{33} + \gamma_i^{33} - 6\}$;
  $\gamma(v_j) = \gamma(v_j) + \min\{\gamma_i^0, \gamma_i^1, \gamma_i^2, \gamma_i^3\}$;
  $\gamma_j'(02) = \min\{\gamma_j'(02) + \gamma_i^0, \gamma_j'(02) + \gamma_i^1, \gamma(v_j) + \gamma_i^2 + 2, \gamma(v_j) + \gamma_i^3 + 2\}$;
  for $(b \in \{2, 3\}) \wedge (a \in \{1, 2, 3\})$ **do**
    $\gamma_j^{1b} = \min\{\gamma_j^{1b} + \gamma_i^0, \gamma_j^{1b} + \gamma_i^1, \gamma_j^{1b} + \gamma_i^{21} - 1, \gamma_j^{1b} + \gamma_i^{31} - 1\}$;
    $\gamma_j^{2a} = \min\{\gamma_j^{2a} + \gamma_i'(02) - 2, \gamma_j^{2a} + \gamma_i^{12} - 2, \gamma_j^{2a} + \gamma_i^{22} - 2, \gamma_j^{2a} + \gamma_i^{32} - 2\}$;
    $\gamma_j^{3a} = \min\{\gamma_j^{3a} + \gamma(v_i), \gamma_j^{3a} + \gamma_i^{13} - 3, \gamma_j^{3a} + \gamma_i^{23} - 3, \gamma_j^{3a} + \gamma_i^{33} - 3\}$;
return $\min\{\gamma_1^0, \gamma_1^1, \gamma_1^2, \gamma_1^3\}$

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