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Numerical Solution of a Free Boundary Problem from Heat Transfer by the Second Kind Chebyshev Wavelets

B. Babayar-Razlighi*

Department of Mathematics, Faculty of Sciences, Qom University of Technology, Qom, Islamic Republic of Iran

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Abstract

In this paper we reduce a free boundary problem from heat transfer to a weakly Singular Volterra integral equation of the first kind. Since the first kind integral equation is ill posed, and an appropriate method for such ill posed problems is based on wavelets, then we apply the Chebyshev wavelets to solve the integral equation. Numerical implementation of the method is illustrated by two benchmark problems originated from heat transfer. The behavior of the initial and free boundary heat functions along the position axis during the time have been shown through some three dimensional plots. The convergence of the method is pointed in the end of section 2. The numerical examples show the accuracy and applicability of the method from application and programming points of views.

Keywords: Volterra integral equation of the first kind; Heat equation; Numerical solution; Second kind Chebyshev wavelets; Free boundary.

Introduction

In this paper we consider the following free boundary problem from heat transfer in one spatial dimension.

$$u_t = u_{xx}, \quad 0 < x < 1, 0 < t, \tag{1}$$

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{2}$$

$$\int_0^{s(t)} u(x,t)dx = g(t), \quad 0 < s(\mathrm{t}) < 1, 0 < t, \tag{3}$$

$$u(1,t) = h(t), \quad 0 < t. \tag{4}$$

Where $u(x,t)$ is the temperature function and is unknown, and the data and the free boundary function $\mathrm{s(t)}$ are known.

In [1, 2] the authors have solved similar problems with product integration technique, which is a good method on short time intervals and the second kind integral equations [3-9]. The product integration is not efficient for the integral equations of the first kind.

Since the solution of the associated first kind integral equation is in $L^2(0,1)$, which is spans by wavelets, hence we solve this problem by wavelets on $[0,1)$. We show the efficiency of the method by two sample problems.

For more application examples of the Chebyshev wavelets for differential and integral equations see [10, 11].

1. Equivalent Integral Equation

Definition 1.1. The fundamental solution of heat

* Corresponding author: Tel: +982532941965; Fax: +982536641604; Email: babayar@qut.ac.ir

equation is denoted by $K(x,t)=\dfrac{\exp\left\{\dfrac{-x^2}{4t}\right\}}{\sqrt{4\pi t}}$.

Definition 1.2. The theta function is defined as follow

$$\theta(x,t)=\sum_{m=-\infty}^{\infty}K(x+2m,t),\quad t>0.$$

Proposition 1.3. We have the following list of methods for generating solutions of heat equation from other solutions [12].

Linear combinations: If u_1,u_2 are solutions, then $\alpha u_1+\beta u_2$ is a solution, where α,β are constants.

Translations: If $u(x,t)$ is a solution, then so is $u(x-\xi,t-\tau)$, where ξ and τ are translation parameters.

Convolutions: If $u(x,t)$ is a solution, then so are $\int_a^b u(\mathrm{x}-\xi,\mathrm{t})\phi(\xi)d\xi$ and $\int_a^b u(x,t-\xi)\phi(\xi)d\xi$. However $\int_a^t u(x,t-\xi)\phi(\xi)d\xi$ is a solution only if $u(x,0)=0$.

Integrate with Respect to a parameter: If $u(x,t,\alpha)$, is a solution for each α in $a\le\alpha\le b$, then so is $\int_a^b u(x,t,\alpha)d\alpha$.

Affine Transformation: If $u(x,t)$ is a solution, then so is $u(\lambda x,\lambda^2 t)$ for any constant λ.

Integration with Respect to x and t : If $u(x,t)$ is a solution, then so is $\int_{x_0}^{x}u(\xi,t)d\xi$, provided that $u_x(x_0,t)=0$. Also, if $u(x,t)$ is a solution, then so is $\int_a^t u(x,\eta)d\eta$, provided that $u(x,a)=0$.

Lamma 1.4. The bounded solution of

$$\begin{aligned}
&v_t=v_{xx}, && 0<x<1,\ 0<t,\\
&v(x,0)=\mathrm{f}(x), && 0<x<1,\\
&v(0,\mathrm{t})=v(1,\mathrm{t})=0, && 0<t,
\end{aligned}$$

is given by

$$v(x,t)=\int_0^1\{\theta(x-\xi,t)-\theta(x+\xi,t)\}f(\xi)d\xi.$$

Proof. See exercise 3.6. of [12].

Lamma 1.5. For $t>0$,

$$\lim_{t\to 1^-}-2\int_0^t\frac{\partial\theta}{\partial x}(x,t-\tau)g(\tau)d\tau=0,$$

for any Lebesgue integrable g. Moreover, this limit is taken on uniformly with respect to t contained in compact sets.

Proof. See lemma 6.2.5 of [12].

Lamma 1.6. For $t>0$, and piecewise-continuous h,

$$\lim_{t\to 1^-}2\int_0^t\frac{\partial\theta}{\partial x}(\mathrm{x}-1,\mathrm{t}-\tau)h(\tau)d\tau=h(t),$$

is uniform for t belonging to a compact subset of an interval of continuity of h.

Proof. See lemma 6.2.3 and last analysis of section 6.2 from [12].

Theorem 1.7. For continuous f,g,h and s with $g(0)=\int_0^{s(0)}f(\xi)d\xi$, the solution of problem (1)-(4) has the representation

$$\begin{aligned}
u(x,t)=&\int_0^1\{\theta(\mathrm{x}-\xi,\mathrm{t})-\theta(\mathrm{x}+\xi,\mathrm{t})\}f(\xi)d\xi\\
&-2\int_0^t\frac{\partial\theta}{\partial x}(\mathrm{x},\mathrm{t}-\tau)\phi(\tau)d\tau\\
&+2\int_0^t\frac{\partial\theta}{\partial x}(\mathrm{x}-1,\mathrm{t}-\tau)h(\tau)d\tau,
\end{aligned}\tag{5}$$

if and only if ϕ is a piecewise-continuous solution of the integral equation

$$\begin{aligned}
&g(t)-\int_0^{s(t)}\int_0^1\{\theta(\mathrm{x}-\xi,\mathrm{t})-\theta(\mathrm{x}+\xi,\mathrm{t})\}f(\xi)d\xi dx\\
&=2\int_0^t\theta(0,\mathrm{t}-\tau)\phi(\tau)d\tau-2\int_0^t\theta(s(t),\mathrm{t}-\tau)\phi(\tau)d\tau\\
&+2\int_0^t\theta(\mathrm{s(t)}-1,\mathrm{t}-\tau)h(\tau)d\tau-2\int_0^t\theta(-1,\mathrm{t}-\tau)h(\tau)d\tau.
\end{aligned}\tag{6}$$

Proof. We are going to search $u(x,t)=u_1(x,t)+u_2(x,t)+u_3(x,t)$ such that

u_1, u_2 and u_3 satisfy heat equation and each of them meet one of the equations (2)-(4). For this aim let

$$u_1(x,t)=\int_0^1 \{\theta(x-\xi,t)-\theta(x+\xi,t)\} f(\xi)d\xi,$$

$$u_2(x,t)=-2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\phi(\tau)d\tau,$$

$$u_3(x,t)=2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)h(\tau)d\tau.$$

Let $L=\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}$, by the uniformly convergent of the theta function, $L\theta(x,t)=\sum_{m=-\infty}^{\infty} LK(\mathrm{x}+2\mathrm{m},\mathrm{t})=0$.

By the linear combinations and translations of Proposition 1.3, $\{\theta(x-\xi,t)-\theta(x+\xi,t)\}f(\xi)$ is a solution of the heat equation. So according to integrate with respect to a parameter $u_1(x,t)$, satisfies the heat equation. From the convolutions of Proposition 1.3, for the investigation of $u_2(x,t)$ as a solution of heat equation it is sufficient to show that $\lim_{t\to 0^+}\frac{\partial\theta}{\partial x}(x,t)=0.$

$$\lim_{t\to 0^+}\frac{\partial\theta}{\partial x}(x,t)=$$

$$\lim_{t\to 0^+}\left(\begin{array}{l}\frac{\partial K}{\partial x}(x,t)+\sum_{m=1}^{\infty}\frac{-(x+2m)}{2t}K(x+2m,t)+\\ \sum_{m=1}^{\infty}\frac{-(x+2m)}{2t}K(x+2m,t-\tau)\end{array}\right)$$

$$=\lim_{t\to 0^+}\frac{-\mathrm{x}\exp\left\{\frac{-x^2}{2t}\right\}}{4\sqrt{\pi}t^{3/2}}+$$

$$\sum_{m=1}^{\infty}\lim_{t\to 0^+}\frac{-(x+2m)\exp\left\{\frac{-(x+2m)^2}{2t}\right\}}{4\sqrt{\pi}t^{3/2}}$$

$$+\sum_{m=1}^{\infty}\lim_{t\to 0^+}\frac{-(x-2m)\exp\left\{\frac{-(x-2m)^2}{2t}\right\}}{4\sqrt{\pi}t^{3/2}}=0,$$

where the uniform convergence of the series allow us to change limit and sigma in lines 4 and 5. Similar evaluations show that $\lim_{t\to 0^+}\frac{\partial\theta}{\partial x}(x-1,t)=0$, and hence $u_3(x,t)$ is a solution of the heat equation. Substitution of $u(x,t)=u_1(x,t)+u_2(x,t)+u_3(x,t)$ in (3) and application of Fubini's theorem, forces (6).

Now we investigate the equations (2) and (4).

$$u(x,0)=u_1(x,0)+u_2(x,0)+u_3(x,0)=u_1(x,0)$$

$$=\lim_{t\to 0^+}\int_0^1\{\theta(x-\xi,t)-\theta(x+\xi,t)\}f(\xi)d\xi=f(\mathrm{x}),$$

where the last equality obtained from Lemma 1.4, and hence u satisfies (2). Applications of Lemmas 1.4-1.6 yield that

$$u(1,\mathrm{t})=u_1(1,\mathrm{t})+u_2(1,\mathrm{t})+u_3(1,\mathrm{t})=u_3(1,\mathrm{t})$$

$$=\lim_{t\to 1^-}2\int_0^t\frac{\partial\theta}{\partial x}(x-1,t-\tau)\,\mathrm{h}(\tau)d\tau=h(\mathrm{t}),$$

and then u satisfies (4). Since the solution of (1)-(4) is unique (Chapter3 of [12]) then Eq. (5) is the only solution.

2. Chebyshev wavelet technique

Wavelets were first applied in geophysics to analyze data from seismic surveys, which are used in oil and mineral exploration to get "pictures" of layering in the subsurface rock [13]. There are several bases for wavelets, such as Haar wavelet, Daubechies Wavelets, Legendre wavelets, Chebyshev wavelets, and so on [14-17]. In this paper we consider the second kind Chebyshev wavelets. Chebysheve wavelets $\psi_{nm}(t)=\psi(k,n,m,t)$ have 4 arguments where $n=1,2,\ldots,2^{k-1}, k\in\mathbb{Z}^+, m$ is the order of Chebyshev polynomials and t is normalized time. They are defined on the interval $[0,1)$ as follow

$$\psi_{n,m}(t)=\begin{cases}2^{k/2}\sqrt{2/\pi}U_m(2^k t-2n+1) & \frac{n-1}{2^{k-1}}\le t<\frac{n}{2^{k-1}}\\ 0, & otherwise.\end{cases} \quad 7)$$

The coefficient $\sqrt{\frac{2}{\pi}}$ is for the orthonormality, the dilation parameter is $a=2^{k-1}$ and translation parameter is $b=(2n-1)2^{-k}$. Here, $U_m(t)$ are the well- known m^{th} order second kind Chebyshev polynomials with respect to weight function $w(t)=\sqrt{1-t^2}$, which is defined on the interval $[-1,1]$. A function $\phi(\mathrm{t})$ defined over $[0,1)$ can be expressed by the Chebyshev wavelets as

$$\phi(\mathrm{t})=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(\mathrm{t}), \quad (8)$$

where

$$c_{n,m} = \langle \phi, \psi_{n,m} \rangle_{w_n} = \int_0^1 w_n(\mathrm{t})\phi(\mathrm{t})\psi_{n,m}(\mathrm{t})dt, \qquad (9)$$

and $\langle .,. \rangle$ denotes the inner product in Hilbert space $L^2(0,1)$, and $w_n(t) = w\left(2^k t - 2n + 1\right)$.

One can consider the following truncated approximation for series (8)

$$\phi(\mathrm{t}) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(\mathrm{t}) = C^T \Psi(\mathrm{t}). \qquad (10)$$

Where C and $\Psi(\mathrm{t})$ are $(2^{k-1}M \times 1)$ vectors, that can be given by

$$C = \begin{bmatrix} c_{1,0} \\ c_{1,1} \\ \vdots \\ c_{1,M-1} \\ c_{2,0} \\ \vdots \\ c_{2,M-1} \\ \vdots \\ c_{2^{k-1},0} \\ \vdots \\ c_{2^{k-1},M-1} \end{bmatrix}; \Psi(\mathrm{t}) = \begin{bmatrix} \psi_{1,0}(\mathrm{t}) \\ \psi_{1,1}(\mathrm{t}) \\ \vdots \\ \psi_{1,M-1}(\mathrm{t}) \\ \psi_{2,0}(\mathrm{t}) \\ \vdots \\ \psi_{2,M-1}(\mathrm{t}) \\ \vdots \\ \psi_{2^{k-1},0}(\mathrm{t}) \\ \vdots \\ \psi_{2^{k-1},M-1}(\mathrm{t}) \end{bmatrix}, \qquad (11)$$

and for the simplicity of numerical evaluations, we rearrange the indices in the second representation of vectors by the mapping

$$\left(\left[\frac{i-1}{M}\right] + 1, \mathrm{i} - \mathrm{M}\left[\frac{i-1}{M}\right] - 1\right) \to i, \; i = 1, \dots, 2^{k-1}M,$$

where [x] denotes the greatest integer less than or equal to x.

The following theorem gives the convergence and accuracy estimation of the second kind Chebyshev wavelets expansion [18].

Theorem 2.1. Let $\phi(\mathrm{t})$ be a second-order derivative square-integrable function defined on $[0,1)$ with bounded second-order derivative, say $|\phi''(\mathrm{x}) \le \mathrm{B}|$ for some constant B, then

(i) $\phi(\mathrm{t})$ can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converges to $\phi(\mathrm{t})$ uniformly, in the form (8).

(ii) $0 \le \sigma_{\phi,k,M} \le$

$$\frac{\sqrt{\pi}B}{2^3}\left(\sum_{n=2^{k-1}+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(\mathrm{m}-1)^4}\right)^{\frac{1}{2}} \to 0 \, as \; M, k \to \infty,$$

where

$$\sigma_{\phi,k,M} = \left(\int_0^1 \left|\phi(t) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t)\right|^2 w_n(\mathrm{t})\,\mathrm{dt}\right)^{\frac{1}{2}},$$

is the L^2 norm error.

3. Application of the Method

The integral equation (6) in theorem 2.7 is as follow

$$\int_0^t \ker(t,\tau)\phi(\tau)d\tau = r(t), t > 0. \qquad (12)$$

Where

$$r(t) = \mathrm{g(t)} + \sum_{m=0}^{\infty} r_m(\mathrm{t}), \qquad (13)$$

$$r_0(t) = -\int_0^{s(t)}\int_0^1 \left(K(\mathrm{x}-\xi, \mathrm{t}) - K(\mathrm{x}+\xi, \mathrm{t})\right) f(\xi)\,\mathrm{d}\xi\,\mathrm{dx}$$

$$+2\int_0^t \left(K(-1, \mathrm{t}-\tau) - K(\mathrm{s(t)}-1, \mathrm{t}-\tau)\right) h(\tau)d\tau,$$

$$r_m(t) = \qquad m \in \mathbb{N}$$

$$-\int_0^{s(t)}\int_0^1 \begin{pmatrix} K(\mathrm{x}-\xi+2m, \mathrm{t}) + K(\mathrm{x}-\xi-2m, \mathrm{t}) \\ -K(\mathrm{x}+\xi+2m, \mathrm{t}) - K(\mathrm{x}+\xi-2m, \mathrm{t}) \end{pmatrix} f(\xi)\,\mathrm{d}\xi\,\mathrm{dx}$$

$$+2\int_0^t \begin{pmatrix} K(-1+2m, \mathrm{t}-\tau) + K(-1-2m, \mathrm{t}-\tau) \\ -K(\mathrm{s(t)}-1+2m, \mathrm{t}-\tau) - K(\mathrm{s(t)}-1-2m, \mathrm{t}-\tau) \end{pmatrix} h(\tau)d\tau,$$

$$\ker(t,\tau) = \sum_{m=0}^{\infty} \ker_m(\mathrm{t},\tau), \qquad (14)$$

$$\ker_0(t,\tau) = 2\left(K(0, \mathrm{t}-\tau) - K(\mathrm{s(t)}, \mathrm{t}-\tau)\right),$$

$$ker_m(t,\tau) = \qquad m \in \mathbb{N}$$

$$2\begin{pmatrix} K(2m, \mathrm{t}-\tau) + K(-2m, \mathrm{t}-\tau) \\ -K(\mathrm{s(t)}+2m, \mathrm{t}-\tau) - K(\mathrm{s(t)}-2m, \mathrm{t}-\tau) \end{pmatrix}.$$

From the uniform absolute convergence of the series for $\theta(x,t)$ and its partial derivatives, the equation (13) can be approximated by

$$r(t) \approx g(t) + \sum_{m=0}^{N} r_m(t)$$

$$\approx G^T\Psi(t) + \sum_{m=0}^{N} R_m^T \Psi(t) = R^T \Psi(t),$$

where N is sufficiently large positive integer and we apply equation (10) in the second row. Similar evaluations are true for other series of $\theta(x,t)$ and its partial derivatives. Every terms of equations (13) and (14) is evaluable by Mathematica software, and in numerical process the functions in (13) and (14) approximated by some finite terms of sigma, as mentioned for $\mathrm{r(t)}$. Using Eq. (10) for approximate $\phi(\tau) \simeq \Phi^T \Psi(\tau)$ and $\mathrm{r(t)} \simeq R^T \Psi(\mathrm{t})$ in Eq. (12), forces

$$\Phi^T \int_0^t \ker(t,\tau)\Psi(\tau)d\tau = R^T \Psi(\mathrm{t}), t>0. \tag{15}$$

Where $\Phi^T = \left[\phi_1,\dots,\phi_{M\,2^{k-1}}\right]^T$ is unknown vector. Let $w(t)=\int_0^t \ker(t,\tau)\Psi(\tau)d\tau$, then from Eq. (10) we obtain $w(t) \simeq W\,\Psi(\mathrm{t})$, where W is a $2^{k-1}M \times 2^{k-1}M$ known matrix. Substitution of these quantities in (13) yields $\Phi^T W\,\Psi(\mathrm{t}) = \mathrm{R}^T\,\Psi(\mathrm{t})$. Hence the linear system $W^T\Phi = \mathrm{R}$, must be solved.

In the following examples we evaluate matrices R, W by the 16-point Gaussian integration rule.

4. Algorithm of the Method

For illustration of superiority and applicability of the method we give an algorithm for the numerical solution of the problem (1)-(4) by the proposed method.

Step1 Input the known functions f, g, h, s;

Define the fundamental solution of heat equation by

$$K(x,t)=\frac{\exp\left\{\frac{-x^2}{4t}\right\}}{\sqrt{4\pi t}};$$

Solve the linear system $W^T\Phi = \mathrm{R}$ as mentioned in section 3 and obtain $\phi(\tau) \simeq \Phi^T\Psi(\tau)$. Note that we apply the numerical integration such as Gaussian integration rule for the numerical evaluation of R and W.

Step2 Since $\Psi(\tau)$ is a piecewise function, then $\phi(\tau)$ is also a piecewise function. For example with M=5, k=3,

$$\phi(\tau) \simeq \begin{cases} \phi_{10}+\phi_{11}\tau+\phi_{12}\tau^2+\phi_{13}\tau^3+\phi_{14}\tau^4 & 0 \le t < \frac{1}{4} \\ \phi_{20}+\phi_{21}\tau+\phi_{22}\tau^2+\phi_{23}\tau^3+\phi_{24}\tau^4 & \frac{1}{4} \le t < \frac{1}{2} \\ \phi_{30}+\phi_{31}\tau+\phi_{32}\tau^2+\phi_{33}\tau^3+\phi_{34}\tau^4 & \frac{1}{2} \le t < \frac{3}{4} \\ \phi_{40}+\phi_{41}\tau+\phi_{42}\tau^2+\phi_{43}\tau^3+\phi_{44}\tau^4 & \frac{3}{4} \le t < 1 \\ 0 & otherwise, \end{cases}$$

which is obtained from step1, and all of ϕ_{ij} are known constants. The function u_2 in the theorem 1.7 is as follow

$$u_2(x,t) = -2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\phi(\tau)d\tau$$

$$\simeq \frac{1}{2}u_{2,0}(x,t)+\sum_{m=1}^{N} u_{2,\mathrm{m}}(x,t),$$

where

$$u_{2,\mathrm{m}}(x,t)=\int_0^t \begin{pmatrix} \frac{x+2m}{t-\tau}K(x+2m,t-\tau)+ \\ \frac{x-2m}{t-\tau}\mathrm{K}(x-2m,t-\tau) \end{pmatrix} \phi(\tau)d\tau,$$

and N is a sufficiently large integer. For evaluation of $u_{2,m}$, first of all, evaluate on $0 \le t < \frac{1}{4}$, then for $\frac{1}{4} \le t < \frac{1}{2}$ evaluate

$$u_{2,\mathrm{m}}(x,t)=\int_0^{\frac{1}{4}} \begin{pmatrix} \frac{x+2m}{t-\tau}K(x+2m,t-\tau)+ \\ \frac{x-2m}{t-\tau}K(x-2m,t-\tau) \end{pmatrix} \phi(\tau)d\tau$$

$$+\int_{\frac{1}{4}}^{t} \begin{pmatrix} \frac{x+2m}{t-\tau}K(x+2m,t-\tau)+ \\ \frac{x-2m}{t-\tau}K(x-2m,t-\tau) \end{pmatrix} \phi(\tau)d\tau,$$

and so on. Since $\phi(\tau)$ is piecewise polynomial, then these evaluations are straight forward.

The function u_3 in the theorem 1.7 is

$$u_3(x,t)=2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)h(\tau)d\tau$$

$$-\int_0^t \frac{x-1}{t-\tau}K(x-1,t-\tau)h(\tau)d\tau$$

$$-\sum_{m=1}^{N}\int_0^t \frac{x-1+2m}{t-\tau}K(x-1+2m,t-\tau)h(\tau)d\tau$$

$$-\sum_{m=1}^{N}\int_0^t \frac{x-1-2m}{t-\tau}K(x-1-2m,t-\tau)h(\tau)d\tau.$$

Suppose $u_c(x,t)=-\int_0^t \frac{K(x-1,t-\tau)}{t-\tau}h(\tau)d\tau$,

then

$$u_c(x,t)=-\int_0^t \frac{K(x-1,\tau)}{\tau}h(\mathrm{t}-\tau)d\tau$$

$$-\frac{1}{\sqrt{4\pi}}\int_0^t \frac{\exp\left(\frac{-y}{4\tau}\right)}{\tau^{3/2}}h(\mathrm{t}-\tau)d\tau\,|_{y=(\mathrm{x}-1)^2}.$$

The last integral is evaluable by any software such as Mathematica . Thus we let

$$u_3(x,t)\approx(\mathrm{x}-1)u_c(x,t)$$

$$+\sum_{m=1}^{N}(x-1+2m)u_c(x-1+2m,t)$$

$$+\sum_{m=1}^{N}(x-1-2m)u_c(x-1-2m,t).$$

The function u_1 in the theorem 1.7 is

$$u_1(x,t)=\int_0^1\{\theta(x-\xi,t)-\theta(x+\xi,t)\}f(\xi)d\xi,$$

$$\approx\frac{1}{2}u_{1,0}(x,t)+\sum_{m=1}^{N}u_{1,\mathrm{m}}(x,t),$$

where

$$u_{1,\mathrm{m}}(x,t)=$$

$$\int_0^1\begin{pmatrix}K(x-\xi+2m,t)+K(x-\xi-2m,t)\\ -K(x+\xi+2m,t)-K(x+\xi-2m,t)\end{pmatrix}f(\xi)d\xi,$$

are evaluable by any software.

Step3 Print and plot $\tilde{u}(x,t)=\tilde{u}_1(x,t)+\tilde{u}_2(x,t)+\tilde{u}_3(x,t)$, where $\tilde{u}_j(x,t), \mathrm{j}=1,2,3$ are the truncated approximation of $u_j(x,t)$ untile sufficiently large integer N, evaluated in step 2.

5. Numerical Examples

Example 5.1. In the problem (1)-(4), for $s(t)=\frac{1}{3}+\frac{1}{2+t}$, $g(t)=\mathrm{e}^{-t}\sin\left[\frac{1}{3}+\frac{1}{2+t}\right]$,

$\mathrm{f}(x)=\cos(\mathrm{x})$, $\mathrm{h}(t)=\mathrm{e}^{-t}\cos(1)$, The exact solution is $u(x,t)=\mathrm{e}^{-t}\cos(\mathrm{x})$. With this data, we solve the integral equation (12) by Chebyshev wavelets technique with M=5, k=3, and then we put this solution in the representation formula (5) to obtain $\tilde{u}$ as approximated solution. Table 1 shows the absolute error of $\tilde{u}$ in the points $(0.15\mathrm{i}, 0.15\,\mathrm{j}), i,j=1,\ldots,6$. Figure 1 shows the three dimensional plot of $\tilde{u}(\mathrm{x},\mathrm{t})$ on $[0,1]\times[0,1]$.

Example 5.2 In the problem (1)-(4), for $s(t)=\frac{1}{2}+\frac{1}{3+t}$, $g(t)=-\mathrm{e}^{-t}\left(-1+\cos\left[\frac{1}{2}+\frac{1}{3+t}\right]\right)$,

$\mathrm{f}(x)=\sin(\mathrm{x})$, $\mathrm{h}(t)=\mathrm{e}^{-t}\sin(1)$, the exact solution is $u(x,t)=\mathrm{e}^{-t}\sin(\mathrm{x})$. With this data we solve the

Table 1. The (i,j)th element of the matrix is $|u(0.15\mathrm{i},0.15\,\mathrm{j})-\tilde{u}(0.15\,\mathrm{i},0.15\,\mathrm{j})|$ in example 5.1

2.7×10^{-9}	2.2×10^{-8}	2.4×10^{-9}	3.3×10^{-8}	5.2×10^{-9}	1.5×10^{-8}
3.1×10^{-10}	5.2×10^{-8}	3.7×10^{-8}	1.5×10^{-8}	3.4×10^{-8}	2.3×10^{-8}
3.4×10^{-10}	3.6×10^{-8}	7.6×10^{-9}	1.9×10^{-8}	1.7×10^{-8}	9.7×10^{-10}
2.2×10^{-10}	2.5×10^{-8}	1.2×10^{-8}	5.7×10^{-9}	1.3×10^{-8}	1.4×10^{-8}
5.9×10^{-11}	1.5×10^{-8}	1.7×10^{-9}	4.5×10^{-9}	2.5×10^{-8}	5.0×10^{-9}
1.6×10^{-11}	2.8×10^{-8}	1.6×10^{-8}	1.6×10^{-8}	3.7×10^{-8}	1.0×10^{-8}

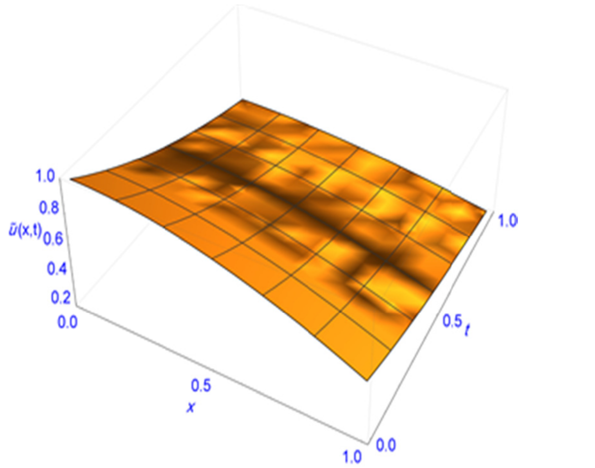

Figure 1. Variation of the $\tilde{u}(x,t)$ as a function of (x,t) for Example 5.1.

Table 2. The (i,j)th element of the matrix is $|u(0.15\,\mathrm{i},0.15\,\mathrm{j})-\tilde{u}(0.15\,\mathrm{i},0.15\,\mathrm{j})|$ in example 5.2.

$$\begin{bmatrix} 5.8\times10^{-15} & 6.5\times10^{-15} & 7.6\times10^{-15} & 9.5\times10^{-15} & 2.4\times10^{-14} & 9.6\times10^{-15} \\ 1.2\times10^{-15} & 1.2\times10^{-15} & 4.2\times10^{-15} & 9.2\times10^{-15} & 5.9\times10^{-15} & 7.5\times10^{-15} \\ 4.6\times10^{-15} & 1.1\times10^{-15} & 4.8\times10^{-15} & 9.2\times10^{-15} & 1.2\times10^{-14} & 4.2\times10^{-15} \\ 5.2\times10^{-15} & 2.2\times10^{-16} & 3.6\times10^{-15} & 3.6\times10^{-15} & 9.1\times10^{-15} & 1.1\times10^{-15} \\ 4.0\times10^{-15} & 2.2\times10^{-16} & 2.3\times10^{-15} & 3.3\times10^{-16} & 1.3\times10^{-15} & 5.6\times10^{-17} \\ 2.0\times10^{-15} & 1.0\times10^{-15} & 1.7\times10^{-16} & 2.0\times10^{-15} & 4.2\times10^{-15} & 5.6\times10^{-16} \end{bmatrix}$$

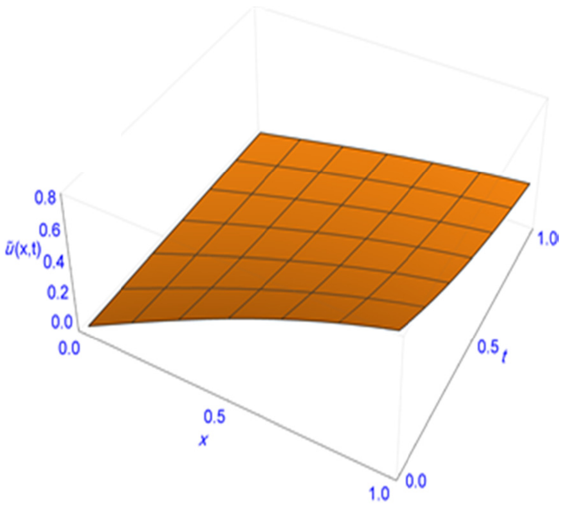

Figure 2. Variation of the $\tilde{u}(x,t)$ as a function of (x,t) for Example 5.2

integral equation (12) by Chebyshev wavelets technique with M=5, k=3, and then we put this solution in the representation formula (5) to obtain $\tilde{u}$ as approximated solution. Similar to the previous example Table 2 shows the absolute error of $\tilde{u}$ in the points $(0.15\,\mathrm{i},0.15\,\mathrm{j}), i,j=1,\ldots,6$. Figure 2 shows the three dimensional plot of $\tilde{u}(\mathrm{x},\mathrm{t})$ on $[0,1]\times[0,1]$.

Results

In this study, we consider the theta function by the following form [12]

$$\theta(x,t)=K(x,t)+\sum_{m=0}^{\infty}\{K(x+2m,t)+K(x-2m,t)\}.$$

θ is a uniform absolute convergence in its domain and hence series and integral can be interchanged (for example see the corollary of theorem 7.16 of Rudin [19], so we can write the equations (13),(14) and these relations help us to apply the proposed method for such free boundary problems from heat transfer phenomenon. There are many heat transfer problems which reduce to the system of Volterra integral equations of the first kind and theta function representation [12]. Many of such problems solved directly by some numerical approaches such as Runge-Kutta methods. Applicability of wavelets for such problems caused the high accuracy for the solution of the system as we have shown in two benchmark sample problems from heat transfer.

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