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Max-Min averaging operator: fuzzy inequality systems and resolution

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ABSTRACT

Minimum and maximum operators are two well-known t-norm and s-norm used frequently in fuzzy systems. In this paper, two different types of fuzzy inequalities are simultaneously studied where the convex combination of minimum and maximum operators is applied as the fuzzy relational composition. Some basic properties and theoretical aspects of the problem are derived and four necessary and sufficient conditions are presented. Moreover, an algorithm is proposed to solve the problem and an example is described to illustrate the algorithm.

Keyword: Fuzzy relation, fuzzy relational inequality, fuzzy compositions and fuzzy averaging operator.

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1 Introduction

In this paper, we study the following fuzzy system in which the constraints consist of the intersection of two types fuzzy relational inequalities defined by "Fuzzy Max-Min"

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averaging operator:

$$\begin{array}{l} A \diamond x \leq b^1 \\ D \diamond x \geq b^2 \\ x \in [0,1]^n \end{array} \tag{1}$$

where $I_1 = \{1,2,..,m_1\}$, $I_2 = \{m_1+1, m_1+2,.., m_1+m_2\}$ and $J = \{1,2,..,n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \leq a_{ij} \leq 1$ ($\forall i \in I_1$ and $\forall j \in J$) and $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1–dimensional fuzzy vector in $[0,1]^{m_1}$ (i.e., ${}_{0 \leq b_i^1 \leq 1}, \forall i \in I_1$) and $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2–dimensional fuzzy vector in $[0,1]^{m_2}$ (i.e., ${}_{0 \leq b_i^2 \leq 1}, \forall i \in I_2$). Moreover, "$\diamond$" is the max-$\diamond$ composition where $\diamond$ is "Fuzzy Max-Min" averaging operator, that is,

$$\diamond(x,y) = \lambda \min\{x,y\} + (1-\lambda)\max\{x,y\}$$

in which $\lambda \in [0,1]$. Furthermore, let $S(A,b^1)$ and $S(D,b^2)$ denote the feasible solutions sets of inequalities type1 $A \diamond x \leq b^1$ and type2 $D \diamond x \geq b^2$, respectively, that is, $S(A,b^1) = \left\{x \in [0,1]^n \ : \ A \diamond x \leq b^1\right\}$ and $S(D,b^2) = \left\{x \in [0,1]^n \ : \ D \diamond x \geq b^2\right\}$. Also, let $S(A,D,b^1,b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S(A,D,b^1,b^2) = S(A,b^1) \bigcap S(D,b^2)$.
By these notations, problem (1) can be also expressed as follows:

$$\begin{array}{l} \max\limits_{j \in J}\{\diamond(a_{ij},x_j)\} \leq b_i^1 \ , \ i \in I_1 \\ \max\limits_{j \in J}\{\diamond(d_{ij},x_j)\} \geq b_i^2 \ , \ i \in I_2 \\ x \in [0,1]^n \end{array} \tag{2}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max-"Fuzzy Max-Min" fuzzy relational equations.
The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [54]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [50]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [40,48].
The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Actually, The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the

minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [47]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2,3,12,13,22 – 30,43,44,47].
Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [22–30,49,51,52,55,57,58,60,63,66]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)[12,13,15 –20,21,31,32,41,65].
The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1,8,9,11 – 30,38,42,45,53,56,59,61,65]. The topic of the linear optimization problem was also investigated with max-product operation [11,34,46]. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [14,37,61], max-Discontinuous t-norms composition [29], max-monotone operators composition [30] and max-t-norm composition [15 – 20, 22 – 28,35,42,56].
Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [4,6,10,22 – 28,32,39,45,62].
The optimization problem subjected to various versions of FRI could be found in the literature as well [12,13,15 – 21,29 – 32,64,65]. Yang [64] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Xiao et al. [65] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.
It is well – known that for any membership values $\mu_A(x)$ and $\mu_B(x)$ of arbitrary fuzzy sets A and B, the membership value of their union $A \bigcup B$ (defined by any S-norm) lies in the interval $[\max\{\mu_A(x), \mu_B(x)\}, S_{ds}\{\mu_A(x), \mu_B(x)\}]$. Similarly, the membership value of the intersection $A \bigcap B$ (defined by any T-norm) lies in the interval

$$\left[T_{dp}\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \mu_B(x)\}\right]$$

Therefore, the union and intersection operators cannot cover the interval between $\min\{\mu_A(x), \mu_B(x)\}$ and $\max\{\mu_A(x), \mu_B(x)\}$. The operators that cover the interval

$$[\min\{\mu_A(x), \mu_B(x)\}, \max\{\mu_A(x), \mu_B(x)\}]$$

are called averaging operators. Similar to the S-norms and T-norms, an averaging operator is a function from $[0,1] \times [0,1]$ to $[0,1]$. Many averaging operators were proposed in the literature [7]. In this paper, problem (1) was investigated where $\diamond$ is "Fuzzy

Max-Min" averaging operator. Clearly, the Max-Min averages cover the whole interval $[\min\{\mu_A(x), \mu_B(x)\}, \max\{\mu_A(x), \mu_B(x)\}]$ as the parameter λ changes from 0 to 1. The remainder of the paper is organized as follows. In section 2, some basic properties and the shape of the feasible solutions set of the type1 "Fuzzy Max-Min"-Inequalities have been attained. It is proved that the set is formed by a unique minimum and a unique maximum solution. Also, two necessary and sufficient conditions for the feasibility of this type of fuzzy systems are presented. The shape of the feasible region of the type2 "Fuzzy Max-Min"-Inequalities is investigated in section 3. It is shown that this region is determined as a union of the finite number of minimal solutions and a unique maximum solution. Moreover, two necessary and sufficient conditions for the feasibility of this type of fuzzy systems are presented. In section4, the intersection of these two fuzzy systems is studied. A necessary and sufficient condition is proposed to determine the feasibility of the main problem and an algorithm is presented to resolve Problem (1). Finally, in section 5 an example is described to illustrate.

2. Basic properties of type1 "Fuzzy Max-Min" – Inequalities

This section describes the structural properties concerning system $A \diamond x \le b^1$. This fuzzy system consists of m_1 inequalities $\max_{j\in J}\{\diamond(a_{ij}, x_j)\} \le b_i^1$ $(\forall i \in I_1)$. For this purpose, we firstly investigate corresponding partial inequalities $\diamond(a_{ij}, x_j) \le b_i^1$, $i \in I_1$ and $j \in J$. As before, for each $i \in I_1$, let $S(a_i, b_i^1) = \left\{x \in [0,1]^n \,:\, \max_{j\in J}\{\diamond(a_{ij}, x_j)\} \le b_i^1\right\}$. Similarly, let$S(a_{ij}, b_i^1) = \left\{x_j \in [0,1] \,:\, \diamond(a_{ij}, x_j) \le b_i^1\right\}$ that is, set $S(a_{ij}, b_i^1)$ includes all solutions $x_j \in [0,1]$ such that

$$\diamond(a_{ij}, x_j) = \lambda \min\left\{a_{ij}, x_j\right\} + (1-\lambda)\max\left\{a_{ij}, x_j\right\} \le b_i^1 \quad , i \in I_1 \ , \ j \in J$$

Definition 1. For each $i \in I_1$ and each $j \in J$, define

$$\underline{W}_{ij}(\lambda) = \frac{b_i^1 - (1-\lambda)a_{ij}}{\lambda}$$

The following four lemmas are easily verified for each $i \in I_1$ and each $j \in J$, and are very useful for some next proofs.

Lemma 1. Suppose that $\lambda > 0$. Then, $a_{ij} \le b_i^1 \Leftrightarrow a_{ij} \le \underline{W}_{ij}(\lambda)$.

Lemma 2. Suppose that $\lambda < 1$. Then, $a_{ij} \le b_i^1 \Leftrightarrow a_{ij} \le \underline{W}_{ij}(1-\lambda)$.

Also, Lemmas 1 and 2 are true if "$\le$" is replaced by "$<$", "$\ge$" or "$>$".

Lemma 3. Suppose that $\lambda > 0$. Then, $\underline{W}_{ij}(\lambda) \ge 0 \Leftrightarrow a_{ij} = 0 \;\; or \;\; \frac{b_i^1 - a_{ij}}{-a_{ij}} \le \lambda \le 1$.

Lemma 4. Suppose that $\lambda < 1$. Then, $\underline{W}_{ij}(1-\lambda) \le 1 \Leftrightarrow a_{ij} = 1 \;\; or \;\; 0 \le \lambda \le \frac{b_i^1 - 1}{a_{ij} - 1}$.

Lemma 5 below determines set $S(a_{ij}, b_i^1)$ where $a_{ij} \le b_i^1$.

Lemma 5. Suppose that $a_{ij} \le b_i^1$. Then,

$$S(a_{ij}, b_i^1) = \begin{cases} \left[0\,,\, \min\left\{\underline{W}_{ij}(1-\lambda), 1\right\}\right] & , 0 \le \lambda < 1 \\ [0\,,1] & , \lambda = 1 \end{cases} .$$

Proof. By $a_{ij} \le b_i^1$, $\lambda < 1$ and Lemma2, we have ${}_{0\le a_{ij}\le \underline{W}_{ij}(1-\lambda)}$. Thus, ${}_{\underline{W}_{ij}(1-\lambda)\ge 0}$. Assume that $\lambda < 1$ and $x_j \in \left[0\,,\, \min\left\{\underline{W}_{ij}(1-\lambda), 1\right\}\right]$. If ${}_{a_{ij}=1}$ or ${}_{0\le\lambda\le(b_i^1-1)/(a_{ij}-1)}$, then by Lemma4, $x_j \in \left[0\,,\, \underline{W}_{ij}(1-\lambda)\right]$. Therefore, in this case we have $\diamond(a_{ij}, x_j) \le \diamond(a_{ij}, \underline{W}_{ij}(1-\lambda)) = \lambda\, a_{ij} + (1-\lambda)\,\underline{W}_{ij}(1-\lambda) = b_i^1$, i.e., $x_j \in S(a_{ij}, b_i^1)$. If ${}_{a_{ij}<1}$ and ${}_{\lambda>(b_i^1-1)/(a_{ij}-1)}$, then by Lemma4, $x_j \in [0, 1]$. In this case, we have

$$\begin{aligned}\diamond(a_{ij}, x_j) &\le \diamond(a_{ij}, 1) = \lambda\, a_{ij} + (1-\lambda) = \lambda\,(a_{ij}-1)+1\\ &< \left(\tfrac{b_i^1-1}{a_{ij}-1}\right)(a_{ij}-1)+1 = b_i^1\end{aligned}$$

Thus, $x_j \in S(a_{ij}, b_i^1)$. Moreover, if $\lambda = 1$, then $\diamond(a_{ij}, x_j) \le b_i^1$ is converted into $\min\left\{a_{ij}, x_j\right\} \le b_i^1$. In this case, we have trivially $x_j \in S(a_{ij}, b_i^1)$, $\forall x_j \in [0, 1]$.
Lemma 6 below determines set $S(a_{ij}, b_i^1)$ where $a_{ij} > b_i^1$.
Lemma 6. Suppose that $a_{ij} > b_i^1$. Then,

$$S(a_{ij}, b_i^1) = \begin{cases} \left[0\,,\, \underline{W}_{ij}(\lambda)\right] & , \left(\left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)\right) \le \lambda \le 1\\ \emptyset & , otherwise\end{cases}$$

Proof. Note that in this case we have $a_{ij} > 0$ and $0 < \left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right) < 1$. Since $a_{ij} > b_i^1$, Lemma1 implies that ${}_{\underline{W}_{ij}(\lambda)<a_{ij}\le 1}$. Thus, ${}_{\underline{W}_{ij}(\lambda)<1}$. Also, by Lemma3 we have ${}_{\underline{W}_{ij}(\lambda)\ge 0}$. Now, assume that $\left(\left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)\right) \le \lambda \le 1$ and $x_j \in \left[0\,,\, \underline{W}_{ij}(\lambda)\right]$. Hence, $\diamond(a_{ij}, x_j) \le \diamond(a_{ij}, \underline{W}_{ij}(\lambda)) = \lambda\, \underline{W}_{ij}(\lambda) + (1-\lambda)a_{ij} = b_i^1$ that means $x_j \in S(a_{ij}, b_i^1)$. On the other hand, if $x_j < 0$, then $x_j \notin S(a_{ij}, b_i^1)$. If $\left(\left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)\right) \le \lambda \le 1$ and $x_j > \underline{W}_{ij}(\lambda)$, then $b_i^1 = \diamond(a_{ij}, \underline{W}_{ij}(\lambda)) < \diamond(a_{ij}, x_j)$, i.e., $x_j \notin S(a_{ij}, b_i^1)$. Finally, if $\lambda < \left(\left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)\right)$, then we have

$$\diamond(a_{ij}, x_j) \ge \diamond(a_{ij}, 0) = (1-\lambda)a_{ij} > \left(1 - \frac{b_i^1 - a_{ij}}{-a_{ij}}\right)\frac{a_{ij}}{2} = b_i^1$$

that implies $x_j \notin S(a_{ij}, b_i^1)$.
Corollary 1. For each $i \in I_1$ and each $j \in J$,

$$S(a_{ij}, b_i^1) = \begin{cases} \left[0\,,\, \min\left\{\underline{W}_{ij}(1-\lambda), 1\right\}\right] & , a_{ij} \le b_i^1\,,\ 0 \le \lambda < 1\\ [0, 1] & , a_{ij} \le b_i^1\,,\ \lambda = 1\\ \left[0\,,\, \underline{W}_{ij}(\lambda)\right] & , a_{ij} > b_i^1\,,\ \left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right) \le \lambda \le 1\\ \emptyset & , a_{ij} > b_i^1\,,\ 0 \le \lambda < \left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)\end{cases}$$

The following theorem gives a necessary and sufficient condition for the feasibility of inequality.
Theorem 1. Let $i \in I_1$. $S(a_i, b_i^1) \ne \emptyset$ iff either $a_{ij} \le b_i^1$ or $\lambda \ge \left(b_i^1 - a_{ij}\right)\big/\left(-a_{ij}\right)$, $\forall j \in J$.

Proof. For an arbitrary $x \in [0,1]^n$, $x \in S(a_i, b_i^1)$ if and only if $\max_{j\in J}\{\diamond(a_{ij}, x_j)\} \le b_i^1$. Also, the last inequality holds true iff $\diamond(a_{ij}, x_j) \le b_i^1$, $\forall j \in J$. Therefore, $S(a_i, b_i^1) \neq \emptyset$ iff $S(a_{ij}, b_i^1) \neq \emptyset$, $\forall j \in J$. Now, the result follows from Corollary1.

Definition 2. Suppose that $S(a_i, b_i^1) \neq \emptyset$. We define $\overline{X}(i) = \left[\overline{X}(i)_1, \overline{X}(i)_2, ..., \overline{X}(i)_n\right]$ where

$$\overline{X}(i)_j = \begin{cases} \min\left\{\underline{W}_{ij}(1-\lambda), 1\right\} & , a_{ij} \le b_i^1 \ , \ 0 \le \lambda < 1 \\ 1 & , a_{ij} \le b_i^1 \ , \ \lambda = 1 \\ \underline{W}_{ij}(\lambda) & , a_{ij} > b_i^1 \ , \ \left(b_i^1 - a_{ij}\right)\Big/\left(-a_{ij}\right) \le \lambda \le 1 \end{cases}$$

By Theorem2 below, the solutions set $S(a_i, b_i^1)$ is completely determined. The theorem shows that $S(a_i, b_i^1)$ has actually the unique maximum solution, $\overline{X}(i)$, and the unique minimum solution, $\mathbf{0}$, where $\mathbf{0}$ is an n–dimensional zero vector.

Theorem 2. Suppose that $S(a_i, b_i^1) \neq \emptyset$. Then, $S(a_i, b_i^1) = \left[\mathbf{0}, \overline{X}(i)\right]$, $\forall i \in I_1$.

Proof. Similar to the proof of Theorem1, for each $x \in [0,1]^n$, $x \in S(a_i, b_i^1)$ iff $x_j \in S(a_{ij}, b_i^1)$, $\forall j \in J$. Thus, from Corollary1 and Definition2, for each $j \in J$ we have $x_j \in \left[0, \overline{X}(i)_j\right]$. Therefore, $x \in \left[0, \overline{X}(i)_1\right] \times \left[0, \overline{X}(i)_2\right] \times \cdots \times \left[0, \overline{X}(i)_n\right] = \left[\mathbf{0}, \overline{X}(i)\right]$.

Definition 3. Let $\overline{X}(i)$ be as in Definition2, $\forall i \in I_1$. We define $\overline{X} = \min_{i\in I_1}\left\{\overline{X}(i)\right\}$.

According to Theorem2 and the fact that $S(A, b^1) = \bigcap_{i\in I_1} S(a_i, b_i^1)$, the following theorem is attained.

Theorem 3. Suppose that $S(a_i, b_i^1) \neq \emptyset$, $\forall i \in I_1$. Then, $S(A, b^1) = \left[\mathbf{0}, \overline{X}\right]$.

Proof. by Theorem2, we have $S(A, b^1) = \bigcap_{i\in I_1} S(a_i, b_i^1) = \bigcap_{i\in I_1}\left[\mathbf{0}, \overline{X}(i)\right] = \left[\mathbf{0}, \min_{i\in I_1}\left\{\overline{X}(i)\right\}\right]$. Now, the result is obtained from Definition3.

Theorem3 determines the solutions set $S(A, b^1)$ as an n–dimensional interval $\left[\mathbf{0}, \overline{X}\right]$ with $\mathbf{0}$ as the unique minimum and $\overline{X}$ as the unique maximum solutions. The following Corollary gives a necessary and sufficient condition for the feasibility of general inequalities $A \diamond x \le b^1$.

Corollary 2. $S(A, b^1) \neq \emptyset$ iff $\mathbf{0} \in S(A, b^1)$.

3. Basic properties of type2 "Fuzzy Or" – Inequalities

In this section, the properties of system $D \diamond x \ge b^2$are investigated. This fuzzy system consists of m_2 inequalities $\max_{j\in J}\{\diamond(d_{ij}, x_j)\} \ge b_i^2$ ($\forall i \in I_2$). As the previous section, we firstly investigate corresponding partial inequalities $\diamond(d_{ij}, x_j) \ge b_i^2$, $i \in I_2$ and $j \in J$. For each $i \in I_2$, let $S(d_i, b_i^2) = \left\{x \in [0,1]^n \ : \ \max_{j\in J}\{\diamond(d_{ij}, x_j)\} \ge b_i^2\right\}$. Also, let$S(d_{ij}, b_i^2) = \left\{x_j \in [0,1] \ : \ \diamond(d_{ij}, x_j) \ge b_i^2\right\}$.

Definition 4. For each $i \in I_2$ and each $j \in J$, define

$$\overline{W}_{ij}(\lambda) = \frac{b_i^2 - (1-\lambda)d_{ij}}{\lambda}$$

The following four lemmas are easily verified for each $i \in I_2$ and each $j \in J$, and are very useful for some next proofs.

Lemma 7. Suppose that $\lambda>0$. Then, $d_{ij} \le b_i^2 \Leftrightarrow d_{ij} \le \overline{W}_{ij}(\lambda)$.

Lemma 8. Suppose that $\lambda<1$. Then, $d_{ij} \le b_i^2 \Leftrightarrow d_{ij} \le \overline{W}_{ij}(1-\lambda)$.

Also, Lemmas 7 and 8 are true if "$\le$" is replaced by "$<$", "$\ge$" or "$>$".

Lemma 9. Suppose that $\lambda>0$. Then,

$$\overline{W}_{ij}(\lambda) \ge 0 \Leftrightarrow d_{ij} = 0 \;\; or \;\; \left(b_i^2 - d_{ij}\right) \Big/ \left(-d_{ij}\right) \le \lambda \le 1$$

Lemma 10. Suppose that $\lambda<1$. Then,

$$\overline{W}_{ij}(1-\lambda) \le 1 \Leftrightarrow d_{ij} = 1 \;\; or \;\; 0 \le \lambda \le \left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right)$$

Lemma 11 below determines set $S(d_{ij}, b_i^2)$ where $d_{ij} < b_i^2$.

Lemma 11. Suppose that $d_{ij} < b_i^2$. Then,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\overline{W}_{ij}(1-\lambda)\,,\,1\right] & , 0 \le \lambda \le \left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right) \\ \emptyset & , otherwise \end{cases}$$

Proof. It is easy to verify that $d_{ij} < 1$ and $\left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right) < 1$. Also, by $d_{ij} < b_i^2$, $\lambda < 1$ and Lemma8 we have $0 \le d_{ij} < \overline{W}_{ij}(1-\lambda)$. Thus, $\overline{W}_{ij}(1-\lambda) > 0$. Additionally, Lemma10 implies $\overline{W}_{ij}(1-\lambda) \le 1$. Now, assume that $0 \le \lambda \le \left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right)$ and $x_j \in \left[\overline{W}_{ij}(1-\lambda)\,,\,1\right]$. So, $b_i^2 = \diamond(d_{ij}, \overline{W}_{ij}(1-\lambda)) \le \diamond(d_{ij}, x_j)$, i.e., $x_j \in S(d_{ij}, b_i^2)$. On the other hand, if $x_j > 1$, then x_j does not clearly belong to$S(d_{ij}, b_i^2)$. If $0 \le \lambda \le \left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right)$ and $x_j < \overline{W}_{ij}(1-\lambda)$, then it can be easily calculated $\diamond(d_{ij}, x_j) < \diamond(d_{ij}, \overline{W}_{ij}(1-\lambda)) = \lambda\, d_{ij} + (1-\lambda)\overline{W}_{ij}(1-\lambda) = b_i^2$ that implies $x_j \notin S(d_{ij}, b_i^2)$. If $\lambda > \left(b_i^2 - 1\right) \Big/ \left(d_{ij} - 1\right)$, then

$$\begin{aligned} \diamond(d_{ij}, x_j) \le \diamond(d_{ij}, 1) &= \lambda\, d_{ij} + (1-\lambda) = 1 + (d_{ij} - 1)\lambda \\ &< 1 + (d_{ij} - 1)\left(\left(b_i^2 - 1\right) \Big/ (d_{ij} - 1)\right) = b_i^2 \end{aligned}$$

, that is, $x_j \notin S(d_{ij}, b_i^2)$.

Lemma 12 below determines set $S(d_{ij}, b_i^2)$ where $d_{ij} \ge b_i^2$.

Lemma 12. Suppose that $d_{ij} \ge b_i^2$. Then,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\max\left\{0, \overline{W}_{ij}(\lambda)\right\}\,,\,1\right] & , 0 < \lambda \le 1 \\ [0, 1] & , \lambda = 0 \end{cases}$$

Proof. At first, we note $\left(b_i^2 - d_{ij}\right) \Big/ \left(-d_{ij}\right) \ge 0$. Since $d_{ij} \ge b_i^2$ and $\lambda > 0$, Lemma7 implies that $\overline{W}_{ij}(\lambda) \le d_{ij} \le 1$. Thus, $\overline{W}_{ij}(\lambda) \le 1$. Assume that $x_j \in \left[\max\left\{0, \overline{W}_{ij}(\lambda)\right\}\,,\,1\right]$. If either $d_{ij} =$

0 or $\left(b_i^2 - d_{ij}\right)/\left(-d_{ij}\right) \le \lambda \le 1$, then by Lemma9, $x_j \in \left[\overline{W}_{ij}(\lambda)\,,\,1\right]$. In this case, we have $\diamond(d_{ij}, x_j) \ge \diamond(d_{ij}, \overline{W}_{ij}(\lambda)) = \lambda\overline{W}_{ij}(\lambda) + (1-\lambda)d_{ij} = b_i^2$ that means $x_j \in S(d_{ij}, b_i^2)$. Furthermore, if $d_{ij} > 0$ and $\lambda < \left(b_i^2 - d_{ij}\right)/\left(-d_{ij}\right)$, $x_j \in [0,1]$ from Lemma9. In this case, we have

$$\diamond(d_{ij}, x_j) \ge \diamond(d_{ij}, 0) = (1-\lambda)\, d_{ij} > \left(1 - \left(\left(b_i^2 - d_{ij}\right)/\left(-d_{ij}\right)\right)\right) d_{ij} = b_i^2$$

, that is, $x_j \in S(d_{ij}, b_i^2)$. On the other hand, if $x_j > 1$ or $x_j < \max\left\{0, \overline{W}_{ij}(\lambda)\right\} = 0$, then obviously $x_j \notin S(d_{ij}, b_i^2)$. If $x_j < \max\left\{0, \overline{W}_{ij}(\lambda)\right\} = \overline{W}_{ij}(\lambda)$, then $\diamond(d_{ij}, x_j) < \diamond(d_{ij}, \overline{W}_{ij}(\lambda)) = b_i^2$, i.e., $x_j \notin S(d_{ij}, b_i^2)$. Moreover, if $\lambda = 0$, then $\diamond(d_{ij}, x_j) \ge b_i^2$ is converted into $\max\left\{d_{ij}, x_j\right\} \ge b_i^2$. In this case, we have trivially $x_j \in S(a_{ij}, b_i^1)$, $\forall x_j \in [0,1]$.

Corollary 3. For each $i \in I_2$ and each $j \in J$,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\max\left\{0, \overline{W}_{ij}(\lambda)\right\}\,,\,1\right] & d_{ij} \ge b_i^2\,,\ 0 < \lambda \le 1 \\ [0\,,\,1] & d_{ij} \ge b_i^2\,,\ \lambda = 0 \\ \left[\overline{W}_{ij}(1-\lambda)\,,\,1\right] & d_{ij} < b_i^2\,,\ 0 \le \lambda \le \left(b_i^2 - 1\right)/\left(d_{ij} - 1\right) \\ \emptyset & d_{ij} < b_i^2\,,\ \lambda > \left(b_i^2 - 1\right)/\left(d_{ij} - 1\right) \end{cases}$$

The following theorem gives a necessary and sufficient condition for the feasibility of inequality.

Theorem 4. Let $i \in I_2$. $S(d_i, b_i^2) \ne \emptyset$ iff there exists some $j \in J$ such that either $d_{ij} \ge b_i^2$ or $0 \le \lambda \le \left(b_i^2 - 1\right)/\left(d_{ij} - 1\right)$.

Proof. For an arbitrary $x \in [0,1]^n$, $x \in S(d_i, b_i^2)$ if and only if $\max\limits_{j\in J}\{\diamond(d_{ij}, x_j)\} \ge b_i^2$. Therefore, $x \in S(d_i, b_i^2)$ iff $\diamond(d_{ij}, x_j) \ge b_i^2$, for some $j \in J$. Therefore, $S(d_i, b_i^2) \ne \emptyset$ iff $S(d_{ij}, b_i^2) \ne \emptyset$, for some $j \in J$. Now, the result follows from Corollary3.

Definition 5. Suppose that $S(d_i, b_i^2) \ne \emptyset$. Let

$$J_1 = \left\{j \in J\ :\ d_{ij} \ge b_i^2\,,\ \lambda > 0\right\},\ J_2 = \left\{j \in J\ :\ d_{ij} \ge b_i^2\,,\ \lambda = 0\right\}$$

and

$$J_3 = \left\{j \in J\ :\ d_{ij} < b_i^2\,,\ \lambda \le \left(b_i^2 - 1\right)/\left(d_{ij} - 1\right)\right\}$$

Definition 6. Suppose that $S(d_i, b_i^2) \ne \emptyset$. For each $j \in J_1 \bigcup J_2 \bigcup J_3$, we define $\underline{X}(i,j) = [\underline{X}(i,j)_1, \underline{X}(i,j)_2, ..., \underline{X}(i,j)_n]$ where

$$\underline{X}(i,j)_k = \begin{cases} \max\left\{0, \max\left\{0, \overline{W}_{ij}(\lambda)\right\}\right\} & ,k = j\,,\ j \in J_1 \\ 0 & ,k = j\,,\ j \in J_2 \\ \overline{W}_{ij}(1-\lambda) & ,k = j\,,\ j \in J_3 \\ 0 & ,\ otherwise \end{cases}$$

By Theorem5 below, the solutions set $S(d_i, b_i^2)$ is completely determined. The theorem shows that $S(d_i, b_i^2)$ has actually the finite number of minimal solutions, $\underline{X}(i,j)$, and the unique maximum solution, $\mathbf{1}$, where $\mathbf{1}$ is an n–dimensional unite vector.

Theorem 5. Suppose that $S(d_i, b_i^2) \neq \emptyset$. Then, $S(d_i, b_i^2) = \bigcup_{j\in J_1 \bigcup J_2 \bigcup J_3} [\underline{X}(i,j), \mathbf{1}]$, $\forall i \in I_2$.

Proof. According to the proof of Theorem4, for each $x \in [0,1]^n$, $x \in S(d_i, b_i^2)$ iff $x_j \in S(d_{ij}, b_i^2)$, for some $j \in J$. Therefore, $S(d_i, b_i^2) = \bigcup_{j\in J} S(d_{ij}, b_i^2)$. Thus, from Corollary3 and Definition5, we have $S(d_i, b_i^2) = \bigcup_{j\in J_1 \bigcup J_2 \bigcup J_3} S(d_{ij}, b_i^2)$. Now, the result is attained from Corollary3 and Definition6.

Definition 7. Let $e : I_2 \to J_1 \bigcup J_2 \bigcup J_3$ so that $e(i) = j \in J_1 \bigcup J_2 \bigcup J_3$, $\forall i \in I_2$, and let E_D be the set of all vectors e. For the sake of convenience, we represent each $e \in E_D$ as an m_2–dimensional vector $e = [j_1, j_2, ..., j_{m_2}]$ in which $j_k = e(k)$, $k = 1, 2, ..., m_2$.

Definition 8. Let $e = [j_1, j_2, ..., j_{m_2}] \in E_D$. Let $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, ..., \underline{X}(e)_n]$, where $\underline{X}(e)_j = \max_{i\in I_2}\left\{\underline{X}(i, e(i))_j\right\} = \max_{i\in I_2}\left\{\underline{X}(i, j_i)_j\right\}$, $\forall j \in J$.

Based on Theorem 5 and Definition8, we have the following theorem characterizing the feasible region of the general inequalities $D \diamond x \geq b^2$.

Theorem 6. Suppose that $S(d_i, b_i^2) \neq \emptyset$, $\forall i \in I_2$. Then, $S(D, b^2) = \bigcup_{e\in E_D} [\underline{X}(e), \mathbf{1}]$.

Proof. Since $S(D, b^2) = \bigcap_{i\in I_2} S(d_i, b_i^2)$, Theorem5 implies that

$$S(D, b^2) = \bigcap_{i\in I_2} \bigcup_{j\in J_1 \bigcup J_2 \bigcup J_3} [\underline{X}(i,j), \mathbf{1}].$$

Therefore, we have

$$S(D, b^2) = \bigcup_{j\in J_1 \bigcup J_2 \bigcup J_3} \bigcap_{i\in I_2} [\underline{X}(i,j), \mathbf{1}] = \bigcup_{e\in E_D} \bigcap_{i\in I_2} [\underline{X}(i, e(i)), \mathbf{1}] = \bigcup_{e\in E_D} \left[\max_{i\in I_2} \{\underline{X}(i, e(i))\}, \mathbf{1}\right]$$

Now, the result follows from Definition8.

Theorem6 determines the solutions set $S(D, b^2)$ as the union of the finite number of n–dimensional interval $[\underline{X}(e), \mathbf{1}]$ with $\underline{X}(e)$ as the minimal and $\mathbf{1}$ as the unique maximum solutions. The following Corollary gives a necessary and sufficient condition for the feasibility of general inequalities $D \diamond x \geq b^2$.

Corollary 4. $S(D, b^2) \neq \emptyset$ iff $\mathbf{1} \in S(D, b^2)$.

4. The resolution of Problem (1)

In this section, a necessary and sufficient condition is derived to determine the feasibility of the main problem. As is shown, the feasible region is completely found by the finite number of closed convex cells.

Lemma 13. $S(A, D, b^1, b^2) \neq \emptyset$ iff there exists some $e \in E_D$ such that $[\mathbf{0}, \overline{X}] \bigcap [\underline{X}(e), \mathbf{1}] \neq \emptyset$.

Proof. Since $S(A, D, b^1, b^2) = S(A, b^1) \bigcap S(D, b^2)$, from Theorems 3 and 6 we have

$$S(A, D, b^1, b^2) = \left[\mathbf{0}, \overline{X}\right] \bigcap \bigcup_{e\in E_D} [\underline{X}(e), \mathbf{1}] = \bigcup_{e\in E_D} \left(\left[\mathbf{0}, \overline{X}\right] \bigcap [\underline{X}(e), \mathbf{1}]\right)$$

This completes the proof.

The following Corollary gives a necessary and sufficient condition for the feasibility of the intersection of general inequalities $A \diamond x \le b^1$and $D \diamond x \ge b^2$.

Corollary 5. Assume that $S(A, b^1) \neq \emptyset$ and $S(D, b^2) \neq \emptyset$. Then, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\overline{X} \in S(D, b^2)$.

Proof. According to Lemma13, $S(A, D, b^1, b^2) \neq \emptyset$ iff $[\mathbf{0}, \overline{X}] \bigcap [\underline{X}(e'), \mathbf{1}] \neq \emptyset$ for some $e' \in E_D$. Thus, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\overline{X} \in [\underline{X}(e'), \mathbf{1}]$ that means $\overline{X} \in \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}]$. The ore, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\overline{X} \in S(D, b^2)$, from Theorem6.

The following theorem characterizes the feasible region of Problem (1). The theorem determines the solutions set $S(A, D, b^1, b^2)$ as the union of the finite number of closed convex intervals.

Theorem 7. Suppose that $S(A, D, b^1, b^2) \neq \emptyset$. Then $S(A, D, b^1, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}]$.

Proof. According to the proof of Lemma13, we have

$$S(A, D, b^1, b^2) = \bigcup_{e \in E_D} \Big(\big[\mathbf{0}, \overline{X}\big] \bigcap [\underline{X}(e), \mathbf{1}] \Big).$$

Now, the required equality is resulted from Corollary5.

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))

Given problem (1):

1. If for some $i \in I_1$ and $j \in J$, $a_{ij} > b_i^1$ and $\lambda < \left(b_i^1 - a_{ij}\right) / \left(-a_{ij}\right)$, then stop; $S(a_i, b_i^1)$ is infeasible (Theorem1).

2. If $\mathbf{0} \notin S(A, b^1)$, then stop; $S(A, b^1)$ is infeasible (Corollary2).

3. If for some $i \in I_2$ and each $j \in J$, $d_{ij} < b_i^2$ and $\lambda > \left(b_i^2 - 1\right) / \left(d_{ij} - 1\right)$, then stop; $S(d_i, b_i^2)$ is infeasible (Theorem4).

4. If $\mathbf{1} \notin S(D, b^2)$, then stop; $S(D, b^2)$ is infeasible (Corollary4).

5. Compute vectors $\overline{X}(i)$ $(\forall i \in I_1)$ from Definition2, and then vector $\overline{X}$ from Definition 3.

6. If $\overline{X} \notin S(D, b^2)$, then stop; $S(A, D, b^1, b^2)$ is infeasible (Corollary5).

7. Compute vectors $\underline{X}(e)$ $(\forall e \in E_D)$ from Definition8.

8. Find the feasible solutions set $S(A, D, b^1, b^2)$ as $\bigcup_{e \in E_D} [\underline{X}(e), \overline{X}]$ (Theorem7).

5. Numerical example

Consider the following problem formed as the intersection of two fuzzy systems defined by "Fuzzy Max-Min"-Inequalities:

$$\begin{array}{l}
\begin{bmatrix} 0.4 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.3 \end{bmatrix} \diamond x \le \begin{bmatrix} 0.8 \\ 0.7 \\ 0.4 \end{bmatrix} \\
\begin{bmatrix} 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \diamond x \ge \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix} \\
x \in [0,1]^n
\end{array}$$

Step1: for $i = 1,2$ and $j = 1,2,3$, we have $a_{ij} \le b_i^1$. Then, from Theorem1 $S(a_1, b_1^1) \ne \emptyset$ and $S(a_2, b_2^1) \ne \emptyset$. Also, $0.5 = \lambda \ge \left(b_3^1 - a_{31}\right) \Big/ (-a_{31}) = 0.2$, $0.5 = \lambda \ge \left(b_3^1 - a_{32}\right) \Big/ (-a_{32}) = 0.2$ and $a_{33} \le b_3^1$ that imply $S(a_3, b_3^1) \ne \emptyset$.
Step2: The following calculation shows that $\mathbf{0} \in S(A, b^1)$.

$$\begin{bmatrix} 0.4 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4000 \\ 0.3500 \\ 0.2500 \end{bmatrix} \le \begin{bmatrix} 0.8 \\ 0.7 \\ 0.4 \end{bmatrix}$$

Therefore, $S(A, b^1) \ne \emptyset$, from Corollary2.
Step3: Since $d_{1j} \ge b_1^2$ for each $j \in J$, then $S(d_1, b_1^2) \ne \emptyset$ from Theorem4. Also, $d_{21} \ge b_2^2 d_{23} \ge b_2^2$, and $0.5 = \lambda \le \left(b_2^2 - 1\right) \Big/ (d_{22} - 1) = 0.875$ that imply $S(d_2, b_2^2) \ne \emptyset$. Finally, since $0.5 = \lambda \le \left(b_3^2 - 1\right) \Big/ (d_{31} - 1) = 0.75$, $0.5 = \lambda \le \left(b_3^2 - 1\right) \Big/ (d_{33} - 1) = 0.8571$ and $d_{32} \ge b_3^2$, then $S(d_3, b_3^2) \ne \emptyset$.
Step4: According to the calculation below, $\mathbf{1} \in S(D, b^2)$. Hence, from Corollary4, $S(D, b^2) \ne \emptyset$.

$$\begin{bmatrix} 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9000 \\ 0.9500 \\ 0.7500 \end{bmatrix} \ge \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Step5: From Definition2, we have

$$\begin{aligned} \overline{X}(1) &= [1.0000 \quad 0.8000 \quad 1.0000] \\ \overline{X}(2) &= [0.7000 \quad 1.0000 \quad 0.9000] \\ \overline{X}(3) &= [0.3000 \quad 0.3000 \quad 0.5000] \end{aligned}$$

Therefore, from Definition3, we attain $\overline{X} = [0.3 \quad 0.3 \quad 0.5]$.
Step6: From Corollary5, since $\overline{X} \in S(D, b^2)$, then $S(A, D, b^1, b^2) \ne \emptyset$. It can be easily verified as follows:

$$\begin{bmatrix} 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 0.3 \\ 0.3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.7 \\ 0.4 \end{bmatrix} \ge \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Step7: From Definition8, the feasible vectors $\underline{X}(e)$ (i.e., $\underline{X}(e) \le \overline{X}$) are resulted as follows:

$$\begin{aligned} e_1 = [1 \ \ 1 \ \ 2] &\Rightarrow \underline{X}(e_1) = [0 \quad 0.3 \quad 0] \\ e_2 = [1 \ \ 1 \ \ 3] &\Rightarrow \underline{X}(e_2) = [0 \quad 0 \quad 0.5] \end{aligned}$$

Vectors $\underline{X}(e_1)$ and $\underline{X}(e_2)$ are actually minimal solutions of the problem.
Step8: From Theorem7, we attain $S(A, D, b^1, b^2) = [\underline{X}(e_1), \overline{X}] \bigcup [\underline{X}(e_2), \overline{X}]$.

Conclusion

In this paper, we proposed an algorithm to solve the intersection of two types of fuzzy relational inequalities defined by "Fuzzy Max-Min" averaging operator. The feasible solutions set of each type of these fuzzy systems was obtained. Based on the foregoing results, the feasible region of the problem is completely resolved and four necessary and sufficient conditions were presented to determine the feasibility of the problem. As future works, we aim at testing our algorithm in other type of fuzzy systems and linear optimization problems whose constraints are defined as FRI with other averaging operators.

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