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First Principles Derivation of Displacement and Stress Function for Three-Dimensional Elastostatic Problems, and Application to the Flexural Analysis of Thick Circular Plates

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ABSTRACT

In this study, stress and displacement functions of the three-dimensional theory of elasticity for homogeneous isotropic bodies are derived from first principles from the differential equations of equilibrium, the generalized stress – strain laws and the geometric relations of strain and displacement. It is found that the stress and displacement functions must be biharmonic functions. The derived functions are used to solve the elasticity problem of finding stresses and displacement fields in a thick circular plate with clamped edges for the case of uniformly distributed transverse load over the plate domain. Superposition of second to sixth order Legendre polynomials which are biharmonic functions are used in the thick circular plate problem as the stress function with the unknown constants as the parameters to be determined. Use of the stresses and displacement fields derived in terms of the stress and displacement function yielded the stress fields and displacement fields in terms of the unknown constants of the biharmonic stress function. Enforcement of the boundary conditions yielded the unknown constants, leading to a complete determination of the stress and displacement function for the stress fields and the displacement fields. The solutions obtained are comparable to solutions in the technical literature.

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1. Introduction

1.1 Background

The classical three-dimensional (3D) linear elasticity problems of the mathematical theory of elasticity for deformable solids are governed in general by a system of fifteen coupled partial differential equations of equilibrium comprising as follows: three differential equations of equilibrium, six generalized stress – strain equations (describing the material constitutive laws), and six strains – displacement relations (kinematic relations) [1 – 15].

The solution of 3D elasticity problems thus requires the solution of the system of fifteen governing equations subject to the appropriate boundary conditions involving stresses and displacement field components. Simplified formulations of elasticity problems had been presented previously using displacement-based, stress-based and mixed methods. [8 – 11, 16, 17]. Displacement-based methods of formulation entail expressing the field equations of elasticity such that the displacement components are the unknown variables by the

elimination of the six Cauchy stress components, and the six strain components from the governing equations. Consequently, the system of equations reduces to three coupled partial differential equations for 3D problems and two coupled partial differential equations for two-dimensional elasticity problems [14, 18].

Stress-based methods of formulation entail the expression of the field equations of elasticity in terms of the six Cauchy stress components by the elimination of the six strain components and the three displacement components. Consequently, the governing equations reduce to a set of six coupled partial differential equations for 3D problems called the Beltrami – Michell stress equations. For two-dimensional elasticity problems, the stress-based methods are a set of three coupled partial differential equations, which would be easier to solve than the original set of field equations.

Mixed methods of formulation entail the expression of the field equations such that the unknowns are some components of stress and some displacement components. Mixed methods are not commonly applied in the literature. Analytical solutions to

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the partial differential equations of stress formulation of elasticity problems called stress functions had been developed and used by various researchers namely Airy, Maxwell, Navier, Beltrami, [7], [16, 17, 19].

Analytical solutions to the equations of the displacement-based formulations of elasticity problems, commonly called displacement functions had been developed and used by various researchers [8 – 14, 16 – 18], [20, 21].

In this study, stress and displacement functions are developed for 3D elasticity problems, and then used to solve the 3D elasticity problem of thick circular plates under uniformly distributed transverse load.

1.2 Review of circular plates under flexure

Circular plates are extensively used in various structural forms in aerospace, civil, mechanical, structural, naval, marine and geotechnical engineering [21]. They can be subjected to various kinds of static or dynamic loads. They are classified according to the ratio of their diameter, D to thickness, t, as thin plates $(D/t > 100)$, moderately thick plates $(20 < D/t < 100)$, and thick plates $(D/t < 3)$.

They are also classified according to their material properties as: isotropic, homogeneous, non–isotropic, non–homogeneous, anisotropic, orthotropic (transversely isotropic) and laminated plates.

Theories that have been used to study the flexural behaviour of circular plates are; the classical Kirchhoff thin plate theory, Mindlin first order shear deformation plate theory [22 – 36](which is a stress-based plate theory), Von Karman large deformation plate theory [37], Reddy's third order shear deformation plate theory, Shimpi plate theory and theory of elasticity methods for thick plates.

The classical Kirchhoff – Love plate theory is based on the following basic assumptions [19], [24, 25], [30 – 34], [38 – 41].

(i) straight lines that are initially orthogonal to the middle surface of the plate before flexure remain straight and orthogonal to the middle surface after flexural deformation, and they remain unchanged in length.

(ii) the displacement is so small as compared to the thickness of the plate, hence the slope at the deflected middle surface is very small, the square of the slope is negligible when compared to unity.

(iii) the normal stresses and in-plane shear stresses are assumed to be zero at the middle surface of the plate.

(iv) the middle surface is unstrained after flexural deformation, and remains a neutral surface.

(v) transverse normal stress is very small compared to other stress components and may be ignored in the stress – strain relations without introduction of significant errors.

Kirchhoff – Love's classical plate theory for circular plates has been found to give satisfactory results for thin circular plates under small deformations; but gives unsatisfactory results for moderately thick and thick circular plates.

The major inadequacy of the classical Kirchhoff – Love plate theory for circular plates is the disregard for shear deformation in the formulation of the governing equations, which consequently makes the theory incapable of precise description of the flexural behaviours of moderately thick and thick plates, where shear

deformation significantly affects the flexural behaviour [27, 28], [31, 32, 34, 40 – 42].

1.3 Review of the theory of elasticity applied to thick plate problems

In the classical theory of elasticity, the problem of equilibrium of an isotropic thick plate is reduced to the determination of six components of normal and shear stresses σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{yz}, τ_{xz}, and three displacement field components u, v and w which strictly satisfy all the differential equations of the theory of elasticity as well as the boundary conditions on the surface of the plate [43]. The conditions on the lateral surfaces (edges) of the plate are required to be satisfied only approximately.

Timoshenko and Goodier [44] used the rigorous methods of the mathematical theory of elasticity to solve the problem of homogeneous isotropic thick circular plates with simply supported edges and subjected to uniformly distributed load over the entire plate domain.

Ding et al [45] solved the problem of linear elastic, homogeneous, isotropic thick circular plates subjected to axially symmetric loads under two types of edge support conditions. They presented analytical solutions for the two types of problems studied.

Luo et al [46] studied the problem of thick circular plates with clamped edges by considering displacements as the primary variables in a displacement formulation of the three-dimensional elasticity problem. Their analysis involved the solution of ordinary differential equations, which rendered it complicated. Another limitation of their work was that they considered only one type of edge support condition.

Ding et al [47 – 49] studied the elasticity problem of homogeneous, isotropic linear elastic thick beams with both ends fixed for the case of uniformly distributed load over the entire beam span; and presented new solutions for the plane elasticity problem.

Lekhniskii [37] studied the problems in the theory of elasticity concerning the flexural analysis of anisotropic, thick circular plates with clamped and simply supported edges for the case of distributed transverse loads.

Ding et al [47, 48] studied the elasticity problem of the flexural behaviour of linearly elastic, homogeneous, isotropic thick circular plates by using the Love harmonic functions for the case of uniformly distributed load over the entire domain of the plate. They presented numerical solutions which they compared with solutions with the finite element method (for the same problems) and the classical Kirchhoff – Love small deflection plate theory, and made some findings.

Timoshenko and Woinowsky – Krieger [50] used the classical Kirchhoff – Love small deformation thin plate theory to solve the problem of linearly elastic, homogeneous, isotropic thin circular plates subject to a uniformly distributed load over the entire plate domain. They solved the problem for the two cases of simply supported edges and fixed edges.

Tseng and Tarn [51] presented an analysis of axisymmetric flexure of thick circular plates using the mathematical theory of elasticity. From the basic principles of the Hamiltonian state space approach, they found a rigorous analytical solution by the use of the method of separation of variables and symplectic eigenfunction expression in a systematic formulation. They evaluated the effect of thickness on the flexure of circular plates

and the applicability of the classical Kirchhoff – Love circular plate solutions.

Sundara Raja Iyengar et al [52] presented a higher order theory for thick axisymmetric circular plates which is an extension of the Reissner shear deformation theory using variational techniques. They assumed stress and displacement fields that satisfy the physical requirements of the problem as closely as possible. Thus, using an assumed displacement field in the form of a higher order truncated polynomial in the thickness coordinate (the coefficients being arbitrary functions of the other coordinate variables) and a consistent stress field satisfying the stress boundary conditions at the top and bottom surfaces of the plate, the governing domain (field) equations of equilibrium and the associated boundary conditions along the edges were obtained using the variational theorem of Reissner. They obtained resulting equations that are such that the symmetric and antisymmetric parts can be considered separable. Their procedure of considering higher order truncated polynomials for displacements and stress fields appears to be novel for thick plates [52].

Sundara Raja Iyengar et al [53] used the method of initial functions for the analysis of thick circular plates. In the method, the governing equations of thick circular plates are derived from the three-dimensional (3D) equations of the mathematical theory of elasticity in cylindrical polar coordinates system using a Maclaurin series expansion in the thickness coordinate for the unknowns. The resulting formulation is in a form that is specifically amenable to consistent reduction to yield approximate theories of any desired order. They compared numerical results obtained using the method of initial functions with those of the theory of elasticity for problems of simply supported circular plate under self equilibrating normal loads, and found agreement with the theory of elasticity solutions. They also solved the problem of thick circular plates with clamped edges subjected to uniformly distributed loads, and found agreement with the theory of elasticity solutions.

Li et al [54] considered the bending of transversely isotropic circular plates with elastic compliance coefficients being arbitrary functions of the thickness coordinate, subject to distributed transverse load. The differential equation satisfied by stress functions for the particular problem considered was derived. They presented a rigorous analytical technique for deriving these stress functions from which the closed form expressions for the internal forces were found by integration of the differential equations. They then applied the method to solve transversely isotropic functionally graded circular plate carrying uniformly distributed load, illustrating the determination of integration constants from the boundary conditions. They obtained solutions for simply supported and clamped circular plates and found that when degenerated, the solutions become identical with the solutions for homogeneous, isotropic circular case. By appropriate superposition and manipulation of the equations, the governing domain (field) equations in higher order shear stress coefficients were obtained for both symmetric and antisymmetric elasticity problems. The solution to the equations is observed to be complete as all other unknowns can be found or expressed in terms of the shear stress coefficients. They applied the method to uniformly loaded solid and annular circular plates.

Ike et al [26] derived from first principles the governing field equations for the flexure of linearly elastic, isotropic homogeneous thick circular plates under static loading using variational calculus methods. The total potential energy functional for the thick circular plate which is the sum of the given energy functional and the load potential functional was obtained using the generalized stress – strain relations, which account for the shear deformation and the axisymmetrical nature of the load and plate. Euler – Lagrange differential equations of equilibrium were used to obtain the differential equations of equilibrium. The total potential energy functional was extremized to obtain the governing equations and the boundary conditions. The governing partial differential equations were integrated to obtain the deflection of the thick circular plate, which was found to be decomposed into deflection due to pure flexure and deflection due to shear deformation. They found that for shear stress coefficient $k = 4/3$ which corresponds to parabolic distribution of shear stress over the plate thickness, the deflection becomes identical with the deflection obtained using the rigorous methods of the theory of elasticity. They solved the particular problem of thick circular plate with clamped edges for the case of uniformly distributed load and found the maximum deflection to occur at the plate centre. They also found that shear deformation makes a significant contribution to the maximum deflection when the thickness/radius radio is greater than 0.05 (1/20) which agrees with the technical literature.

Ike [21] used the mathematical technique of separation of variables to solve the problem of first order shear deformable circular plate under transverse axisymmetric load. The problem was defined as a boundary value problem of a system of differential equations in terms of the stress resultants and the stress – resultants – displacement relations. The set of equations were considered simultaneously to express them in variable – separable form. The mathematical technique of separation of variables was then employed to obtain the unknown generalized displacements. Specific problems of circular thick plates with simply supported and clamped edges under uniformly distributed load and concentrated load applied at the centre were considered and solved using the same technique of separation of variables. It was found in all cases that the deflection was expressed in terms of flexural and shear deflection components. It was also found that the maximum deflection occurs at the centre as is logical from the symmetrical nature of the problem. The study further found that the shear component of the transverse deflection increased significantly with increase in the ratio of the plate thickness to radius (h/r_0). Other relevant studies on plates and elasticity include Chandrashekhara [55], Elliot [56], Hu [57] and Shimpi [58].

Recent research publications on elasticity theory include: Danesh et al [59], Mohammadi et al [60], Mohammadi and Rastgoo [61, 62], Safarabadi et al [63], Asemi et al [64], Mohammadi et al [65] and Goodarzi et al [66].

1.4 Research aim and objectives

The aim of this research is to present a first principle derivation of stress and displacement functions for three-dimensional elastostatic problems, and then apply the functions derived to the flexural analysis problem of thick circular plates. The research objectives are as follows:

(i)   to derive the displacement functions that simultaneously satisfy the stress – strain relations, the geometric equations of strain and the differential equations of equilibrium for small displacement elasticity problems in three dimensions for homogeneous, isotropic materials.

(ii)   to derive the corresponding stress functions.

(iii)   to apply the stress and displacement functions derived to solve the flexural problem of thick circular plates with

clamped edges subjected to uniformly distributed transverse load, and determine the stresses and displacement fields in the plate due to the applied load.

2. Theoretical framework

The governing field equations of three-dimensional (3D) elasticity for isotropic, linear elastic, homogeneous materials are developed from the three basic requirements, namely: generalized Hooke's stress – strain law, the kinematic relations of strains and displacements, and the differential equations of equilibrium.

2.1 Generalized Hooke's stress strain law

The generalized Hooke's stress – strain law for 3D elasticity problems for linear elastic, isotropic homogeneous materials can be expressed by the six relations:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \mu\sigma_{yy} - \mu\sigma_{zz}) \tag{1}$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \mu\varepsilon_{xx} - \mu\varepsilon_{zz}) \tag{2}$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \mu\sigma_{xx} - \mu\sigma_{yy}) \tag{3}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\tau_{xy}}{E/2(1+\mu)} \tag{4}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\tau_{yz}}{E/2(1+\mu)} \tag{5}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{\tau_{xz}}{E/2(1+\mu)} \tag{6}$$

where ε_{xx}, ε_{yy}, ε_{zz} are normal strains in the *x, y* and *z* Cartesian coordinate directions respectively, γ_{xy}, γ_{yz}, γ_{xz} are shear strains, σ_{xx}, σ_{yy}, σ_{zz} are normal stresses in the *x, y, z* Cartesian coordinate directions respectively. τ_{xy}, τ_{yz}, τ_{xz} are shear stresses. μ is the Poisson's ratio of the elastic material, *E* is the Young's modulus of elasticity of the material, and *G* is the shear modulus, or modulus of rigidity. *G* is expressed in terms of *E* and μ as:

$$G = \frac{E}{2(1+\mu)} \tag{7}$$

2.2 Kinematic relations (equations)

For small displacement elasticity problems, also called linear elasticity or infinitesimal displacement problems of elasticity, the strain – displacement (kinematic) equations are given by the following six linear partial differential equations (PDFs) relating strains and displacement fields:

$$\varepsilon_{xx} = \frac{\partial u(x, y, z)}{\partial x} \tag{8}$$

$$\varepsilon_{yy} = \frac{\partial v(x, y, z)}{\partial y} \tag{9}$$

$$\varepsilon_{zz} = \frac{\partial w(x, y, z)}{\partial z} \tag{10}$$

$$\gamma_{xy} = \frac{\partial u(x, y, z)}{\partial y} + \frac{\partial v(x, y, z)}{\partial x} \tag{11}$$

$$\gamma_{yz} = \frac{\partial v(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial y} \tag{12}$$

$$\gamma_{xz} = \frac{\partial u(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial x} \tag{13}$$

where *u(x, y, z)*, *v(x, y, z)* and *w(x, y, z)* are the *x, y* and *z* Cartesian components of the displacement field, while *x, y, z* are the Cartesian coordinates.

2.3 Differential equations of equilibrium

The differential equations of equilibrium are given for the general case of 3D elasticity problems – in statics and dynamics – by the set of three partial differential equations:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + f_x = \rho\frac{\partial^2 u}{\partial t^2} = \rho\ddot{u} \tag{14}$$

$$\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + f_y = \rho\frac{\partial^2 v}{\partial t^2} = \rho\ddot{v} \tag{15}$$

$$\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + f_z = \rho\frac{\partial^2 w}{\partial t^2} = \rho\ddot{w} \tag{16}$$

where *t* denotes time, f_x, f_y and f_z are the body force components in the *x, y* and *z* Cartesian coordinate directions respectively; the dots over *u, v, w* denote time derivatives of *u, v, w* respectively, and ρ is the density.

For elastostatic problems where the body force components are disregarded, $f_x = f_y = f_z = 0$, and $\ddot{u} = \ddot{v} = \ddot{w} = 0$, the differential equations of equilibrium become:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0 \tag{17}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0 \tag{18}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0 \tag{19}$$

3. First principles derivation of displacement functions for 3D elastostatic problems

3.1 Generalized 3D Hooke's strain – strain equations for homogeneous isotropic bodies

The generalized stress – strain equations for isotropic, homogeneous, linear elastic materials are given by the six equations:

$$\sigma_{xx} = \frac{(1-\mu)E\varepsilon_{xx}}{(1+\mu)(1-2\mu)} + \frac{\mu E\varepsilon_{yy}}{(1+\mu)(1-2\mu)} + \frac{\mu E\varepsilon_{zz}}{(1+\mu)(1-2\mu)} \tag{20}$$

$$\sigma_{yy} = \frac{\mu E\varepsilon_{xx}}{(1+\mu)(1-2\mu)} + \frac{(1-\mu)E\varepsilon_{yy}}{(1+\mu)(1-2\mu)} + \frac{\mu E\varepsilon_{zz}}{(1+\mu)(1-2\mu)} \tag{21}$$

$$\sigma_{zz} = \frac{\mu E\varepsilon_{xx}}{(1+\mu)(1-2\mu)} + \frac{\mu E\varepsilon_{yy}}{(1+\mu)(1-2\mu)} + \frac{(1-\mu)E\varepsilon_{zz}}{(1+\mu)(1-2\mu)} \tag{22}$$

$$\tau_{xy} = G\gamma_{xy} \tag{23}$$

$$\tau_{yz} = G\gamma_{yz} \tag{24}$$

$$\tau_{xz} = G\gamma_{xz} \tag{25}$$

The elastic constants in the stress – strain relations are denoted by E_1 and E_2 where:

$$E_1 = \frac{(1-\mu)E}{(1+\mu)(1-2\mu)} = \frac{2G(1-\mu)}{1-2\mu} \tag{26}$$

$$E_2 = \frac{\mu E}{(1+\mu)(1-2\mu)} = \frac{2G\mu}{1-2\mu} \tag{27}$$

Then the stress – strain relations involving the normal stresses become:

$$\sigma_{xx} = E_1\varepsilon_{xx} + E_2\varepsilon_{yy} + E_2\varepsilon_{zz} \tag{28}$$

$$\sigma_{yy} = E_2\varepsilon_{xx} + E_1\varepsilon_{yy} + E_2\varepsilon_{zz} \tag{29}$$

$$\sigma_{zz} = E_2\varepsilon_{xx} + E_2\varepsilon_{yy} + E_1\varepsilon_{zz} \tag{30}$$

3.2 Stress – displacement equations

The stress – displacement equations are obtained by substitution of Equations (8 – 13) into Equations (28 – 30), and (23 – 25) as the following six PDEs:

$$\sigma_{xx} = E_1\frac{\partial u}{\partial x} + E_2\frac{\partial v}{\partial y} + E_2\frac{\partial w}{\partial z} \tag{31}$$

$$\sigma_{yy} = E_2\frac{\partial u}{\partial x} + E_1\frac{\partial v}{\partial y} + E_2\frac{\partial w}{\partial z} \tag{32}$$

$$\sigma_{zz} = E_2\frac{\partial u}{\partial x} + E_2\frac{\partial v}{\partial y} + E_1\frac{\partial w}{\partial z} \tag{33}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{34}$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{35}$$

$$\tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{36}$$

3.3 Displacement formulation of the differential equations of equilibrium

The differential equations of equilibrium Equations (17 – 19) are expressed in displacement terms using Equations (31 – 36) to obtain, after simplifications:

$$E_1\frac{\partial^2 u}{\partial x^2} + G\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}\left((E_2+G)\frac{\partial v}{\partial y} + (E_2+G)\frac{\partial w}{\partial z}\right) = 0 \tag{37}$$

$$E_1\frac{\partial^2 v}{\partial y^2} + G\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial z^2}\right) + \frac{\partial}{\partial y}\left((E_2+G)\frac{\partial u}{\partial x} + (E_2+G)\frac{\partial w}{\partial z}\right) = 0 \tag{38}$$

$$E_1\frac{\partial^2 w}{\partial z^2} + G\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial}{\partial z}\left((E_2+G)\frac{\partial u}{\partial x} + (E_2+G)\frac{\partial v}{\partial y}\right) = 0 \tag{39}$$

3.4 Derivation of displacement function φ(x, y, z)

We define the displacement components *u* and *v* in terms of the displacement function $\phi(x, y, z)$ as follows:

$$u = -\frac{\partial^2 \phi}{\partial x \partial z} \tag{40}$$

$$v = -\frac{\partial^2 \phi}{\partial y \partial z} \tag{41}$$

We then seek to find the third vertical displacement component in terms of φ(x, y, z) such that the governing field equations are satisfied identically. By substitution of Equations (40) and (41) into the displacement formulation of the differential equations of equilibrium, Equations (37 – 39), we obtain as follows:

From Equation (37),

$$E_1\frac{\partial^2}{\partial x^2}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + G\frac{\partial^2}{\partial y^2}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + G\frac{\partial^2}{\partial z^2}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) +$$
$$(E_2+G)\frac{\partial^2}{\partial x\partial y}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + (E_2+G)\frac{\partial^2 w}{\partial x\partial z} = 0 \tag{42}$$

Simplifying,

$$(E_2+G)\frac{\partial^2 w}{\partial x\partial z} = E_1\frac{\partial^4\phi}{\partial x^3\partial z} + G\frac{\partial^4\phi}{\partial x\partial y^2\partial z} + G\frac{\partial^4\phi}{\partial x\partial z^3} +$$
$$(E_2+G)\frac{\partial^4\phi}{\partial x\partial y^2\partial z} \tag{43}$$

Integrating, we obtain:

$$w = \left(\frac{E_1}{E_2+G}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left(\frac{G}{E_2+G}\right)\frac{\partial^2\phi}{\partial z^2} \tag{44}$$

Alternatively,

$$w = 2(1-\mu)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + (1-2\mu)\frac{\partial^2\phi}{\partial z^2} \tag{45}$$

$$w = 2(1-\mu)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) - \frac{\partial^2\phi}{\partial z^2} \tag{46}$$

$$w = 2(1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2} \tag{47}$$

Identical result as Equation (47) is obtained from Equation (38). Using Equations (40), (41) and (47) in Equation (39) we obtain the biharmonic equation:

$$\nabla^4\phi(x, y, z) = 0 \tag{48}$$

3.5 Derivation of the stress functions

The stress functions are derived by substitution of the displacement function expressions for the three displacement components *u*, *v* and *w* into the corresponding stress displacement relations. Then,

$$\sigma_{xx} = E_1\frac{\partial}{\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + E_2\frac{\partial}{\partial y}\left(-\frac{\partial^2\phi}{\partial y\partial z}\right) +$$
$$E_2\frac{\partial}{\partial z}\left\{\frac{E_1}{E_2+G}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G}{E_2+G}\frac{\partial^2\phi}{\partial z^2}\right\} \tag{49}$$

Simplifying,

$$\sigma_{xx} = \frac{E_1 E_2}{E_2+G}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{E_2 G}{E_2+G}\frac{\partial^3\phi}{\partial z^3} - E_1\frac{\partial^3\phi}{\partial x^2\partial z} - E_2\frac{\partial^3\phi}{\partial y^2\partial z} \tag{50}$$

$$\sigma_{xx} = \frac{4G\mu(1-\mu)}{1-2\mu}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + 2G\mu\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial z^2} - \frac{2G(1-\mu)}{1-2\mu}\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial x^2} - \frac{2G\mu}{1-2\mu}\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial y^2} \tag{51}$$

Further simplifications yield:

$$\sigma_{xx} = 2G\left(\mu\frac{\partial}{\partial z}\nabla^2\phi - \frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial x^2}\right) = 2G\frac{\partial}{\partial z}\left(\mu\nabla^2\phi - \frac{\partial^2\phi}{\partial x^2}\right) \quad (52)$$

Also,

$$\sigma_{yy} = E_2\frac{\partial}{\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + E_1\frac{\partial}{\partial y}\left(-\frac{\partial^2\phi}{\partial y\partial z}\right) + E_2\frac{\partial}{\partial z}\left(\frac{E_1}{E_2+G}\right.$$
$$\left.\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G}{E_2+G}\frac{\partial^2\phi}{\partial z^2}\right) \quad (53)$$

Simplifying,

$$\sigma_{yy} = \frac{E_1 E_2}{E_2+G}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{E_2 G}{E_2+G}\frac{\partial^3\phi}{\partial z^3} -$$
$$E_2\frac{\partial^3\phi}{\partial x^2\partial z} - E_1\frac{\partial^3\phi}{\partial y^2\partial z} \quad (54)$$

$$\sigma_{yy} = \frac{4G\mu(1-\mu)}{1-2\mu}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + 2G\mu\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial z^2} -$$
$$\frac{2G\mu}{1-2\mu}\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial x^2} - \frac{2G(1-\mu)}{1-2\mu}\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial y^2} \quad (55)$$

Further simplifications yield:

$$\sigma_{yy} = 2G\left(\mu\frac{\partial}{\partial z}\nabla^2\phi - \frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial y^2}\right) = 2G\frac{\partial}{\partial z}\left(\mu\nabla^2\phi - \frac{\partial^2\phi}{\partial y^2}\right) \quad (56)$$

$$\sigma_{zz} = E_2\frac{\partial}{\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + E_2\frac{\partial}{\partial y}\left(-\frac{\partial^2\phi}{\partial y\partial z}\right) + E_1\frac{\partial}{\partial z}\left(\frac{E_1}{E_2+G}\right.$$
$$\left.\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G}{E_2+G}\frac{\partial^2\phi}{\partial z^2}\right) \quad (57)$$

Simplifying,

$$\sigma_{zz} = \left(\frac{E_1^2}{E_2+G} - E_2\right)\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{E_1 G}{E_2+G}\frac{\partial^3\phi}{\partial z^3} \quad (58)$$

$$\sigma_{zz} = G(4-2\mu)\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + 2(1-\mu)G\frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial z^2} \quad (59)$$

Further simplifications yield:

$$\sigma_{zz} = 2G\frac{\partial}{\partial z}\left((2-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right) \quad (60)$$

Also,

$$\tau_{xy} = -2G\frac{\partial^3\phi}{\partial x\partial y\partial z} \quad (61)$$

$$\tau_{xz} = G\left\{\frac{\partial}{\partial z}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{E_1}{E_2+G}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G}{E_2+G}\frac{\partial^2\phi}{\partial z^2}\right)\right\} \quad (62)$$

$$\tau_{xz} = \frac{E_1 G}{E_2+G}\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G^2}{E_2+G}\frac{\partial}{\partial x}\frac{\partial^2\phi}{\partial z^2} - G\frac{\partial^3\phi}{\partial x\partial z^2} \quad (63)$$

$$\tau_{xz} = 2G(1-\mu)\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + (1-2\mu)G\frac{\partial^3\phi}{\partial x\partial z^2} - G\frac{\partial^3\phi}{\partial x\partial z^2} \quad (64)$$

Further simplifications yield

$$\tau_{xz} = 2G\frac{\partial}{\partial x}\left((1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right) \quad (65)$$

Also,

$$\tau_{yz} = G\left\{\frac{\partial}{\partial z}\left(-\frac{\partial^2\phi}{\partial y\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{E_1}{E_2+G}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G}{E_2+G}\frac{\partial^2\phi}{\partial z^2}\right)\right\} \quad (66)$$

Simplifying,

$$\tau_{yz} = \frac{E_1 G}{E_2+G}\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{G^2}{E_2+G}\frac{\partial}{\partial y}\frac{\partial^2\phi}{\partial z^2} - G\frac{\partial^3\phi}{\partial y\partial z^2} \quad (67)$$

Alternatively,

$$\tau_{yz} = 2G(1-\mu)\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + (1-2\mu)G\frac{\partial}{\partial y}\frac{\partial^2\phi}{\partial z^2} - G\frac{\partial^3\phi}{\partial y\partial z^2} \quad (68)$$

Further simplifications yield:

$$\tau_{yz} = 2G\frac{\partial}{\partial y}\left((1-\mu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right) \quad (69)$$

3.6 Stress and displacement function for axisymmetric elasticity problems in cylindrical polar coordinates

The stress and displacement functions derived in this paper are expressed in cylindrical polar coordinates system as follows:

$$u_r = -\frac{\partial^2\phi(r,z)}{\partial r\partial z} \quad (70)$$

$$w = u_z = 2(1-\mu)\nabla^2\phi(r,z) - \frac{\partial^2\phi}{\partial z^2}(r,z) \quad (71)$$

$$\tau_{rz} = 2G\frac{\partial}{\partial r}\left((1-\mu)\nabla^2\phi(r,z) - \frac{\partial^2\phi(r,z)}{\partial z^2}\right) \quad (72)$$

$$\sigma_{rr} = 2G\frac{\partial}{\partial z}\left(\mu\nabla^2\phi(r,z) - \frac{\partial^2\phi(r,z)}{\partial r^2}\right) \quad (73)$$

$$\sigma_{\theta\theta} = 2G\frac{\partial}{\partial z}\left(\mu\nabla^2\phi(r,z) - \frac{1}{r}\frac{\partial\phi(r,z)}{\partial r}\right) \quad (74)$$

$$\sigma_{zz} = 2G\frac{\partial}{\partial z}\left((2-\mu)\nabla^2\phi(r,z) - \frac{\partial^2\phi(r,z)}{\partial z^2}\right) \quad (75)$$

where σ_{rr}, is the radial stress component; $\sigma_{\theta\theta}$ is the circumferential (hoop) stress component. σ_{zz} is the vertical stress component; τ_{rz} is the shear stress. u_r is the radial displacement component, $w = u_z$ is the vertical displacement component.

4. Application to the elasticity analysis of thick circular plates

The semi-inverse technique is applied to derive elasticity solutions to thick circular plate problems using the stress function derived and Legendre polynomials as the stress function. Stress function $\phi_2(r,z)$ from the second degree Legendre polynomials given by $f_{21}(r,z)$ and $f_{22}(r,z)$ is:

$$\phi_2(r,z) = c_{21}f_{21}(r,z) + c_{22}f_{22}(r,z) = c_{21}(2z^2 - r^2) + c_{22}(z^2 + r^2) \quad (76)$$

where c_{21} and c_{22} are constants.

The use of Equations (70 – 75) yield the stress and displacement fields for $\phi_2(r,z)$ as:

$$\tau_{rz}(r,z) = 0 \quad (77)$$
$$\sigma_{zz}(r,z) = 0 \quad (78)$$
$$\sigma_{rr}(r,z) = 0 \quad (79)$$
$$\sigma_{\theta\theta}(r,z) = 0 \quad (80)$$
$$u(r,z) = 0 \quad (81)$$
$$w(r,z) = -4c_{21} + c_{22}(10 - 12\mu) = b_2 \quad (82)$$

$$\frac{\partial w(r,z)}{\partial r} = 0 \tag{83}$$

Stress function $\phi_3(r,z)$ from the third degree Legendre polynomial $f_{31}(r,z)$ and $f_{32}(r,z)$ is

$$\phi_3(r,z) = c_{31}f_{31}(r,z) + c_{32}f_{32}(r,z) = c_{31}(2z^3 - 3r^2z) + c_{32}(r^2z + r^3) \tag{84}$$

where c_{31} and c_{32} are constants. The stress and displacement fields are obtained using Equations (70 – 75) as; Equations (77) and (83) and:

$$\sigma_{zz}(r,z) = 2G(-12c_{31} + (14 - 10\mu)c_{32}) \tag{85}$$

$$\sigma_{rr}(r,z) = 2G(6c_{31} + (10\mu - 2)c_{32}) \tag{86}$$

$$\sigma_{\theta\theta}(r,z) = 2G(6c_{31} + (10\mu - 2)c_{32}) \tag{87}$$

$$u_r(r,z) = 6rc_{31} - 2rc_{32} \tag{88}$$

$$w(r,z) = -12zc_{31} + (14 - 20\mu)zc_{32} \tag{89}$$

Stress function $\phi_4(r, z)$ from the fourth degree Legendre polynomials $f_{41}(r, z)$ and $f_{42}(r, z)$ is:

$$\phi_4(r,z) = c_{41}f_{41}(r,z) + c_{42}f_{42}(r,z) = c_{41}(8z^4 - 24r^2z^2 + 3r^4) + c_{42}(2z^4 + r^2z^2 - r^4) \tag{90}$$

The stress and displacement fields from $\phi_4(r, z)$ are obtained using Equations (70 – 75) as:

$$\tau_{rz}(r,z) = 2G(96rc_{41} - 2r(16 - 14\mu)c_{42}) \tag{91}$$

$$\sigma_{zz}(r,z) = 2G(-192zc_{41} + 4z(16 - 14\mu)c_{42}) \tag{92}$$

$$\sigma_{rr}(r,z) = 2G(96zc_{41} + 4z(14\mu - 1)c_{42}) \tag{93}$$

$$\sigma_{\theta\theta}(r,z) = 2G(96zc_{41} + 4z(4\mu - 1)c_{42}) \tag{94}$$

$$u(r,z) = 96rzc_{41} - 4rzc_{42} \tag{95}$$

$$w(r,z) = -(96z^2 - 48r^2)c_{41} + ((32 - 56\mu)z^2 - r^2(30 - 28\mu))c_{42} \tag{96}$$

$$\frac{\partial w}{\partial r}(r,z) = 96rc_{41} + (60 - 56\mu)rc_{42} \tag{97}$$

The stress function $\phi_6(r, z)$ from the sixth degree Legendre polynomials $f_{61}(r, z)$ and $f_{62}(r, z)$ is

$$\phi_6(r,z) = c_{61}f_{61}(r,z) + c_{62}f_{62}(r,z) = c_{61}\left(\frac{16}{3}z^6 - 40z^4r^2 + 30z^2r^4 - \frac{5}{3}r^6\right) + c_{62}(8z^6 - 16z^4r^2 - 21z^2r^4 + 3r^6) \tag{98}$$

The stress fields are obtained as:

$$\tau_{rz}(r,z) = 2G((960rz^2 - 240r^3)c_{61} + ((432 - 264\mu)r^2 + (1056\mu - 672)rz^2)c_{62}) \tag{99}$$

$$\sigma_{zz}(r,z) = 2G\{(960zr^2 - 640z^3)c_{61} + ((448 - 704\mu)z^3 + (1056\mu - 1728)zr^2)c_{62}\} \tag{100}$$

$$\sigma_{rr}(r,z) = 2G\{(320z^3 - 720zr^2)c_{61} + c_{62}(64(2 + 11\mu)z^3 + (504 - 1056\mu)zr^2)\} \tag{101}$$

$$\sigma_{\theta\theta}(r,z) = 2G\{(c_{61}(320z^3 - 240zr^2) + c_{62}(64z^3(2 + 11\mu) + (168 - 1056\mu)zr^2)\} \tag{102}$$

$$u(r,z) = c_{61}(168zr^3 + 128z^3r) + (320z^3r - 240zr^3)c_{62} \tag{103}$$

$$w(r,z) = -c_{61}(160z^4 - 480z^2r^2 + 60r^4) + c_{62}((112 - 352\mu)z^4 + (1056\mu - 864)z^2r^2 + (174 - 132\mu)r^4) \tag{104}$$

$$\frac{\partial w}{\partial r}(r,z) = c_{61}(960z^2r - 240r^3) + c_{62}((2112\mu - 1728)z^2r + 4r^3(174 - 132\mu)) \tag{105}$$

4.1 Application of stress function to the elasticity analysis of thick circular plate with clamped edges subject to uniformly distributed transverse load on the top surface (Solution A)

The stress function derived in this research is applied to the elasticity analysis of the problem of thick circular plate of radius R with clamped edges and under uniformly distributed transverse load of intensity p on the top surface as shown in Figure 1.
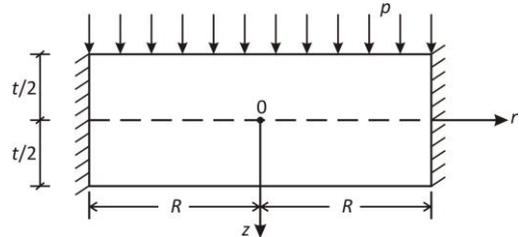

Figure 1: Thick circular plate of radius R subject to uniformly distributed transverse load, p on the top surface

The origin of the cylindrical polar coordinates system is chosen at the centre O of the plate as shown in Figure 1. The circular plate considered has a thickness denoted by t and the elasticity problem is identified to be axisymmetric since the problem and the loading are symmetrical about the z axis.

The boundary conditions are obtained from the deformation and stress conditions and also from the requirement of equilibrium of internal vertical stresses and the applied vertical load as follows:

$$u(r = \pm R, z = 0) = 0 \tag{106}$$

$$w(r = \pm R, z = 0) = 0 \tag{107}$$

$$\frac{\partial w}{\partial r}(r = \pm R, z = 0) = 0 \tag{108}$$

$$\tau_{rz}(r, z = t/2) = 0 \tag{109}$$

$$\tau_{rz}(r, z = -t/2) = 0 \tag{110}$$

$$\sigma_{rr}(r, z = \pm t/2) = 0 \tag{111}$$

$$\sigma_{\theta\theta}(r, z = \pm t/2) = 0 \tag{112}$$

$$\sigma_{zz}(r, z = t/2) = 0 \tag{113}$$

$$\sigma_{zz}(r, z = -t/2) = -p \tag{114}$$

The boundary conditions involving σ_{rr} and $\sigma_{\theta\theta}$ are identical, and we have seven unique equations.

4.2 Stress function $\Omega(r, z)$

The stress function $\Omega(r, z)$ used to solve the problem is obtained from a superposition of stress functions obtained from Legendre polynomials of degree from two to six as follows:

$$\Omega(r,z) = \phi_2(r,z) + \phi_3(r,z) + \phi_4(r,z) + \phi_6(r,z) \tag{115}$$

$$\Omega(r,z) = c_{21}f_{21}(r,z) + c_{22}f_{22}(r,z) + c_{31}f_{31}(r,z) +$$

$$c_{32}f_{32}(r,z)+c_{41}f_{41}(r,z)+c_{42}f_{42}(r,z)+$$
$$c_{61}f_{61}(r,z)+c_{62}f_{62}(r,z) \qquad (116)$$

where Equation (82) has reduced the unknown constants by one, and we have a total of seven unknown constants and seven boundary conditions.

The stress and displacement fields for $\Omega(r, z)$ are obtained as follows:

$$\tau_{rz}(r,z) = 2G\{96rc_{41}-2r(16-14\mu)c_{42}+(960rz^2-$$
$$240r^3)c_{61}+c_{62}((432-264\mu)r^3+(1056\mu-672)rz^2)\}$$
$$\qquad \ldots(117)$$

$$\sigma_{zz}(r,z) = 2G\{-12c_{31}+(14-10\mu)c_{32}-192zc_{41}+$$
$$4z(16-14\mu)c_{42}+(960zr^2-640z^3)c_{61}+c_{62}((448-$$
$$704\mu)z^3+(1056\mu-1728)zr^2)\} \qquad (118)$$

$$\sigma_{rr}(r,z) = 2G\{6c_{31}+(10\mu-2)c_{32}+96zc_{41}+4z(14\mu-1)c_{42}+$$
$$(320z^3-720zr^2)c_{61}+((128+704\mu)z^3+$$
$$(504-1056\mu)zr^2)c_{62}\} \qquad (119)$$

$$\sigma_{\theta\theta}(r,z) = 2G\{6c_{31}+(10\mu-2)c_{32}+96zc_{41}+4z(14\mu-$$
$$1)c_{42}+(320z^3-240zr^2)c_{61}+((128+704\mu)z^3+$$
$$(168-1056\mu)zr^2)c_{62}\} \qquad (120)$$

$$\sigma_{rr}\left(r,z=\pm\frac{t}{2}\right)=0=6c_{31}+(10\mu-2)c_{32}+48tc_{41}+$$
$$2t(14\mu-1)c_{42}+(40t^3-360tr^2)c_{61}+$$
$$((16+88\mu)t^3+(252-528\mu)tr^2)c_{62}=0 \qquad (121)$$

$$u(r,z)=6rc_{31}-2rc_{32}+96rzc_{41}-4rzc_{42}+(320z^3r-$$
$$240zr^3)c_{61}+(168zr^3+128z^3r)c_{62} \qquad (122)$$

$$w(r,z)=-4c_{21}+(10-12\mu)c_{22}-12zc_{31}+(14-$$
$$20\mu)zc_{32}+(48r^2-96z^2)c_{41}+c_{42}((32-56\mu)z^2-$$
$$(30-28\mu)r^2)-c_{61}(160z^4-480z^2r^2+60r^4)+$$
$$c_{62}((112-352\mu)z^4+(1056\mu-864)z^2r^2+$$
$$(174-132\mu)r^4) \qquad (123)$$

$$w(r,z)=b_2+(14-20\mu)zc_{32}-12zc_{31}+(48r^2-$$
$$96z^2)c_{41}+c_{42}((32-56\mu)z^2-(30-28\mu)r^2)-$$
$$c_{61}(160z^4-480z^2r^2+60r^4)+c_{62}((112-352\mu)z^4+$$
$$(1056\mu-864)z^2r^2+(174-132\mu)r^4) \qquad (124)$$

$$\frac{\partial w}{\partial r}(r,z)=96rc_{41}+(60-56\mu)rc_{42}+(960z^2r-$$
$$240r^3)c_{61}+c_{62}((2112\mu-1728)z^2r+$$
$$(1096-528\mu)r^3) \qquad (125)$$

4.3 Enforcement of boundary conditions

From $u(r=R,z=0)=6Rc_{31}-2Rc_{32}=0 \qquad (126)$

$$c_{32}=3c_{31} \qquad (127)$$

$$\frac{\partial w}{\partial r}(R,z=0)=96c_{41}+(60-56\mu)c_{42}-240R^2c_{61}+$$
$$(1096-528\mu)R^2c_{62}=0 \qquad (128)$$

$$\tau_{rz}(r,z=\pm t/2)=0=$$

$$96c_{41}-(32-28\mu)c_{42}+(240t^2-240r^2)c_{61}+$$
$$c_{62}((432-264\mu)r^2+(264\mu-168)t^2)=0 \qquad (129)$$

$$\sigma_{zz}(r,z=t/2)=0=$$
$$(30-30\mu)c_{31}-96tc_{41}+(32-28\mu)tc_{42}+(480r^2t-80t^3)c_{61}+$$
$$c_{62}((56-88\mu)t^3+(528\mu-864)r^2t)=0 \qquad (130)$$

$$\sigma_{zz}(r,z=-t/2)=0=(30-30\mu)c_{31}-96tc_{41}-$$
$$(32-28\mu)tc_{42}-(480r^2t-80t^3)c_{61}-c_{62}((56-88\mu)t^3+$$
$$(528\mu-864)r^2t)=\frac{-p}{2G} \qquad (131)$$

Then from Equations (130) and (131) we obtain:

$$c_{31}=\frac{-p}{60(1-\mu)2G}=\frac{-p}{120G(1-\mu)} \qquad (132)$$

Hence,

$$c_{32}=\frac{-p}{20(1-\mu)2G}=\frac{-p}{40G(1-\mu)} \qquad (133)$$

The enforcement of all the boundary conditions, and solving for the unknown constants yield the following solutions:

$$b_2=\frac{3(1-\mu)}{32G}\frac{pR^4}{t^3}=\frac{3(1-\mu)}{32G}\left(\frac{R}{t}\right)^4pt \qquad (134)$$

$$c_{41}=\left(\frac{8-7\mu}{896}\frac{R^2}{t^3}p-\frac{15-14\mu}{896(1-\mu)}\frac{p}{t}\right)\frac{1}{2G} \qquad (135)$$

$$c_{41}=\frac{1}{1792G}\left((8-7\mu)\frac{R^2}{t^3}-\left(\frac{15-14\mu}{(1-\mu)}\right)\frac{1}{t}\right)p \qquad (136)$$

$$c_{42}=\frac{3}{224G}\left(\frac{R^2p}{t^3}-\frac{1}{1-\mu}\frac{p}{t}\right) \qquad (137)$$

$$c_{61}=\left(\frac{18-11\mu}{7040G}\right)\frac{p}{t^3} \qquad (138)$$

$$c_{62}=\frac{1}{704G}\frac{p}{t^3} \qquad (139)$$

4.4 Stress fields

The stress fields are then found as:

$$\tau_{rz}(r,z)=(960rz^2-240r^3)\frac{18-11\mu}{3520}\frac{p}{t^3}+\frac{1}{352}\frac{p}{t^3}((432-$$
$$264\mu)r^3+(1056\mu-672)rz^2)+96r\left(\frac{8-7\mu}{896}\frac{R^2p}{t^3}-$$
$$\frac{15-14\mu}{896(1-\mu)}\frac{p}{t}\right)-2r(16-14\mu)\left(\frac{3}{112}\frac{R^2p}{t^3}-$$
$$\frac{3}{112(1-\mu)}\frac{p}{t}\right) \qquad (140)$$

$$\sigma_{zz}(r,z)=(960zr^2-640z^3)\left(\frac{18-11\mu}{3520}\right)\frac{p}{t^3}+$$
$$\frac{p}{352t^3}((448-704\mu)z^3+(1056\mu-1728)zr^2)+$$

$$(64-56\mu)z\left(\frac{3}{112}\frac{R^2 p}{t^3}-\frac{3}{112(1-\mu)}\frac{p}{t}\right)-$$

$$194z\left(\left(\frac{8-7\mu}{896}\right)\frac{R^2 p}{t^3}-\frac{15-14\mu}{896(1-\mu)}\frac{p}{t}\right)-\frac{p}{2} \quad (141)$$

$$\sigma_{rr}(r,z)=(320z^3-720zr^2)\left(\frac{18-11\mu}{3520}\right)\frac{p}{t^3}+$$

$$\frac{p}{352t^3}(64(2+11\mu)z^3+(504-1056\mu)zr^2)+$$

$$96z\left(\left(\frac{8-7\mu}{896}\frac{R^2 p}{t^3}-\left(\frac{15-14\mu}{896(1-\mu)}\right)\frac{p}{t}\right)\right)+4z(14\mu-$$

$$1)\frac{3}{112}\left(\frac{R^2 p}{t^3}-\frac{p}{(1-\mu)t}\right)-\frac{\mu p}{2(1-\mu)} \quad (142)$$

$$\sigma_{\theta\theta}=(320z^3-240zr^2)\left(\frac{18-11\mu}{3520}\right)\frac{p}{t^3}+\frac{p}{352t^3}(64z^3(2+$$

$$11\mu)+(168-1056\mu)zr^2)+96z\left(\frac{8-7\mu}{896}\frac{R^2 p}{t^3}-\right.$$

$$\left.\left(\frac{15-14\mu}{896(1-\mu)}\right)\frac{p}{t}\right)+4z(14\mu-1)\frac{3}{112}\left(\frac{R^2 p}{t^3}-\right.$$

$$\left.\frac{p}{(1-\mu)t}\right)-\frac{\mu p}{2(1-\mu)} \quad (143)$$

$$u=\frac{1}{2G}\left\{\frac{p}{352t^3}(168zr^3+128z^3 r)+(320z^3 r-240zr^3)\right.$$

$$\left(\frac{18-11\mu}{3520}\right)\frac{p}{t^3}+96rz\left(\frac{8-7\mu}{896}\frac{R^2 p}{t^3}-\right.$$

$$\left.\left(\frac{15-14\mu}{896(1-\mu)}\right)\frac{p}{t}\right)-4rz\frac{3}{112}\left(\frac{R^2 p}{t^3}-\frac{p}{(1-\mu)t}\right)\right\} \quad (144)$$

$$w(r,z)=\frac{1}{2G}\left\{-\left(\frac{18-11\mu}{3520}\right)\frac{p}{t^3}(160z^4-480z^2 r^2+\right.$$

$$60r^4)+\frac{p}{352t^3}((112-352\mu)z^4+(1056\mu-$$

$$864)z^2 r^2+(174-132\mu)r^4)+\frac{3}{112}\left(\frac{R^2 p}{t^3}-\right.$$

$$\left.\frac{p}{(1-\mu)t}\right)((32-56\mu)z^2-(30-28\mu)r^2)+(48r^2-$$

$$96z^2)\left(\frac{8-7\mu}{896}\frac{R^2 p}{t^3}-\frac{15-14\mu}{896(1-\mu)}\right)\frac{p}{t}+\frac{12zp}{60(1-\mu)}-$$

$$\frac{(14-20\mu)zp}{20(1-\mu)}+\frac{3(1-\mu)}{16}\frac{pR^4}{t^3}\right\} \quad (145)$$

4.5 Deflection of the middle surface of the plate w(r, z = 0)

The deflection of the middle surface of the plate is obtained by substitution of $z = 0$ into the expression for $w(r, z)$ and after simplifications, as:

$$w(r,0)=\frac{3(1-\mu)}{32}\left(1-\left(\frac{r}{R}\right)^2\right)\left(\frac{R}{t}\right)^4\frac{pt}{G} \quad (146)$$

The deflection at the centre $w(0, 0)$ is obtained as:

$$w(0,0)=w_c=\frac{3(1-\mu)}{32}\left(\frac{R}{t}\right)^4\frac{pt}{G} \quad (147)$$

and is presented in Table 1 as $w(0,0)$ vs R/t.

The radial stress at $r = 0$, $z = t/2$ is:

$$\sigma_{rr}\left(0,\frac{t}{2}\right)=p\left\{\frac{3}{8}(1+\mu)\left(\frac{R}{t}\right)^2-\frac{(1+\mu)(4+\mu)}{8(1-\mu)}\right\} \quad (148)$$

Table 1: Variation of deflection at the centre with the ratio of radius to thickness (R/t) for $\mu = 0.25$ and $\mu = 0.30$ (Solution A)

R/t	$w(0,0)(\mu=0.25)$ $(\times pt/G)$	$w(0,0)(\mu=0.30)$ $\times pt/G$
20	11250	10,500
15	3559.57	3322.266
10	703.125	656.25
8	288	268.8
6	91.125	85.05
5	43.945	41.016
4	18	16.8
3	5.695	5.3156
2	1.125	1.05
0.5	4.3945×10^{-3}	4.1016×10^{-3}
0.25	2.7466×10^{-4}	2.5635×10^{-4}
0.20	1.125×10^{-4}	1.05×10^{-4}
0.10	7.03125×10^{-6}	6.5625×10^{-6}
0.05	4.3945×10^{-7}	4.1016×10^{-7}
0.02	1.125×10^{-8}	1.05×10^{-8}
0.01	7.03125×10^{-10}	6.5625×10^{-10}
0.005	4.3945×10^{-11}	4.1016×10^{-11}

Table 2: Variation of non-dimensional radial stress with the ratio of radius R to thickness $t(R/t)$ for $\mu = 0.25$, and for $\mu = 0.30$ for Solution A
Solution A

R/t	$(\mu=0.25)$ $\sigma_{rr}(0,t/2)/p$	$(\mu=0.30)$ $\sigma_{rr}(0,t/2)/p$
20	186.6146	194.0018
15	104.5833	108.6893
10	45.9896	47.7518
8	29.1146	30.2018
6	15.9896	16.5518
5	10.8333	11.1893
4	6.6146	6.8018
3	3.3333	3.3893
2	0.9896	0.9518
1	-0.4167	-0.5107
0.5	-0.7682	-0.8763
0.25	-0.8561	-0.9677
0.20	-0.8667	-0.9787
0.10	-0.8807	-0.9933
0.05	-0.8842	-0.9970
0.02	-0.8852	-0.9980
0.01	-0.8854	-0.9982
0.005	-0.8854	-0.9982

4.6 Alternative solution from second type of boundary conditions (Solution B)

For the same problem considered as shown in Figure 1, if the boundary conditions are given by replacing $\frac{\partial w}{\partial r}(r=R,z=0)$

with $\frac{\partial u}{\partial z}(r=R,z=0)$, then the stress and displacement fields are obtained (if all other boundary conditions are unchanged) as follows:

$$\sigma_{zz}(r,z) = \left(-\frac{2z^3}{t^3} + \frac{3R^2z}{2t^3} - \frac{1}{2}\right)p \qquad (149)$$

$$\tau_{rz}(0,z) = \frac{3r}{t}\left(\frac{z^2}{t^2} - \frac{1}{4}\right)p \qquad (150)$$

$$\sigma_{rr}(r,z) = p\left\{(2+\mu)\frac{z^3}{t^3} - \frac{3(3+\mu)}{4}\frac{zr^2}{t^3} + \right.$$
$$\left. \frac{3}{4}\left((1+\mu)\frac{R^2}{t^3} + \frac{2}{(1-\mu)t}\right)z - \frac{\mu}{2(1-\mu)}\right\} \qquad (151)$$

$$\sigma_{\theta\theta} = p\left\{(2+\mu)\frac{z^3}{t^3} - \frac{3(1+3\mu)zr^2}{4t^3} + \right.$$
$$\left. \frac{3}{4}\left((1+\mu)\frac{R^2}{t^3} + \frac{2}{t(1-\mu)}\right)z - \frac{\mu}{2(1-\mu)}\right\} \qquad (152)$$

$$u(r,z) = \frac{p}{2G}\left\{(2-\mu)\frac{z^3r}{t^3} - \frac{3(1-\mu)}{4}\frac{zr^3}{t^3} + \frac{3(1-\mu)}{4}\frac{R^2rz}{t^3}\right\} \qquad (153)$$

$$w(r,z) = \frac{p}{2G}\left\{-\frac{(1+\mu)}{2}\frac{z^4}{t^3} + \frac{3\mu z^2r^2}{2t^3} + \frac{3(1-\mu)r^4}{16t^3} - \right.$$
$$\frac{3}{4}\left(\frac{\mu R^2}{t^3} - \frac{(1-2\mu)}{(1-\mu)t}\right)z^2 - \frac{3}{8}\left((1-\mu)\frac{R^2r^2}{t^3} + \frac{2r^2}{t}\right) - $$
$$\left. \frac{(1-2\mu)z}{2(1-\mu)} + \frac{3}{4}\frac{R^2}{t} + \frac{3(1-\mu)}{16}\frac{R^4}{t^3}\right\} \qquad (154)$$

$$w(r, z=0) = \frac{p}{2G}\left\{\frac{3(1-\mu)r^4}{16t^3} - \frac{3}{8}\left(\frac{(1-\mu)}{t^3}R^2r^2 + \frac{2r^2}{t}\right) + \right.$$
$$\left. \frac{3}{4}\frac{R^2}{t} + \frac{3(1-\mu)R^4}{16t^3}\right\} \qquad (155)$$

$$w(r,0) = \frac{p}{2G}\left\{\frac{3(1-\mu)}{16t^3}(r^4 - 2R^2r^2 + R^4 + \frac{3}{8t}(2R^2 - 2r^2)\right\} \qquad (156)$$

$$w(r,0) = \frac{p}{2G}\left\{\frac{3(1-\mu)}{16t^3}(R^2 - r^2)^2 + \frac{3}{4t}(R^2 - r^2)\right\} \qquad (157)$$

$$w(r,0) = \frac{p}{2G}\left\{\frac{3(1-\mu)}{16t^3}\left(\frac{(R^2-r^2)}{R^2}\right)^2 R^4 + \frac{3R^2}{4t}\left(\frac{R^2-r^2}{R^2}\right)\right\}$$

$$w(r,0) = \frac{p}{2G}\left\{\frac{3(1-\mu)}{16}\left(1-\left(\frac{r}{R}\right)^2\right)^2\left(\frac{R^4}{t^3}\right) + \frac{3R^2}{4t}\left(1-\left(\frac{r}{R}\right)^2\right)\right\} \qquad (158)$$

$$w(r,0) = \left\{\frac{3(1-\mu)}{32}\left(1-\left(\frac{r}{R}\right)^2\right)\left(\frac{R}{t}\right)^4 + \frac{3}{8}\left(\frac{R}{t}\right)^2\left(1-\left(\frac{r}{R}\right)^2\right)\right\}\frac{pt}{G} \qquad (159)$$

The deflection at the centre is given by:

$$w(0,0) = \left\{\frac{3(1-\mu)}{32}\left(\frac{R}{t}\right)^4 + \frac{3}{8}\left(\frac{R}{t}\right)^2\right\}\frac{pt}{G} \qquad (160)$$

and presented in Table 2 as $w(0,0)$ *vs* (R/t).

Table 3: Variation of deflection at the centre with the ratio of radius to thickness (R/t) for $\mu = 0.25$ and $\mu = 0.30$ (Solution B)

R/t	($\mu = 0.25$) $w(0,0)$ ($\times pt/G$)	($\mu = 0.30$) $w(0,0)\times pt/G$
20	11400	10650
15	3643.945	3406.6406
10	740.625	693.75
8	312	292.8
6	104.625	98.55
5	53.320	50.390625
4	24	22.8
3	9.0703	8.690625

2	2.625	2.55
1	0.4453	0.440625
0.5	0.098145	0.097852
0.25	0.0237122	0.023694
0.20	0.0151125	0.015105
0.10	3.7570×10^{-3}	3.7566×10^{-3}
0.05	9.3794×10^{-4}	9.3791×10^{-4}
0.02	1.5001×10^{-4}	1.500105×10^{-4}
0.01	3.75007×10^{-5}	3.75007×10^{-5}
0.005	9.3750×10^{-6}	9.37504×10^{-6}

Table 4: Variation of non-dimensional radial stress at the plate centre with the ratio of radius R to thickness t (R/t) for $\mu = 0.25$, and for $\mu = 0.30$ for Solution B

R/t	($\mu = 0.25$) $\sigma_{rr}(0, t/2)/p$	($\mu = 0.30$) $\sigma_{rr}(0, t/2)/p$
20	188.6146	196.1446
15	106.5833	110.8321
10	47.9896	49.8946
8	31.1146	32.3446
6	17.9896	18.6946
5	12.8333	13.3321
4	8.6146	8.9446
3	5.3333	5.5321
2	2.9896	3.0946
1	1.5833	1.6321
0.5	1.2318	1.2665
0.25	1.1439	1.1751
0.20	1.1333	1.1641
0.10	1.1193	1.1495
0.05	1.1158	1.1459
0.02	1.1148	1.1448
0.01	1.1146	1.1447
0.005	1.1146	1.1447

Table 5: Variation of non-dimensional radial stress with R/t for the theory of Kirchhoff circular plate theory

R/t	($\mu = 0.25$) $\sigma_{rr}(0,t/2)/p$	($\mu = 0.30$) $\sigma_{rr}(0,t/2)/p$
20	187.50	195
15	105.46875	109.6875
10	46.875	48.75
8	30	31.20
6	16.875	17.55
5	11.71875	12.1875
4	7.5	7.80
3	4.21875	4.3875
2	1.875	1.9500
1	0.46875	0.4875
0.5	0.1172	0.1219
0.25	0.02930	0.0305
0.20	0.01875	0.0195
0.10	4.6875×10^{-3}	4.875×10^{-3}
0.05	1.1719×10^{-3}	1.21875×10^{-3}
0.02	1.875×10^{-4}	1.95×10^{-4}
0.01	4.6875×10^{-5}	4.875×10^{-5}
0.005	1.1719×10^{-5}	1.21875×10^{-5}

The radial stress σ_{rr} at $r = 0$, $z = t/2$ is

$$\sigma_{rr}\left(r=0, z=\frac{t}{2}\right) = \left\{\frac{3}{8}(1+\mu)\left(\frac{R}{t}\right)^2 + \frac{8-5\mu-\mu^2}{8(1-\mu)}\right\}p \quad 161)$$

The radial stress at the centre $r = 0$, $z = t/2$ obtained from the Kirchhoff – Love classical plate theory is given by:

$$\sigma_{rr}\left(0, \frac{t}{2}\right) = \frac{3(1+\mu)}{8}\frac{R^2}{t^2}p = \sigma_{rr}\left(r=0, z=\frac{t}{2}\right) \qquad (162)$$

5. Discussion

This study has successfully presented first principles derivation of stress functions and displacement functions for three dimensional, small displacement elastostatic problems for homogeneous, isotropic materials. It also has successfully illustrated an application of the derived stress function and displacement function to the flexural analysis of thick isotropic homogeneous circular plates.

The theoretical framework for the formulation and solution of 3D homogeneous, isotropic elasticity problems involves a simultaneous consideration of the six generalized equations of stress – strain, Equations (1 – 6), the six kinematic relations for small – displacement assumptions – Equations (8 – 13) – and the three differential equations of equilibrium given by Equations (14 – 16) for dynamic problems and Equations (17 – 19) for static problems.

The generalized 3D Hooke's stress–strain equations for homogeneous isotropic elasticity problems given by Equations (20 – 25) were used with the kinematic relations of small displacement elasticity and the differential equations of equilibrium to obtain a displacement formulation in 3D Cartesian coordinates system of 3D elastostatic problems as a set of three coupled partial differential equations given by Equations (37 – 39) which are in terms of the Cartesian displacement field components $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ as the unknowns.

Displacement function $\phi(x, y, z)$ is derived to apriori satisfy the set of three coupled differential equations, Equations (37 – 39) by assuming that the two Cartesian components of the displacement field in the x and y coordinate directions could be obtained as Equations (40) and (41). The vertical component of the displacement field $w(x, y, z)$ was then obtained in terms of the displacement function $\phi(x, y, z)$ by integration of the displacement equation of equilibrium – Equations (32 – 38) – as Equation (44) in generic form in terms of E_1, E_2 and G. The vertical displacement field component was obtained in simplified form as Equation (47). The condition for the three displacement field components obtained to identically satisfy the three coupled displacement equations of equilibrium was obtained as Equation (48), a biharmonic equation in terms of the displacement function. The displacement function derived was thus found to be a solution of the fourth order partial differential equation of the biharmonic problem to be eligible to be a solution to the system of coupled differential equations of equilibrium of the displacement formulation.

The stress functions are derived by substitution of the displacement function expressions derived for the three displacement field components into the corresponding stress - displacement relations. Thus, after algebraic processes and after simplifications, the stresses are found to be derivable from $\phi(x, y, z)$ as Equations (52), for σ_{xx}, (56) for σ_{yy}, (60) for σ_{zz}, (61) for τ_{xy}, (65) for τ_{xz}, and (69) for τ_{yz}. Hence $\phi(x, y, z)$ is shown to be a stress function as well since the stress components are derived from it. The stress and displacement components

derived are expressed in cylindrical polar coordinates system as Equations (70 – 75).

An illustration of the application of the stress and displacement function derived in this paper is presented by considering the derivation of elasticity solutions to the flexural problem of homogeneous, isotropic thick circular plates. The particular problem considered, which is shown in Figure 1 is the theory of elasticity analysis of the flexural problem of thick circular plate of radius R and the thickness t with clamped (fixed) edges subject to uniformly distributed load on the top surface. The plate material is assumed to be homogeneous and isotropic.

The semi-inverse method in the theory of elasticity is applied to derive elasticity solutions to the flexural problem of isotropic, homogeneous thick circular plate using the stress function derived in this study. The stress function derived is required to be a biharmonic function from Equation (48), and Legendre polynomials were shown by Timoshenko and Goodier (1970) to be biharmonic functions. Hence, Legendre polynomials were chosen as the biharmonic stress and displacement functions to analyze and solve the thick circular plate problem. For second degree Legendre polynomials given by Equation (76) as the biharmonic stress and displacement function, the use of Equations (70 – 75) yielded the stress and displacement fields as Equations (77 – 82). The use of third degree Legendre polynomials – Equation (84) – as the stress and displacement functions yielded the stress and displacement field components as Equations (85 – 89), (77).

The use of fourth degree Legendre polynomials – Equation (90) – in the equations derived yielded the stress and displacement field components as Equations (91 – 96). The use of the sixth degree Legendre polynomials – Equation (98) – in the equations derived yielded the stress and displacement field components as Equations (99) – (104).

The problem considered is a linear elasticity problem, making superposition principle valid. The stress function $\Omega(x, z)$ used to solve the isotropic, homogeneous thick circular plate problem is obtained by a superposition of stress functions obtained by using the Legendre polynomials of degree from two to six as given by Equations (115) and (116), where c_{21}, c_{22}, …, c_{61}, c_{62} are unknown constants. The boundary conditions obtained from the stress and deformation conditions and also from the requirements of equilibrium of internal vertical stresses and the applied load are given by Equations (106 – 114).

The stresses and displacement fields obtained by using the Legendre polynomial in Equation (116) as the biharmonic function $\Omega(r, z)$ are obtained in terms of the unknown constants as Equations (117), (118), (119), (120), (122) and (123). The enforcement of the boundary conditions – Equations (106 – 114) – yielded the values of the unknown constants as Equations (132), (133), (134), (135), (137), (138) and (139). Substitution of the constants and simplification gave the stress fields as Equations (140), (141), (142), (143), (144) and (145). The deflection of the middle (neutral) surface of the plate $w(r, z = 0)$ was found to depend on the ratio of radius to thickness as given by Equation (146). The deflection of the centre $w(0, 0)$ of the plate was similarly found to depend on the ratio (R/t) as presented in Equation (147). The radial stress at $r = 0$, $z = t/2$ was obtained as Equation (148). The variation of deflection at the centre with the ratio R/t for values of the Poisson ratio $\mu = 0.25$ and $\mu = 0.30$, and for various values of R/t are presented in Table 1. Similarly, the variation of non-dimensional radial stress

at $r = 0$, $z = t/2$ for values of the Poisson ratio $\mu = 0.25$ and $\mu = 0.30$, and for various values of R/t are presented in Table 2.

The boundary conditions were slightly modified by replacing $\frac{\partial w}{\partial r}(r = R, z = 0) = 0$ with $\frac{\partial u}{\partial z}(r = R, z = 0) = 0$, and the solutions obtained for the same isotropic, homogeneous thick circular plate problem with clamped edges and subjected to uniformly distributed transverse load was found as Equations (149 – 154). The transverse displacement $w(r, z = 0)$ for the second type of boundary condition was obtained as Equation (159). The deflection at the centre $w(0, 0)$ was found as Equation (160), which was presented in tabular form in Table 3 as $w(0, 0)$ *vs* (R/t).

For the second type of boundary condition, at the clamped edge, the radial stress at $r = 0$, $z = t/2$ was found as Equation (161), and presented in tabular form in Table 4 for value of the Poisson's ratio $\mu = 0.25$ and $\mu = 0.30$, and for various values of R/t. The radial stress at $r = 0$, $z = t/2$ obtained from the Kirchhoff – Love classical plate theory for circular plates is given by Equation (162) which is presented for values of Poisson ratio $\mu = 0.25$ and $\mu = 0.30$, and various values of R/t as Table 5.

6. Conclusion

The conclusions of the present study are as follows:

(i) The displacement function derived simultaneously satisfies the generalized 3D stress–strain relations, the kinematic relations of strain to displacement and the differential equations of equilibrium for infinitesimal displacement elasticity problems for homogeneous, isotropic materials.

(ii) The stress function derived also simultaneously satisfies the generalized 3D material constitutive relations, the strain – displacement equations, and the differential equations of equilibrium for small displacement elasticity problems involving homogeneous isotropic materials.

(iii) The derived stress and displacement function has been successfully used to find the stresses and displacement fields in a thick circular plate with clamped edges and subjected to uniformly distributed transverse load over the entire plate domain.

Conflict of interest

The authors declare that they have no conflict of interest.

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