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Direct Limit of Krasner (m, n)-Hyperrings

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Abstract

The purpose of this paper is the study of direct limits in category of Krasner (*m, n*)-hyperrings. In this regards we introduce and study direct limit of a direct system in category (*m, n*)-hyperrings. Also, we consider fundamental relation Γ^*, as the smallest equivalence relation on an (*m, n*)-hyperring R such that the quotient space R/Γ^* is an (*m, n*)-ring, to introduce the fundamental functor from category of Krasner (*m, n*)-hyperrings to the category of (*m, n*)-rings. Finally, we study the relationship between fundamental functor and direct limit on Krasner (*m, n*)-hyperrings. In particular, we prove that the fundamental functor is exact and obtain some its basic properties.

Keywords: Krasner (m, n)-hyperrings; Direct system; Fundamental functor; Direct limit.

Mathematics Subject Classification 2010: 20N20, 20N25.

Introduction

An $n-$ary hyperoperation is a mapping $f:\underbrace{H\times\cdots\times H}_{n}\to \mathrm{P}^*(H)$, where $\mathrm{P}^*(H)$ is the set of all nonempty subsets of H . In this case, (H, f) is said to be an $n-$ary hypergroupoid. This is a generalization of a hypergroupoid (when $n = 2$), that was defined by Marty in [1] as founder of hyperstructure theory (for more details refer to [2], [3], [4], [5], [6], [7], [8] and [9]. An $n-$ary hyperoperation initiated an $n-$ary hyperstructure. Nowadays, $n-$ary hyperstructures is a well-known field of researches on hyperstructures theory(for more see [10], [11], [12], [13], [14], [15], [16] and [17]). Also, recently some researchers studied direct systems and direct limit on (fuzzy) hyperstructures (for instance see [18], [19], [20], [21], [22] and [23]).

In this paper, we consider category of Krasner (m, n)-hyperrings ([11]), introduce, and study direct limit of a direct system in this category. In this regards, we introduce the fundamental functor from category of (m, n)-hyperrings into category of (m, n)-rings, via the fundamental relation. In particular, we prove that this functor preserves all direct limits.

Preliminaries

In this section, we present some basic concepts of $n-$ary hyperstructures which we need to development our paper.

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In dealing with $n-$ary hyperstructures, for abbreviation, we will show the sequence $x_i, x_{i+1}, \ldots, x_j$ by x_i^j, and we put $x_i^j = \varnothing$ for $j < i$. Hence $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Also, if $y_{i+1} = \cdots = y_j = y$, then it is shown by $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$. Moreover, if f is an n-ary hyperoperation and $t = l(n-1)+1$, for some $l \geq 0$, then t-ary hyperoperation $f_{(l)}$ is given by $f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f(f(\ldots,f(f}_{l}(x_1^n), x_{n+1}^{2n-1}),\ldots), x_{(l-1)(n-1)+1}^{l(n-1)+1})$. For nonempty subsets $A_1, \ldots, A_n$ of H we define $f(A_1^n) = \bigcup\{f(x_1^n) \mid x_i \in A_i, i = 1, \ldots, n\}$. We say (H, f) is an $n-$ary semihypergroup if f is associative, that is, $f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$, holds, for every $1 \leq i < j \leq n$ and all $x_1, x_2, \ldots, x_{2n-1} \in H$ If (H, f) is an $n-$ary semihypergroup and $f(x_1^{i-1}, H, x_{i+1}^n) = H$ for all $x_1^n \in H$ and $1 \leq i \leq n$, then (H, f) is called an $n-$ary hypergroup. (H, f) is said to be commutative, if for all $\sigma \in S_n$ and for every $a_1^n \in H$, we have $f(a_1^n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. moreover, a nonempty subset B of an $n-$ary hypergroup (H, f) is called an $n-$ary subhypergroup of H, if (B, f) is an $n-$ary hypergroup.

Let (H, f) be a commutative $n-$ary hypergroup. (H, f) is called a canonical $n-$ary hypergroup ([12]), if

i) there exists a unique $e \in H$ such that for every $x \in H, f(x, e^{n-1}) = \{x\}$;

ii) for all $x \in H$ there exists a unique $x^{-1} \in H$ such that $e \in f(x, x^{-1}, e^{n-2})$;

iii) if $x \in f(x_1^n)$ then, for all $1 \leq i \leq n$ we have $x_i \in f(x, x^{-1}, \cdots, x_{i-1}^{-1}, x_{i+1}^{-1}, \cdots, x_n^{-1})$.

Definition [16] (R, f, g) is said to be an (m, n)-hyperring, if:

i) (R, f) is an m-ary hypergroup.

ii) (R, g) Is an n-ary semihypergroup.

The n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$ $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n))$. A nonempty subset S of R is called an (m, n)-subhyperring, if (R, f, g) is an (m, n)-hyperring. Let $i \in \{1, \ldots, n\}$. An i-hyperideal I of R is an (m, n)-subhyperring of R such that $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$ for every $x_1^n \in R$. I is called a hyperideal, if I is a i-hyperideal, for all $1 \leq i \leq n$.

An (m, n)-hyperring (R, f, g) is said to be Krasner if (R, f) is a canonical n-ary hypergroup and (R, g) is an n-ary semigroup such that 0 is a zero element (absorbing element) of the n-ary operation g, i.e. for all $x_2^n \in R$ we have $g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0)$.

Example [16] Suppose that $(L, \vee, \wedge)$ is a relatively complemented distributive lattice. Define f and g on L as follows: $f(a_1, a_2) = \{c \in L \mid a_1 \wedge c = a_2 \wedge c = a_1 \wedge a_2\}$, and $g(a_1^n) = \vee_{i=1}^n a_i, \forall a_1^n \in L$.

It is easy to verify that (L, f, g) is a Krasner $(2, n)$-hyperring.

Definition [24] A category consists of the following data:

- Objects A, B, C
- Arrows: f, g, h
- For each arrow f, there are given objects $dom(f), cod(f)$ called the domain and codomain of f. We write $f: A \to B$ to indicate that $A = dom(f)$ and $B = cod(f)$.
- Given arrows $f: A \to B$ and $g: B \to C$, that is, with $cod(f) = dom(g)$ there is given an

arrow $g \circ f : A \to C$ called the composite of f and g.

- For each object A, there is given an arrow $1_A : A \to A$ called the identity arrow of A. These data are required to satisfy the following laws:
- Associatively: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B, g : B \to C,\ h : C \to D$.
- Unit: $f \circ 1_A = f = 1_A \circ f$ for all $f : A \to B$.

Example [24] group and group homomorphisms is a category.

Definition [24] In any category C, an object

- 0 is initial if for any object c there is a unique morphism $0 \to c$
- 1 is terminal if for any object c there is a unique morphism $c \to 1$.

Definition [24] A functor $F : C \to D$ between categories C and D is a mapping of objects and arrows to arrows, in such a way that

- $F(f : A \to B) = F(f) : F(A) \to F(B),$
- $F(1_A) = 1_{F(A)}$,
- $F(g \circ f) = F(g) \circ F(f)$.

Definition [24] Let J and C be categories. A diagram of type J in C is a functor $F : J \to C$.

We write the objects in the index category J lower case, $i, j, \ldots$ and the values of the functor $D : J \to C$ in the form D_i, D_j, etc.

A cone to a diagram D consists of an object c in C and a family of arrows in C, $c_j : C \to D_j$ one for each object $j \in J$, such that for each arrow $\alpha : i \to j$ in J, the following triangle commutes:

$$\begin{tikzcd} C \arrow[r, "c_j"] \arrow[d, "c_i"'] & D_j \\ D_i \arrow[ur, "D_\alpha"'] & \end{tikzcd}$$

A morphism of cones $\vartheta : (C, c_j) \to (C', c'_j)$ is an arrow ϑ in C making each triangle,

$$\begin{tikzcd} C \arrow[r, "\vartheta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

commute. Thus we have evident category $Cone(D)$ of cones to D.

Definition [24] A limit for a diagram $D : J \to C$ is a terminal object in category $Cone(D)$.

We often denote a limit in the form $p_i : \varprojlim D_j \to D_i$.

$$\begin{tikzcd} C \arrow[rr, dotted, "u"] \arrow[dr, "c_i"'] \arrow[drr] & & \varprojlim_j D_j \arrow[dl] \arrow[d, "p_j"] \\ & D_i \arrow[r, "D_\alpha"'] & D_j \end{tikzcd}$$

In general, a colimit for a diagram $D : J \to C$ is of course, an initial object in the category of cocones. We write such a colimit in the form $\varinjlim D_j$.

Results

Direct System (Limit) of Krasner (m, n)-Hyperrings

We say a partially ordered set I is a directed set if for each $(i, j) \in I \times I$ there exists $k \in I$ such that $i \le k$ and $j \le k$. Let I be a directed set and $(m, n) - \mathrm{KH}_r$ the category of Krasner (m, n)-hyperrings with strong homomorphisms. Let $(R_i, f_i, g_i)_{i \in I}$ be a family of Krasner (m, n)-hyperrings indexed by I. For each pair $i, j \in I$ such that $i \le j$, let $\phi_j^i : R_i \to R_j$ be a strong homomorphism where ϕ_i^i is the identity homomorphism for all $i \in I$ and $\phi_k^i = \phi_k^j \circ \phi_j^i$ for $i \le j \le k$. Then $R = (R_i, \phi_j^i)$ is said to be a direct system over the direct set I.

The direct limit of a direct system $R = (R_i, \phi_j^i)$ in $(m, n) - \mathrm{KH}$ denoted by $\varinjlim R_i$, is a Krasner (m, n)-hyperring and a family of strong homomorphisms

$\alpha_i : R_i \to \varinjlim R_i \mid \alpha_i = \alpha_j \phi_j^i ; i \leq j\}$, which for every Krasner (m,n)-hyperrings K and every family of strong homomorphisms $\{\mu_i : R_i \to K \mid \mu_i = \mu_j \phi_j^i ; i \leq j\}$ there is a unique strong homomorphism $\beta : \varinjlim R_i \to K$ such that the following diagram is commutated:

$$\begin{tikzcd}
R_i \arrow[rr, "\phi_j^i"] \arrow[dr, "\alpha_i"] \arrow[ddr, "\mu_i"'] & & R_j \arrow[dl, "\alpha_j"'] \arrow[ddl, "\mu_j"] \\
& \varinjlim R_i \arrow[d, dashed, "\beta"] & \\
& K &
\end{tikzcd}$$

Let $X = \bigcup_{i \in I} R_i$ and define the following equivalence relation on X for $a_i \in R_i$ and $a_j \in R_j$ $: a_i \sim a_j \Leftrightarrow \phi_k^i a_i = \phi_k^j a_j$ for $k \geq i, j$. Suppose that $\overline{X} = \{[a_i] \mid a_i \in X\}$ where $[a_i]$ is the equivalent class of a_i. It is clear that $a_i \sim \phi_j^i a_j$ for $j \geq i$ in $\overline{X}$.

Theorem 1.1. $(\overline{X}, F)$ is an m-ary canonical hypergroup where $\left\{[x] \mid x \in f(a_{1k}^{mk}); a_{lk} = \phi_k^l a_l\right\}$ and $a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m$.

Proof. First we show that F is well-defined. Let $([a_1], \cdots, [a_m]) = ([b_1], \cdots, [b_m])$, then $[a_i] = [b_i]$ for all $1 \leq i \leq m$. Thus, for every $1 \leq i \leq m$ there exist $k_i \geq i$ such that $\phi_{k_i}^i a_i = \phi_{k_i}^i b_i$ Hence, we have $[x] \in F([a_1], \cdots, [a_m]) \Leftrightarrow x \in f(\phi_{k_{(m+1)}}^1 a_1, \cdots, \phi_{k_{(m+1)}}^m a_m)$ for $k_{(m+1)} \geq 1, \cdots, m \Leftrightarrow x \in f(\phi_k^1 a_1, \cdots, \phi_k^m a_m)$ for $k \geq k_1, \cdots, k_m, k_{(m+1)} \Leftrightarrow$
$x \in f(\phi_k^{k_1} \phi_{k_1}^1 a_1, \cdots, \phi_k^{k_m} \phi_{k_m}^m a_m) \Leftrightarrow$
$x \in f(\phi_k^{k_1} \phi_{k_1}^1 b_1, \cdots, \phi_k^{k_m} \phi_{k_m}^m b_m) \Leftrightarrow$
$x \in f(\phi_k^1 b_1, \cdots, \phi_k^m b_m) \Leftrightarrow [x] \in F([b_1], \cdots, [b_m])$.
So, F is well-defined. Now, let $[a]_1^{2m-1} \in \overline{X}$ and $k \geq 1, \cdots, 2m-1$. Also, let $[x] \in F([a]_1^{i-1}, F([a]_i^{m+i-1}), [a]_{m+i}^{2m-1})$ for all $1 \leq i \leq m$. So, there exists $[y] \in F([a]_i^{m+i-1})$ such that $[x] \in F([a]_1^{i-1}, [y], [a]_{m+i}^{2m-1})$ Therefore $y \in f(a_{ik}, \cdots, a_{(m+i-1)k})$ for some $k \geq i, \cdots, m+i-1$ such that $\phi_k^i a_i = a_{ik}, \cdots, \phi_k^{m+i-1} a_{m+i-1} = a_{(m+i-1)k}$. Also $x \in f(a_{1k}, \cdots, a_{(i-1)k}, b_k, a_{(m+i)k}, \cdots, a_{(2m-1)k})$ for some $k \geq 1, \cdots, i-1, m+i, \cdots, 2m-1$ such that $\phi_k^1 a_1 = a_{1k}, \cdots, \phi_k^{i-1} a_{i-1} = a_{(i-1)k}, \cdots \phi_k^{m+i} a_{m+i} = a_{(m+i)k}, \cdots, \phi_k^{2m-1} a_{2m-1} = a_{(2m-1)k}$. Since $y \in R_k$, $b_k = \phi_k^k y = y$. Thus by associativity of R_k we have $x \in f\left(a_{1k}, \cdots, a_{(i-1)k}, b_k, a_{(m+i)k}, \cdots, a_{(2m-1)k}\right)$
$= f\left(a_{1k}, \cdots, a_{(i-1)k}, y, a_{(m+i)k}, \cdots, a_{(2m-1)k}\right)$
$\subseteq f\left(a_{1k}, \cdots, a_{(i-1)k}, f(a_{ik}, \cdots, a_{(m+i-1)k}), a_{(m+i)k}, \cdots, a_{(2m-1)k}\right) = f\left(a_{1k}, \cdots, a_{(j-1)k}, f(a_{jk}, \cdots, a_{(m+j-1)k}), a_{(m+j)k}, \cdots, a_{(2m-1)k}\right)$. So there exist $y' \in f(a_{jk}, \cdots, a_{(m+j-1)k})$ such that $x \in f(a_{1k}^{(j-1)k}, y', a_{(m+j)k}^{(2m-1)k})$. Then, $[y'] \in F([a]_j^{m+j-1})$ and since $\phi_k^k y' = y'$, we have $[x] \in F([a]_1^{j-1}, [y'], [a]_{m+j}^{2m-1})$ Hence $[x] \in F\left([a]_1^{j-1}, F([a]_j^{m+j-1}), [a]_{m+j}^{2m-1}\right)$ and then for any $m \geq j > i \geq 1$ $F\left([a]_1^{i-1}, F([a]_i^{m+i-1}), [a]_{m+i}^{2m-1}\right)$
$\subseteq F\left([a]_1^{j-1}, F([a]_j^{m+j-1}), [a]_{m+j}^{2m-1}\right)$. Similarly, we can prove the converse of above inclusion. So $(\overline{X}, F)$ is associative.

Now, we prove the reproduction axiom. It is clear that

$F([a]_1^{i-1}, \overline{X}, [a]_{i+1}^m) \subseteq \overline{X}$, for all $[a]_1^m \in \overline{X}$.

Assume that $[a]_1^m \in \overline{X}$. Then

$\phi_k^j a_j = a_{jk} \in R_k$ for any $k \geq 1, \cdots, m$ where $1 \leq j \leq m$. Since R_k is an m-ary hypergroup, so for all $1 \leq i \leq m$, $a_{ik} \in R_k = f(a_{1k}, \cdots, a_{(i-1)k}, R_k, a_{(m+i)k}, \cdots, a_{mk})$.

Then, there exists $a_k^{'} \in R_k$ such that $a_{ik} \in f\,(a_{1k}^{(i-1)k}, a_k^{'}, a_{(m+i)k}^{mk})$ and $\phi_k^k a_k^{'} = a_k^{'}$. Thus $[a_i] \in F([a]_1^{i-1}, [a_k^{'}], [a]_{i+1}^m)$ $\subseteq F([a]_1^{i-1}, \bar{X}, [a]_{i+1}^m)$. Thus, $\bar{X} = F([a]_1^{i-1}, \bar{X}, [a]_{i+1}^m)$ for all $[a]_1^{i-1}, [a]_{i+1}^m \in \bar{X}$ So $(\bar{X}, F)$ is an m-ary hypergroup.

Since (R_k, f) is a canonical n-ary hypergroup, there exists a unique element $0_k \in R_k$ such that $\{a_k\} = f\,(a_k, 0_k^{(m-1)})$ for all $a_k \in R_k$. Now let $[a_i] \in \bar{X}$. Then there exists $k \geq i$ such that $\phi_k^i a_i = a_k \in R_k$, since (R_k, f) is canonical. Hence $\{a_k\} = f\,(a_k, 0_k^{(m-1)})$ such that $\phi_k^k 0_k = 0_k$. Since 0_k is unique, so $\{[a_i]\} = F([a_i], [0_k]^{(m-1)})$. Also, since (R_k, f) has canonical property, there exists uniquely $a^{'} \in R_k$ such that $0_k \in f\,(a_k, a_k^{'}, 0_k^{(m-2)})$.

Since $\phi_k^k a_k^{'} = a_k^{'}$, and hence $[0_k] \in F([a_i], [a_k^{'}], [0_k]^{(m-2)})$. Therefore, we show $[a_k^{'}]$ as the inverse of $[a_i]$ is in $\bar{X}$. Finally, we investigate the reversibility property. Let $[x] \in F([a]_1^m)$. Then there exist $k \geq 1, \cdots, n$ such that $x \in f\,(a_{1k}^{mk})$ and $\phi_k^1 a_1 = a_{1k}, \cdots, \phi_k^m a_m = a_{mk}$. Since $x \in R_k$, so $\phi_k^k x = x$ and since (R_k, f) is canonical, we have $a_{ik} \in f\,(a_{1k}^{'}, \cdots, a_{(i-1)k}^{'}, x, a_{(i+1)k}^{'}, a_{mk}^{'})$ for all $1 \leq i \leq m$. Hence $[a_i] \in F([a_1]^{'}, \cdots, [a_{i-1}]^{'}, [x], [a_{i+1}]^{'}, \cdots, [a_m]^{'})$.

Theorem 1.2. $(\bar{X}, F, G)$ is a Krasner (m, n)-hyperring where $G([b_1], \cdots, [b_n]) = [g\,(b_{1k}^{nk})]$ such that $b_{lk} = \phi_k^l b_l$ for $k \geq l$ and $1 \leq l \leq n$.

Proof. By Theorem 1.1. $(\bar{X}, F)$ is an m-ary canonical hypergroup. Similarly, we can prove $(\bar{X}, G)$ is an n-ary semigroup. We prove that G is distributive with respect to F. Assume that $[a]_1^{i-1}, [x]_1^m, [a]_{i+1}^n \in \bar{X}$ and for any $1 \leq i \leq n$, $[y] \in G([a]_1^{i-1}, F([x]_1^m), [a]_{i+1}^n)$. Then there exists $[b] \in F([x]_1^m)$ such that $[y] = G([a]_1^{i-1}, [b], [a]_{i+1}^n)$. Hence, $b \in f\,(x_{1k}, \cdots, x_{mk})$ for $\phi_k^1 x_1 = x_{1k}, \cdots, \phi_k^m x_m = x_{mk}$ with $k \geq 1, \cdots, m$. Also $y = g\,(a_{1k}^{(i-1)k}, c_k, a_{(i+1)k}^{nk})$, such that $\phi_k^1 a_1 = a_{1k}, \cdots, \phi_k^{i-1} a_{i-1} = a_{(i-1)k}$ $, \phi_k^{i+1} a_{i+1} = a_{(i+1)k}, \cdots, \phi_k^n a_n = a_{nk}$ for some $k \geq 1, \cdots, i-1, i+1, \cdots, n$. Since, $b \in R_k$, $c_k = \phi_k^k b = b$. Therefore, by distributivity on R_k, we obtain that $y = g\,(a_{1k}^{(i-1)k}, c_k, a_{(i+1)k}^{nk})$

$$= g\,(a_{1k}^{(i-1)k}, b, a_{(i+1)k}^{nk})$$
$$\in g\left(a_{1k}^{(i-1)k}, f\,(x_{1k}^{mk}), a_{(i+1)k}^{nk}\right)$$
$$= f\left(g\,(a_{1k}^{(i-1)k}, x_{1k}, a_{(i+1)k}^{nk}), \cdots, g\,(a_{1k}^{(i-1)k}, x_{mk}, a_{(i+1)k}^{nk})\right),$$

for all $1 \leq i \leq n$. Since, $x_{1k}, \cdots, x_{mk} \in R_k$, $\phi_k^k x_i = x_i$. It follows that $[y] \in F\left(G([a]_1^{i-1}, [x_1], [a]_{i+1}^n), \cdots, G([a]_1^{i-1}, [x_m], [a]_{i+1}^n)\right)$.

Similarly, the converse of above inclusion can be obtained.

Theorem 1.3. $\bar{X}$ is $\varinjlim R_i$ in (R_i, ϕ_j^i), the direct system of Krasner $(m,\ n)$-hyperrings indexed by I.

Proof. Consider $\alpha_i : R_i \to \bar{X}$ defined by $\alpha_i(a_i) = [a_i]$ and the following diagram:

$$\begin{tikzcd} R_i \arrow[rr, "\phi_j^i"] \arrow[dr, "\alpha_i"'] & & R_j \arrow[dl, "\alpha_j"] \\ & \bar{X} & \end{tikzcd}$$

For all $a_i \in R_i$ we have $\alpha_j \circ \phi_j^i(a_i) = [\phi_j^i a_i] = [a_i] = \alpha_i(a_i)$ and so the diagram is commutative. Now, let R be a Krasner (m, n)-hyperring and $\{\mu_i \mid \mu_i : R_i \to R\}$ a family of strong homomorphism with $\mu_i = \mu_j \circ \phi_j^i$. Define $\beta : \bar{X} \to R$ by

$\beta([a_i]) = \mu_i(a_i)$. We show that β is a strong homomorphism and so the universal mapping property holds. First we show that β is well defined. Let $[a_i] = \left[b_j\right]$ then there exists $k \geq i, j$ such that

$\phi_k^i a_i = \phi_k^j b_j$. Thus $\mu_k(\phi_k^i a_i) = \mu_k(\phi_k^j b_j)$, and so $\mu_i(a_i) = \mu_j(b_j)$. Hence, β is well defined. Now let $[a_1],\cdots,[a_m] \in \bar{X}$ then $\beta\big(F([a]_1^m)\big) =$ $\{\beta([x]) \mid x \in f(a_{1k}^{mk});$
$a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m\}$
$= \mu_k\big(f(a_{1k}^{mk})\big); a_{lk} = \phi_k^l a_l$ for $k \geq l, 1 \leq l \leq m$
$= f\big(\mu_k(a_{1k}),\cdots,\mu_k(a_{mk})\big)$ (since μ_k is strong homomorphism) $= f\big(\mu_k(\phi_k^1 a_1),\cdots,\mu_k(\phi_k^m a_m)\big)$
$= f\big(\mu_1(a_1),\cdots,\mu_m(a_m)\big) = f\big(\beta([a_1]),\cdots,\beta([a_m])\big)$.
similarly, we can show that $\beta\big(G([b]_1^n)\big) = g\big(\beta([a_1]),\cdots,\beta([a_n])\big)$ So, β is a strong homomorphism such that $\beta \circ \alpha_i = \mu_i$.

Γ^**-Relation and Direct systems*

Let (R, f, g) be an Krasner (m,n)-hyperring. Mirvakili and Davvaz in [16] introduced the strongly compatible relation Γ^* on (m,n)-hyperrings as follow:

For every $k \in \mathbf{Z}^+$ and $l_1^s \in \mathbf{Z}^+$, where $s = k(m-1)+1$ we have $x\,\Gamma_{k;l_1^s}\,y$ if and only if $\{x, y\} \subseteq f_{(k)}(u_1,\ldots,u_s)$ with $u_i = g_{(l_i)}(x_{i1}^{it_i})$ for some $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$ and $1 \leq i \leq s$. Now, set $\Gamma_k = \bigcup_{l_1^s \in N} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}^*} \Gamma_k$. It is shown that the transitive closure of Γ, denoted by Γ^*, is the smallest strongly compatible relation on R, such that $(R/\Gamma^*, F, G)$ is an (m,n)-ring, where the m-ary and n-ary operations F and G in R/Γ^* are defined as follows: $F(\Gamma^*(x)_1^m) = \Gamma^*(c)$, for all $c \in f(\Gamma^*(x)_1^m)$; $G(\Gamma^*(x)_1^n) = \Gamma^*(d)$ for all $d \in g(\Gamma^*(x)_1^n)$. Consider $\phi: R \to R/\Gamma^*$ and suppose $\omega_R = \{x \in R \mid \phi(x) = 0_{R/\Gamma^*}\}$, where 0 is the unit element of m-ary group $(R/\Gamma^*, F)$. Then the unit element of $(R/\Gamma^*, F)$ is equal to ω_R, i.e., $F(\Gamma^*(x), \omega_R^{(m-1)}) = \Gamma^*(x)$ for all $x \in R$.

Proposition 2.1. If (R_i, ϕ_j^i) is a direct system of Krasner (m,n)-hyperrings indexed by a direct set I, then $(R_i/\Gamma_{R_i}^*, \phi_j^{*i})$ is a direct system of (m,n)-rings, where $\phi_j^{*i}: R_i/\Gamma_{R_i}^* \to R_j/\Gamma_{R_j}^*$ defined by $\phi_j^{*i}\big(\Gamma_{R_i}^*(a_i)\big) = \Gamma_{R_j}^*(\phi_j^i a_i)$.

Proof. It is clear that $(R_i/\Gamma_{R_i}^*, \phi_j^{*i})$ is a family of (m,n)-rings and strong homomorphisms. Clearly, ϕ_i^{*i} is the identity for all $i \in I$. Now, for $i \leq j \leq k$, we have
$(\phi_k^j \circ \phi_j^i)^*\big(\Gamma_{R_i}^*(a_i)\big) = \phi_k^{*i}\big(\Gamma_{R_i}^*(a_i)\big)$
$= \Gamma_{R_k}^*(\phi_k^i a_i) = \Gamma_{R_k}^*\big(\phi_k^j \circ \phi_j^i(a_i)\big)$
$= \Gamma_{R_k}^*\big(\phi_k^j(\phi_j^i a_i)\big) = \phi_k^{*j}\big(\Gamma_{R_j}^*(\phi_j^i a_i)\big)$
$= \phi_k^{*j} \circ \phi_j^{*i}\big(\Gamma_{R_i}^*(a_i)\big)$.

Therefore, one concludes $(\phi_k^j \circ \phi_j^i)^* = \phi_k^{*i} = \phi_k^{*j} \circ \phi_j^{*i}$.

Proposition 2.2. For $[a_i],[b_j] \in \bar{X}$ let $[a_i]\theta[b_j]$ if $\phi_k^i a_i \Gamma_{R_k} \phi_k^j b_j$ for $k \geq i, j$. Then $\theta = \Gamma_{\bar{X}}$.

Proof. Let $[a_{j'}]\Gamma_{\bar{X}}[b_j]$, then there exist $[x]_{i1}^{it_i} \in \bar{X}$ and $h, l_1^s \in \mathbf{Z}^+$ where $t_i = l_i(m-1)$ and $1 \leq i \leq s$ such that $\{[a_{j'}],[b_j]\} \subseteq F_{(h)}([u_1],\cdots,[u_s])$ where $[u_i] = G_{(l_i)}\big([x]_{i1}^{it_i}\big)$. Suppose $x_{i1} \in R_{i1},\cdots,x_{it_i} \in R_{it_i}$ where $1 \leq i \leq s$. Since $G_{(l_i)}\big([x]_{i1}^{it_i}\big) = [g_{(l_i)}\big(x_{i1k}^{it_ik}\big)]$ such that $x_{i1k} = \phi_k^{i1} x_{i1},\ldots,x_{it_ik} = \phi_k^{it_i} x_{it_i}$ for some $k \geq i1,\cdots,it_i, j', j$, then $\{[a_{j'}],[b_j]\} = \{[\phi_k^{j'} a_{j'}],[\phi_k^j b_j]\} \subseteq$ $F_{(h)}\big(G_{(l_1)}\big([x]_{11}^{1t_1}\big),\cdots,G_{(l_s)}\big([x]_{s1}^{st_s}\big)\big)$.

Hence for some $n \geq k$ we obtain $\{\phi_n^{j'} a_{j'}, \phi_n^j b_j\} \subseteq$ $F_{(h)}\big(G_{(l_1)}\big([x]_{11}^{1t_1}\big),\cdots,G_{(l_s)}\big([x]_{s1}^{st_s}\big)\big)$, which implies that $\phi_n^{j'} a_{j'} \Gamma_{R_n} \phi_n^j b_j$ Thus $[a_{j'}]\theta[b_j]$. Conversely, if $[a_{j'}]\theta[b_j]$ then there exists $k \geq j', j$ such that $\phi_k^{j'} a_{j'} \Gamma_{R_k} \phi_k^j b_j$ and so there exist $x_{i1}^{it_i} \in R_k \subseteq \bigcup R_i$ and $h, l_1^s \in \mathbb{N}$ where $t_i = l_i(m-1)$ and $1 \leq i \leq s$ such that $\{\phi_k^{j'} a_{j'}, \phi_k^j b_j\} \subseteq f_{(h)}(u_1, \cdots, u_s)$, where $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Thus $\{[\phi_k^{j'} a_{j'}], [\phi_k^j b_j]\} \subseteq F_{(h)}([u_1], \cdots, [u_s])$ where $[u_i] = G_{(l_i)}([x]_{i1}^{it_i})$. Hence $[a_{j'}]\Gamma_{\bar{X}}[b_j]$, and so $\Gamma_{\bar{X}} = \theta$.

Theorem 2.3. Let (R_i, ϕ_j^i) be a direct system of Krasner (m, n)-hyperrings indexed by a direct set I, and let Γ^* be the fundamental relation of $\varinjlim R_i$. Therfore, $\varinjlim(R_i / \Gamma_{R_i}^*) \cong (\varinjlim R_i)/\Gamma^*$.

Proof. Define $\mu: \varinjlim(R_i / \Gamma_{R_i}^*) \to (\varinjlim R_i)/\Gamma^*$ by $[\Gamma_{R_i}^*(a_i)] \mapsto \Gamma^*([a_i])$ for any $a_i \in R_i$. Since

$\Gamma^*([a_i]) = \Gamma^*([b_j]) \Leftrightarrow \phi_k^i a_i \Gamma_{R_k}^* \phi_k^j b_j$ for some $k \geq i, j$

$\Leftrightarrow \Gamma_{R_k}^*(\phi_k^i a_i) = \Gamma_{R_k}^*(\phi_k^j b_j)$

$\Leftrightarrow \phi_k^{*i} \Gamma_{R_i}^*(a_i) = \phi_k^{*j} \Gamma_{R_j}^*(b_j)$

$\Leftrightarrow [\Gamma_{R_i}^*(a_i)] = [\Gamma_{R_j}^*(b_j)]$,

μ is well defined and one to one. Now, we show that μ is a strong homomorphism: $\mu(F([\Gamma_{R_i}^*(a_i)]_1^m)) = \mu([F(\phi_k^{*i}\Gamma_{R_i}^*(a_i)_1^m)])$ for some $k \geq i_1, \cdots, i_m = \mu([F(\Gamma_{R_k}^*(\phi_k^i a_i)_1^m)])$ $= \mu([\Gamma_{R_k}^*(f((\phi_k^i a_i)_1^m))]) := \mu([\Gamma_{R_k}^*(x)])$, for $x \in f((\phi_k^i a_i)_1^m) = \Gamma^*([x])$ where $x \in f((\phi_k^i a_i)_1^m)$. On the other hands, one obtains $F(\mu([\Gamma_{R_i}^*(a_i)])_1^m) = F(\Gamma^*([a_i])_1^m) = \Gamma^*([x])$, $[x] \in f([a_i]_1^m) = \Gamma^*([x])$, where $x \in f((\phi_k^i a_i)_1^m)$ for some $k \geq i_1, \cdots, i_m$. So $\mu(F([\Gamma_{R_i}^*(a_i)]_1^m)) = F(\mu([\Gamma_{R_i}^*(a_i)])_1^m)$. Similarly, $\mu(G([\Gamma_{R_i}^*(a_i)]_1^n)) = G(\mu([\Gamma_{R_i}^*(a_i)])_1^n)$.

Proposition 2.4. If $a_i \in \omega_{R_i}$ then $[\Gamma_{R_i}^*(a_i)]$ is the zero element in $\varinjlim(R_i / \Gamma_{R_i}^*)$ and $[a_i] \in \omega_{\varinjlim R_i}$.

Proof. Let $F([\Gamma_{R_i}^*(a_i)]^{(m-1)}, [\Gamma_{R_j}^*(b_j)]) = [X]$, where $X = F((\phi_k^{*i}\Gamma_{R_i}^*(a_i))^{(m-1)}, \phi_k^{*j}\Gamma_{R_j}^*(b_j))$ for $k \geq i, j$. So $((\phi_k^{*i}(\omega_{R_i}))^{(m-1)}, \phi_k^{*j}\Gamma_{R_j}^*(b_j))$ $= F((\omega_{R_k})^{(m-1)}, \phi_k^{*j}\Gamma_{R_j}^*(b_j)) = \phi_k^{*j}\Gamma_{R_j}^*(b_j)$ Thus $F([\Gamma_{R_i}^*(a_i)]^{(m-1)}, [\Gamma_{R_j}^*(b_j)]) = [\phi_k^{*j}\Gamma_{R_j}^*(b_j)] = [\Gamma_{R_j}^*(b_j)]$. Hence $[\Gamma_{R_i}^*(a_i)]$ is the zero element in $\varinjlim(R_i / \Gamma_{R_i}^*)$. By Theorem 3.3. it concludes $\Gamma^*([a_i])$ is the zero element in $(\varinjlim R_i)/\Gamma^*$ and so $[a_i] \in \omega_{\varinjlim R_i}$.

It is easy to verify that if (R, f, g) is an Krasner (m, n)-hyperring, then ω_R, is an (m, n)-subhyperring.

Corollary 2.5. Let (R_i, ϕ_j^i) be a direct system of Krasner (m, n)-hyperrings indexed by a direct set I. Then $(\omega_{R_i}, \phi_j^i |_{\omega_{R_i}})$ is a direct system and $(\varinjlim \omega_{R_i})/\Gamma^*$ is a zero (m, n)-ring.

Proof. Using Theorem 2.3. $(\varinjlim \omega_{R_i})/\Gamma^* \cong \varinjlim(\omega_{R_i} / \Gamma_{R_i}^*) = 0$

Lemma 2.6. $\omega_{\varinjlim R_i} = \{[a_i] \mid a_i \in \bigcup R_i,\ \phi_k^i a_i \in \omega_{R_k} \text{ for some } k \geq i\}$.

Proof. Set $D = \{[a_i] \mid a_i \in \bigcup R_i, \phi_k^i a_i \in \omega_{R_k} \text{ for some } k \geq i\}$ Clearly, $D \subseteq \omega_{\varinjlim R_i}$. Let $[a_i] \in \omega_{\varinjlim R_i}$, then $a_i \in \bigcup R_i$ and $\Gamma^*([a_i]) = 0_{(\varinjlim R_i)/\Gamma^*}$.

By Theorem 2.3, it concludes that $[\Gamma^*(a_i)] = [\Gamma^*(0_{R_i})]$. Thus $[\phi_k^i a_i] = [\Gamma^*(0_{R_k})]$ for some $k \geq i$ and so $\phi_k^i a_i \in \omega_{R_k}$ for some $k \geq i$. Therefore, $\omega_{\varinjlim R_i} \subseteq D$.

Theorem 2.7. Let $Dir(I)$ be the category of all direct systems of Krasner (m, n)-hyperrings and strong homomorphisms over direct set I and $(m, n)-\mathrm{KH}_r$ be the category of Krasner (m, n)-hyperrings and strong homomorphisms. Suppose $\varinjlim : Dir(I) \to (m, n)-\mathrm{KH}_r$ by

$\{R_i,\phi_j^i\}\mapsto\{[a]\mid a\in\bigcup R_i\}$ and whenever $t:\{R_i,\phi_j^i\}\to\{H_i,\phi_j^{'i}\}$ is a morphism in $Dir(I)$ $\varinjlim t=\vec{t}:\varinjlim R_i\to\varinjlim H_i$ where $a=a_i\in R_i$ and $\vec{t}([a])=[t_i a_i]$, where $t_i:R_i\to H_i$. Then $\varinjlim$ is an exact functor.

Proof. It is easy to see that $\varinjlim$ is a functor. We prove that if $\{A_i,\phi_j^i\}\to\{B_i,\phi_j^{'i}\}\to\{C_i,\phi_j^{''i}\}$ is a sequence of morphisms of direct systems over I, such that $\omega_{A_i}\xrightarrow{n_i}A_i\xrightarrow{t_i}B_i\xrightarrow{s_i}C_i\xrightarrow{u_i}\omega_{C_i}$ (*) is exact for any $i\in I$, then $\omega_{\varinjlim A_i}\xrightarrow{\vec{n}}\varinjlim A_i\xrightarrow{\vec{t}}\varinjlim B_i\xrightarrow{\vec{s}}\varinjlim C_i\xrightarrow{\vec{u}}\omega_{\varinjlim C_i}$ is an exact sequence of Krasner (m,n)-hyperrings. Hence,

(i) $Ker\vec{t}=Im\vec{n}$; suppose $[a]\in Ker\vec{t}$, then $\vec{t}[a]\in\omega_{\varinjlim B_i}$. Let, $a=a_i\in A_i$, then $\vec{t}[a]=[t_i a_i]\in\omega_{\varinjlim B_i}$ which implies $\phi_k^{'i}(t_i a_i)\in\omega_{\varinjlim B_k}$ for some $k\ge i$, and so $\phi_k^{'i}(t_i a_i)=t_k\phi_k^i(a_i)\in\omega_{B_k}$ Since the sequence (*) is exact for every $i\in I$, so $Ker\ t_k=Im\ n_k=\omega_{A_k}$ and then $t_k\phi_k^i(a_i)\in Im\ n_k=\omega_{A_k}$.

Hence $[a_i]=[\phi_k^i a_i]\in\varinjlim\omega_{A_k}=\varinjlim Im\ n_k=Im\ \vec{n}$. Therefore, $Ker\vec{t}\subseteq Im\vec{n}$. Conversely, if $[a]\in Im\vec{n}=\varinjlim\omega_{A_i}$, then $a_i\in\omega_{A_i}$ and so $t_i a_i\in\omega_{B_i}$ which implies $\vec{t}[a_i]=[t_i a_i]\in\omega_{\varinjlim B_i}$. Thus $[a_i]\in Ker\vec{t}$.

(ii) $Ker\vec{s}=Im\vec{t}$; let $[x]=[b_i]\in Ker\vec{s}$, then $\vec{s}[b_i]=[s_i b_i]\in\omega_{\varinjlim C_i}$.

Thus $\phi_k^{''i}(s_i b_i)\in\omega_{\varinjlim C_k}$ for some $k\ge i$. We have $\phi_k^{''i}(s_i b_i)=s_k\phi_k^{'i}(b_i)\in\omega_{C_k}$. Since (*) is exact for every $i\in I$, we obtain that $\phi_k^{'i}(b_i)\in Ker\ s_k=Im\ t_k$ and so there exists $a_k\in A_k$ such that $t_k a_k=\phi_k^{'i}(b_i)$. Thus $\vec{t}[a_k]=[t_k a_k]=[\phi_k^{'i}b_i]=[b_i]$ and so $[b_i]\in Im\vec{t}$. Conversely, $\vec{s}\circ\vec{t}[a_i]=\vec{s}([t_i a_i])=[s_i\circ t_i a_i]$. Since (*) is exact for any $i\in I$, then $s_i\circ t_i a_i\in\omega_{C_i}$, and hence $[s_i\circ t_i a_i]\in\omega_{\varinjlim C_i}$. Thus $\vec{s}\circ\vec{t}[a_i]\in\omega_{\varinjlim C_i}$ and $Im\vec{t}\subseteq Ker\vec{s}$.

(iii) $Ker\vec{u}=Im\vec{s}$; the proof is similar to (ii).

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