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Sign Test for Fuzzy Random Variables

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ABSTRACT

This paper extends the sign test to the case where data are observations of fuzzy random variables, and the hypotheses are imprecise rather than crisp. In this approach, first a new notion of fuzzy random variables is introduced. Then, the α-level sets of the imprecise observations are transacted to extend the usual method of sign test. To do this, the concepts of fuzzy median and fuzzy sample median are defined. We also develop a well-known large sample property of the classical sample median. In addition, the test statistic is extended for investigating fuzzy hypothesis. After that, applying an index called credibility degree, the degree that the observed fuzzy test statistics belongs to the critical region is evaluated. The result provides a fuzzy test function which leads to some degrees to accept or to reject the fuzzy null hypothesis. A numerical example is provided to clarify the discussions made in this paper.

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1 Introduction

Nonparametric tests are statistical tests used to analyze data for which an underlying probability distribution is not assumed [19, 23, 35]. They have advantages over their parametric counterparts because they have fewer underlying assumptions (e.g. data normality, equal variance, etc). A particular class of nonparametric tests is composed of median tests. These procedures are commonly based on, for example, crispness of the observations and the hypotheses of interest.

But, in the real world, different elements in environmental sciences may be imprecisely observed or defined. In many studies we are faced with the problem of handling imprecision, e.g., in cluster analysis of ecological data [45, 43], or in fuzzy rule-based classification models when the soil quality has to be defined based on soil microbial biomass [48]. Also in habitat suitability models, the information is usually only verbally described, so linguistic fuzzy models are employed [49, 50], and in ecosystem management, fuzzy rule-based models for decision support are used [1]. Fuzzy data have also been treated in statistics for a long time [51, 52, 53]. More recently, fuzzy data referred to meteorological variables were analyzed in [12]. Another example of imprecise variable in an environmental context is introduced and analyzed in [7], and [17]. In many situations, we may also not be able to make precise formulation of the underlying hypotheses. For such cases, it is more appropriate to model the imprecise hypotheses using fuzzy quantities [7, 22, 54].

To achieve suitable statistical methods dealing with imprecise data and imprecise hypotheses, we need to model the imprecise information and extend the usual approaches to imprecise environments. The introduction of the fuzzy set theory in the area of statistics encouraged many researchers to look for the generalizations of statistical procedures to the fuzzy environments. After introducing fuzzy set theory, specially in parametric statistical inferences, there have been a lot of attempts for developing fuzzy statistical methods [2, 3, 5, 6, 8, 13, 18, 33, 38, 53]. But, as the authors know, there have been few studies on nonparametric statistical approaches in fuzzy environment. Concerning the purposes of this paper, let us briefly review some of the literature on this topic.

Kahranam et al. [31] proposed some algorithms for fuzzy nonparametric rank-sum tests based on fuzzy random variables. Grzegorzewski [24] introduced a method to inference about the median of a population using fuzzy random variables. He [25] demonstrated also a straightforward generalization of some classical nonparametric tests for fuzzy random variables based on a metric in the space of fuzzy numbers. Moreover, he [26, 28] studied some nonparametric median fuzzy tests based on the necessity index of strict dominance suggested by Dubois and Prade [15] for fuzzy observations showing a degree of possibility and a degree of necessity for evaluating the underlying hypotheses. Grzegorzewski [27] also proposed a modification of the classical sign test to cope with fuzzy data which was so-called bi-robust test, i.e. a test which is both distribution-free and which does not depend so heavily on the shape of the membership functions used for modeling fuzzy data. Using a fuzzy partial ordering on closed intervals, Denoeux et al. [14] extended the nonparametric rank-sum tests based on fuzzy data. For evaluating the hypotheses of interest at a crisp or a fuzzy significance level, they employed the concepts of fuzzy

p-value and degree of rejection of the null hypothesis quantified by a degree of possibility and a degree of necessity. Hryniewicz [30] investigated the fuzzy version of the Goodman-Kruskal γ-statistic described by ordered categorical data. Lin et al. [39] considered the problem of two-sample Kolmogorov-Smirnov test for continuous fuzzy intervals based on a crisp test statistic. Taheri and Hesamian [46] introduced a fuzzy version of the Goodman-Kruskal γ-statistic for two-way contingency tables when the observations were crisp but the categories were described by fuzzy sets. In this approach, a method was also developed for testing of independence in the two-way contingency tables. Taheri and Hesamian [47] also extended the Wilcoxon signed-rank test to the case where the available observations are imprecise and underlying hypotheses are crisp. They generalized the concept of critical value to the case when the significance level is given by a fuzzy number. For more on statistical methods with fuzzy observations, the reader is referred to the relevant literatures, for example [3, 53].

The present paper aims to develop the sign test test for fuzzy random variables in which the underlying hypotheses are imprecise. To do this, after introducing a new concept of fuzzy random variable, we define fuzzy median and fuzzy sample median. Then, we introduce the concepts of fuzzy test statistic. For making a decision rule to reject or to accept the null hypothesis, we employ an index called credibility degree with a nominal crisp significance level which provides a fuzzy test function. A real-data problem in life-time testing is used to illustrate the applicability of the introduced method in fuzzy environment.

This paper is organized as follows: In the next section, we briefly review the classical sign test. In the same section, some definitions from fuzzy numbers and some results from credibility theory are also presented. In Section 3, we propose a new definition of fuzzy random variable. In Section 4, we generalize the concept of fuzzy median and fuzzy sample median for fuzzy random variables. We also extend a well-known large sample property of sample median to fuzzy environments. Then, by introducing the concepts of fuzzy test statistic, we provide an approach to test the imprecise hypothesis when the available data are observations of fuzzy random variables. A numerical example is also provided in this section to clarify the discussions made in this paper and to show the applicability of the proposed method in fuzzy environment. Finally, a brief conclusion is provided in Section 5.

2 Preliminaries

2.1 Sign test: a brief review

Suppose we have a random sample $X_1, X_2, \ldots, X_n$, with observed value $x_1, \ldots, x_n$, from a continuous and symmetric random variable X with median M_X. A null hypothesis about the population median is written as $H_0 : M_X = M_0$, where M_0 is a known real number. If $r(.)$ is the rank of an observation, the modified sign statistic [19] can be written symbolically as

$$T^{+} = \sum_{i=1}^{n} r(|d_i|)I[d_i > 0], \tag{1}$$

where $d_i = x_i - M_0$, and I is the indicator function,

$$I[\rho] = \begin{cases} 1 & if\ \rho\ is\ true, \\ 0 & if\ \rho\ is\ false. \end{cases}$$

The sampling distribution of T^{+} under the assumption of $H_0 : M_X = M_0$ is given, for example, in [19]. The appropriate rejection regions at the significance level δ to test the null hypothesis $H_0 : M_X = M_0$ are shown at Table 1. For a large sample size, the appropriate rejection region can be found by using normal approximation [19].

Table 1: Rejection regions for sign test

H_1 :	Rejection Region
1) $M_X > M_0$	$T^{+} \in \{C_\delta, C_\delta + 1, \ldots, n(n+1)/2\}$, where $\mathbf{P}_{H_0}(T^{+} \geq C_\delta) = \delta$
2) $M_X < M_0$	$T^{+} \in \{0, 1, \ldots, C'_\delta\}$, where $\mathbf{P}_{H_0}(T^{+} \leq C'_\delta) = \delta$
3) $M_X \neq M_0$	$T^{+} \in (\{0, 1, \ldots, C'_{\delta/2}\} \cup \{n(n+1)/2 - C'_{\delta/2}, \ldots, n(n+1)/2\})$

In Section 4, we will extend the sign test to the case when the available observations as well as hypotheses are provided as fuzzy quantities, rather than crisp quantities.

2.2 Fuzzy numbers

A fuzzy set $\widetilde{A}$ of the universal set $\mathbb{X}$ is defined by its membership function $\mu_{\widetilde{A}} : \mathbb{X} \to [0, 1]$. In this paper, we consider $\mathbb{R}$ (the real line) as the universal set. We denote by $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\}$ the α-level set (α-cut) of the fuzzy set $\widetilde{A}$ of $\mathbb{R}$, for every $\alpha \in (0, 1]$, and $\widetilde{A}[0]$ is the closure of the set $supp(\widetilde{A}) = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if for every $\alpha \in [0, 1]$, the set $\widetilde{A}[\alpha]$ is a non-empty compact interval, $\mu_{\widetilde{A}}(x)$ is continuous at any $x \in \mathbb{R}$, and there exists a unique $x^* \in \mathbb{R}$ such that $\mu_{\widetilde{A}}(x^*) = 1$. Such an interval will be denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. We denote by $\mathcal{F}(\mathbb{R})$, the set of all fuzzy numbers of $\mathbb{R}$.
The imprecision or vagueness can be treated by means of a particular kind (family) of fuzzy numbers, the LR-fuzzy numbers. These are very useful in practice since they can be characterized by means of three real numbers: the center, the left spread, and the right spread. The term LR is due to the left (L) and the right (R) shape of the membership function referred to the fuzzy set [38, 53]. A special type of LR-fuzzy numbers is the so-called triangular fuzzy numbers. By a triangular fuzzy number, we mean the fuzzy

number fully determined by the triple $(a^l, a, a^r)_T$ of crisp numbers with $a^l < a < a^r$, and the shape functions $L(x) = R(x) = \max\{0, 1 - |x|\}$, $x \in \mathbb{R}$. The membership function of triangular fuzzy number $\widetilde{A} = (a^l, a, a^r)_T$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & x < a^l, \\ \frac{x-a^l}{a-a^l} & a^l \leq x < a, \\ \frac{a^r-x}{a^r-a} & a \leq x \leq a^r, \\ 0 & x > a^r. \end{cases}$$

In the following, we introduce an index to compare the fuzzy number $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ and crisp value $x \in \mathbb{R}$. The index is used for making fuzzy decision rule (here fuzzy test function) to accept or to reject the null hypothesis with some degree.

Definition 1[[40]] Let $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ and $B \subseteq \mathbb{R}$. The index

$$D : \mathcal{F}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0, 1],$$

which is defined by

$$\begin{aligned} D(\widetilde{A} \in B) &= \frac{Pos\ (\widetilde{A} \in B) + Nec\ (\widetilde{A} \in B)}{2} \\ &= \frac{\sup_{y \in B} \mu_{\widetilde{A}}(y) + 1 - \sup_{y \notin B} \mu_{\widetilde{A}}(y)}{2}, \end{aligned} \tag{2}$$

shows the credibility degree that "$\widetilde{A}$ belongs to B".

Remark 1 It is worth noting that

1. if $B_1 \subseteq B_2$, then $D(\widetilde{A} \in B_1) \leq D(\widetilde{A} \in B_2)$.
2. $D(\widetilde{A} \notin B) = 1 - D(\widetilde{A} \in B)$.
3. $D(\widetilde{A} \in B) = 1$ if and only if $supp(\widetilde{A}) \subseteq B$.

Remark 2 It is readily seen that if fuzzy number $\widetilde{A}$ reduces to the crisp number a, then $D(\widetilde{A} \in B) = I[a \in B]$.

Remark 3 Let $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in (0, 1]$, then

$$\widetilde{A}_\alpha = \sup\{x \in \mathbb{R} : D(\widetilde{A} \in [x, \infty)) \geq \alpha\}, \tag{3}$$

is called the α-optimistic value of $\widetilde{A}$. It is clear that $\widetilde{A}_\alpha$ is a non-increasing function of $\alpha \in (0, 1]$ [40].

Remark 4 For a given fuzzy number $\widetilde{A}$, it is easy to verify that

$$\widetilde{A}_\alpha = \begin{cases} (\widetilde{A})^U_{2\alpha} & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ (\widetilde{A})^L_{2(1-\alpha)} & \text{for} \quad 0.5 \leq \alpha \leq 1.0. \end{cases}$$

and therefore, the α-cuts of $\widetilde{A}$ is equivalent to

$$\widetilde{A}[\alpha] = [\widetilde{A}_{(1-\frac{\alpha}{2})}, \widetilde{A}_{\frac{\alpha}{2}}],\ \alpha \in (0,1].$$

Example 1 Suppose that $\widetilde{A} = (a^l, a, a^r)_T$ is a triangular fuzzy number, and let $x \in \mathbb{R}$, then

$$D(\widetilde{A} \in [x,\infty)) = \begin{cases} 1 & \text{if} \quad x \leq a^l, \\ \frac{2a-a^l-x}{2(a-a^l)} & \text{if} \quad a^l < x \leq a, \\ \frac{a^r-x}{2(a^r-a)} & \text{if} \quad a < x < a^r, \\ 0 & \text{if} \quad x \geq a^r. \end{cases}$$

In addition, it is easy to verify that

$$\widetilde{A}_\alpha = \begin{cases} a^r - 2\alpha(a^r - a) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ 2a - a^l - 2\alpha(a - a^l) & \text{for} \quad 0.5 \leq \alpha \leq 1.0. \end{cases}$$

For example, let $\widetilde{A} = (-2, 0, 1)_T$, then

$$D(\widetilde{A} \in [x,\infty)) = \begin{cases} 1 & \text{if} \quad x \leq -2, \\ \frac{2-x}{4} & \text{if} \quad -2 < x \leq 0, \\ \frac{1-x}{2} & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x \geq 1, \end{cases}$$

and

$$\widetilde{A}_\alpha = \begin{cases} 1 - 2\alpha & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ 2 - 4\alpha & \text{for} \quad 0.5 \leq \alpha \leq 1.0. \end{cases}$$

Now, we propose a method to order fuzzy numbers, used in Section (5) for defining the hypotheses of interest which is defined as follows:
Definition 2[[54]] Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}(\mathbb{R})$, then

1. $\widetilde{A} = (\neq)\widetilde{B}$, if $\widetilde{A}_\alpha = (\neq)\widetilde{B}_\alpha$ for any $\alpha \in (0,1]$.

2. $\widetilde{A} \prec \widetilde{B}$, if $\widetilde{A}_\alpha < \widetilde{B}_\alpha$ for any $\alpha \in (0,1]$.

3. $\widetilde{A} \succ \widetilde{B}$, if $\widetilde{A}_\alpha > \widetilde{B}_\alpha$ for any $\alpha \in (0,1]$.

3 Fuzzy random variables

In the context of random experiments whose outcomes are not numbers (or vectors in $\mathbb{R}^p$) but they are expressed in inexact terms, the concept of fuzzy random variable turns out to be useful. In this regard, different notions of fuzzy random variable have been introduced and investigated in the literature [9, 16, 20, 21, 32, 33, 36, 37, 41, 42, 44]. In this paper, based on Remark 2.2, we introduce a new notion of fuzzy random variable.
Definition 3 Suppose that a random experiment is described by a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, where Ω is a set of all possible outcomes of the experiment, $\mathcal{A}$ is a σ-algebra of

subsets of Ω and $\mathbf{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{A})$. The fuzzy-valued mapping $\widetilde{X} : \Omega \to \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable if for any $\alpha \in [0,1]$, the real-valued mapping $\widetilde{X}_\alpha : \Omega \to \mathbb{R}$ is a real-valued random variable on $(\Omega, \mathcal{A}, \mathbf{P})$. Throughout this paper, we assume that all random variables have the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ The following example illustrates the notions in the definition 3.
Example 2 Let $\widetilde{X} = \widetilde{\Theta} \oplus \Xi$, where Ξ is a (usual) normal random variable with mean 0 and variance σ^2, i.e. $\Xi \sim N(0, \sigma^2)$, and $\widetilde{\Theta}$ is a constant fuzzy set. This notion of fuzzy random variable is the definition of normality for fuzzy random variables and $\widetilde{X}$ is called the normal (Gaussian) fuzzy random variable in the literature [41, 16]. As an especial case, suppose $\widetilde{\Theta}$ is a triangular fuzzy number, i.e. $\widetilde{\Theta} = (\theta^l, \theta, \theta^r)_T$ with known θ, θ^l, and θ^r. Therefore, $\widetilde{X} = (\Xi + \theta^l, \Xi + \theta, \Xi + \theta^r)_T$ and for each ω, $\widetilde{X}(\omega) = (\Xi(\omega) + \theta^l, \Xi(\omega) + \theta, \Xi(\omega) + \theta^r)_T$ is an observation of $\widetilde{X}$. Now, we have

$$\widetilde{X}_\alpha = \begin{cases} \Xi + \theta^r - 2\alpha(\theta^r - \theta) & \text{for} \quad 0.0 < \alpha \le 0.5, \\ \Xi + 2\theta - \theta^l - 2\alpha(\theta - \theta^l) & \text{for} \quad 0.5 \le \alpha \le 1.0. \end{cases}$$

Therefore, it is clear that $\widetilde{X}_\alpha$ is a random variable for each $\alpha \in (0,1]$ which is distributed as

$$\widetilde{X}_\alpha \sim \begin{cases} N(\theta^r - 2\alpha(\theta^r - \theta), \sigma^2) & \text{for} \quad 0.0 < \alpha \le 0.5, \\ N(2\theta - \theta^l - 2\alpha(\theta - \theta^l), \sigma^2) & \text{for} \quad 0.5 \le \alpha \le 1.0. \end{cases}$$

So, according to Definition 3, $\widetilde{X}$ is a fuzzy random variable.
Definition 4 Two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are said to be independent if $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are independent, for all $\alpha \in [0,1]$. In addition, we say that two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are identically distributed if $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ are identically distributed, for all $\alpha \in [0,1]$. Similar arguments can be used for more than two fuzzy random variables and we say that $\widetilde{X}_1, \ldots, \widetilde{X}_n$ is a fuzzy random sample if $\widetilde{X}_i$'s are independent and identically distributed fuzzy random variables. We denote by $\widetilde{x}_1, \ldots, \widetilde{x}_n$ the observed values of fuzzy random sample $\widetilde{X}_1, \ldots, \widetilde{X}_n$. We also say that the fuzzy random variable $\widetilde{X}$ is continuous if for every $\alpha \in [0,1]$, the crisp random variable $\widetilde{X}_\alpha$ is continuous. In this paper, we assume that the fuzzy random variable $\widetilde{X}$ is continuous.
In addition, we say that the fuzzy random variable $\widetilde{X}$ is symmetric if for every $\alpha \in (0,1]$, $\widetilde{X}_\alpha$ is symmetric.

4 Fuzzy median and fuzzy sample median

In this section, we extend the concepts of population median for fuzzy random variables and median of fuzzy random samples. Let us recall that $M_X \in \mathbb{R}$ is the median of the continuous random variable X if

$$M_X = \inf\{x : F_X(x) \ge 0.5\},$$

where F_X denotes the distribution function of X. Now, using Zadeh's extension principle [34] we may generalize this notion to the fuzzy environment.

Definition 5 The fuzzy median of fuzzy random variable $\widetilde{X}$ is defined as fuzzy set $\widetilde{M}_{\widetilde{X}}$ with the following membership function

$$\mu_{\widetilde{M}_{\widetilde{X}}}(y) = \sup\{\alpha \in [0,1] : y \in [(\widetilde{M}_{\widetilde{X}})^L_\alpha, (\widetilde{M}_{\widetilde{X}})^U_\alpha]\}, \tag{4}$$

where

$$(\widetilde{M}_{\widetilde{X}})^L_\alpha = \inf_{\beta \geq \alpha} \inf\{x : F_{\widetilde{X}_\beta}(x) \geq 0.5\},$$

$$(\widetilde{M}_{\widetilde{X}})^U_\alpha = \sup_{\beta \geq \alpha} \inf\{x : F_{\widetilde{X}_\beta}(x) \geq 0.5\},$$

Remark 5 Since $\widetilde{X}_\alpha$ is decreasing in $\alpha \in (0,1]$, therefore the α-cuts of the fuzzy median $\widetilde{M}_{\widetilde{X}}$ reduce as follows

$$(\widetilde{M}_{\widetilde{X}})^L_\alpha = \inf\{x : F_{\widetilde{X}_1}(x) \geq 0.5\},$$

$$(\widetilde{M}_{\widetilde{X}})^U_\alpha = \inf\{x : F_{\widetilde{X}_\alpha}(x) \geq 0.5\}.$$

Thus, it is clear that the membership function of $\widetilde{M}_{\widetilde{X}}$ reduces as

$$\mu_{\widetilde{M}_{\widetilde{X}}}(y) = \sup\{\alpha \in [0,1] : y = (\widetilde{M}_{\widetilde{X}})^U_\alpha\}. \tag{5}$$

Example 3 Consider Example 3. It is easy to see that the fuzzy median of the fuzzy random variable $\widetilde{X}$ is given by

$$(\widetilde{M}_{\widetilde{X}})^U_\alpha = \begin{cases} \theta^r - 2\alpha(\theta^r - \theta) & \text{for} \quad 0.0 < \alpha \leq 0.5, \\ 2\theta - \theta^l - 2\alpha(\theta - l) & \text{for} \quad 0.5 \leq \alpha \leq 1.0. \end{cases}$$

For instance, by assuming $\widetilde{\Theta} = (-2, 0, 2)_T$, we obtain

$$(\widetilde{M}_{\widetilde{X}})^U_\alpha = 2 - 4\alpha, \ \alpha \in (0,1].$$

which is a triangular fuzzy number as $\widetilde{M}_{\widetilde{X}} = (-2, -2, 2)_T$. Now our task is to obtain a fuzzy point estimator (for more, see [24]) of the fuzzy median $\widetilde{M}_{\widetilde{X}}$. Let us recall that based on a crisp random sample $X_1, X_2, \ldots, X_n$, the sample median is defined as

$$m_n = \begin{cases} X_{(\frac{n+1}{2})} & \text{if} \quad \text{n is odd}, \\ \frac{X_{(n/2)} + X_{(n/2+1)}}{2} & \text{if} \quad \text{n is even}. \end{cases} \tag{6}$$

where $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote order statistics of the sample. Therefore, similar to fuzzy median, we may define a fuzzy estimator of the median.

Definition 6 A fuzzy sample median $\widetilde{m}_n$ from the fuzzy random sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ is a fuzzy set with the following membership function

$$\mu_{\widetilde{m}_n}(y) = \sup\{\alpha \in [0,1] : y \in [(\widetilde{m}_n)^L_\alpha, (\widetilde{m}_n)^U_\alpha]\}, \tag{7}$$

where

$$(\widetilde{m}_n)^L_\alpha = \inf_{\beta \geq \alpha} g_n(\beta),\ (\widetilde{m}_n)^L_\alpha = \sup_{\beta \geq \alpha} g_n(\beta),$$

in which

$$g_n(\beta) = \begin{cases} (\widetilde{X}_\beta)_{(\frac{n+1}{2})} & \text{if} \quad \text{n is odd,} \\ \frac{(\widetilde{X}_\beta)_{(n/2)}+(\widetilde{X}_\beta)_{(n/2+1)}}{2} & \text{if} \quad \text{n is even.} \end{cases}$$

It is easy to check that the membership function of the fuzzy sample median reduces as follows

$$\mu_{\widetilde{m}_n}(y) = \sup\{\alpha \in [0,1] : y = (\widetilde{m}_n)^U_\alpha\}, \tag{8}$$

where

$$(\widetilde{m}_n)^U_\alpha = \begin{cases} (\widetilde{X}_\alpha)_{(\frac{n+1}{2})} & \text{if} \quad \text{n is odd,} \\ \frac{(\widetilde{X}_\alpha)_{(n/2)}+(\widetilde{X}_\alpha)_{(n/2+1)}}{2} & \text{if} \quad \text{n is even.} \end{cases}$$

From the classical statistical inferences, we know that if the sample is drawn from the distribution with the uniquely determined median (e.g. X is a continuous random variable), then the sample median m_n converges to the population median M_X with probability one (briefly, $m_n \to M_X$ w.p.1) [19]. Now we will discuss this large sample properties of the fuzzy random sample. Here we extend this property for fuzzy random variables.

Definition 7 For the sequence $\{\widetilde{X}_n\}_{n=1}^\infty$ of fuzzy random variables and fuzzy number $\widetilde{Z}$, we say $\widetilde{X}_n \to \widetilde{Z}$ w.p.1, if

$$\mathbf{P}(|(\widetilde{X}_n)_\alpha - \widetilde{Z}_\alpha| \to 0) = 1,\ \text{for all } \alpha \in (0,1]. \tag{9}$$

Remark 6 If the fuzzy random variables $\{\widetilde{X}_n\}_{n=1}^\infty$ and $\widetilde{Z}$ reduce to the crisp random variables $\{X_n\}_{n=1}^\infty$ and Z, then it is easy to verify that the above equality reduces to $\mathbf{P}(X_n \to Z) = 1$, which is the definition of convergence w.p.1 of ordinary (non fuzzy) random variables. Definition 4, therefore, is a generalization of convergence w.p.1 to the case of fuzzy random variables. Now we may prove a following theorem.

Theorem 4.1. *Let $\widetilde{X}$ be a continuous fuzzy random variable. Then $\widetilde{m}_n \to \widetilde{M}_X$ w.p.1, i.e.*

$$\mathbf{P}(|(\widetilde{m}_n)_\alpha - (\widetilde{M}_{\widetilde{X}})_\alpha| \to 0) = 1,\ \textit{for every } \alpha \in (0,1]. \tag{10}$$

Proof. Note that from Remark 2.2, we have $(\widetilde{m}_n)_\alpha = (\widetilde{m}_n)^U_{2\alpha}$ (similarly, $(\widetilde{M}_{\widetilde{X}})_\alpha = (\widetilde{M}_{\widetilde{X}})^U_{2\alpha}$) for every $\alpha \in (0,1]$. Therefore, Equation (10) reduces as follows

$$\mathbf{P}(|(\widetilde{m}_n)^U_{2\alpha} - (\widetilde{M}_{\widetilde{X}})^U_{2\alpha}| \to 0) = 1,\ \text{for every } \alpha \in (0,1]. \tag{11}$$

Now, for any fixed $\alpha \in (0,1]$, we know that $(\widetilde{m}_n)^U_{2\alpha}$ is the point estimator of $(\widetilde{M}_{\widetilde{X}})^U_{2\alpha}$ in which $\mathbf{P}(|(\widetilde{m}_n)^U_{2\alpha} - (\widetilde{M}_{\widetilde{X}})^U_{2\alpha}| \to 0) = 1$, which completes the proof. □

Remark 7 It is mentioned that Grzegorzewski [24] also proposed a notion of fuzzy median and a fuzzy estimator for the fuzzy median based on a concept of fuzzy random variables. Then he discussed the problem of crisp hypothesis testing concerning the (crisp) median in the presence of imprecise observations. But in this work, the underlying hypotheses are considered as fuzzy sets rather than crisp. To do this, applying a new definition of fuzzy random variables, we propose a method to rank fuzzy numbers to define fuzzy hypotheses. On the other hand, introducing a notion of convergence w.p.1., Grzegorzewski [24] discussed the statistical large sample property of the fuzzy sample median. His work is relied on the α-cuts of fuzzy random variables. While we used a different method to that of Grzegorzewski's method [24], since it was essentially based on a different notion of fuzzy random variables.

5 Sign test for fuzzy random variables

In this section, we extend the sign test to the case when the underlying hypotheses and the available observations are imprecise rather than crisp.

5.1 Fuzzy hypotheses

Definition 8 Let $\widetilde{X}$ be a continuous and symmetric fuzzy random variable.

1. **Left one-sided fuzzy hypotheses:** We define the hypothesis that $\widetilde{M}_{\widetilde{X}}$ and $\widetilde{M}_0$ are identical against the alternative that $\widetilde{M}_{\widetilde{X}}$ is smaller than $\widetilde{M}_0$ as

$$\begin{cases} \widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0 & \equiv \quad \widetilde{H}_0 : (\widetilde{M}_{\widetilde{X}})_\alpha = (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1], \\ \widetilde{H}_1 : \widetilde{M}_{\widetilde{X}} \prec \widetilde{M}_0 & \equiv \quad \widetilde{H}_1 : (\widetilde{M}_{\widetilde{X}})_\alpha < (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1]. \end{cases} \tag{12}$$

2. **Right one-sided fuzzy hypotheses:** We define the hypothesis that $\widetilde{M}_{\widetilde{X}}$ and $\widetilde{M}_0$ are identical against the alternative that $\widetilde{M}_{\widetilde{X}}$ is larger than $\widetilde{M}_0$ as

$$\begin{cases} \widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0 & \equiv \quad \widetilde{H}_0 : (\widetilde{M}_{\widetilde{X}})_\alpha = (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1], \\ \widetilde{H}_1 : \widetilde{M}_{\widetilde{X}} \succ \widetilde{M}_0 & \equiv \quad \widetilde{H}_1 : (\widetilde{M}_{\widetilde{X}})_\alpha > (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1]. \end{cases} \tag{13}$$

3. **Two-sided fuzzy hypotheses:** We define the hypothesis that $\widetilde{M}_{\widetilde{X}}$ and $\widetilde{M}_0$ are identical against the alternative that $\widetilde{M}_{\widetilde{X}}$ is different from $\widetilde{M}_0$ as

$$\begin{cases} \widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0 & \equiv \quad \widetilde{H}_0 : (\widetilde{M}_{\widetilde{X}})_\alpha = (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1], \\ \widetilde{H}_1 : \widetilde{M}_{\widetilde{X}} \neq \widetilde{M}_0 & \equiv \quad \widetilde{H}_1 : (\widetilde{M}_{\widetilde{X}})_\alpha \neq (\widetilde{M}_0)_\alpha, \quad \forall\alpha \in (0,1]. \end{cases} \tag{14}$$

5.2 Fuzzy test statistic

Suppose that we have a fuzzy sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ from a population related to the continuous and symmetric fuzzy random variable $\widetilde{X}$. In this section, to provide a procedure for testing the fuzzy hypothesis about the fuzzy median of the fuzzy random variable $\widetilde{X}$. For this, we will construct a fuzzy test statistic and a critical region as follows

Definition 9 Consider the problem of the sign test $\widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0$ based on the fuzzy sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$. The fuzzy sign test statistic $\widetilde{T}^+$ is defined to be a fuzzy set with the following membership function

$$\mu_{\widetilde{T}^+}(t) = \sup\left\{\alpha \in [0,1] : t \in \{(\widetilde{T}^+)^L_\alpha, (\widetilde{T}^+)^L_\alpha + 1, \ldots, (\widetilde{T}^+)^U_\alpha\}\right\}, \tag{15}$$

where

$$(\widetilde{T}^+)^L_\alpha = \inf_{\beta \geq \alpha} \sum_{i=1}^{n} r(|(\widetilde{X}_i)_\beta - (\widetilde{M}_0)_\beta|) I[(\widetilde{X}_i)_\beta > (\widetilde{M}_0)_\beta],$$

$$(\widetilde{T}^+)^U_\alpha = \sup_{\beta \geq \alpha} \sum_{i=1}^{n} r(|(\widetilde{X}_i)_\beta - (\widetilde{M}_0)_\beta|) I[(\widetilde{X}_i)_\beta > (\widetilde{M}_0)_\beta].$$

In the following, we denote by $\widetilde{t}^+$, the observed fuzzy test statistics.

Remark 8 It should be mentioned that, if the fuzzy random sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ reduce to the crisp observations $X_1, X_2, \ldots, X_n$ (hence the fuzzy median $\widetilde{M}_{\widetilde{X}}$ reduces to crisp number M_X), then for every $\alpha \in (0,1]$

$$(\widetilde{T}^+)^L_\alpha = (\widetilde{T}^+)^U_\alpha = \sum_{i=1}^{n} r(|X_i - M_0|) I[X_i > M_0], \tag{16}$$

which is the classical sign statistic T^+.

5.3 Critical region

In the classical hypothesis tests, the usual approach for accepting or rejecting the null hypothesis of interest is to compare the observed test statistic with a related critical value. For instance, assume we wish to test the hypothesis $H_0 : M_X = M_0$ against $H_1 : M_X < M_0$. At a given significance level δ, note that the common method rejects H_0 if $T^+ \leq C'_\delta$, where $\mathbf{P}_{H_0}(T^+ \leq C'_\delta) = \delta$ [19]. Now, suppose that we wish to test the fuzzy hypothesis $\widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0$ against $\widetilde{H}_1 : \widetilde{M}_{\widetilde{X}} < \widetilde{M}_0$. Therefore, using the Resolution Identity [55], we may extend the concept of critical value for fuzzy random variables as a fuzzy set given as follows

$$\mu_{\widetilde{C}'_\delta}(t) = \sup_{\alpha \in [0,1]} \alpha I(t \in \{(\widetilde{C}'_{\widetilde{\delta}})^L_\alpha, (\widetilde{C}'_{\widetilde{\delta}})^L_\alpha + 1, \ldots, (\widetilde{C}'_{\widetilde{\delta}})^U_\alpha\}), \tag{17}$$

in which

$$(\widetilde{C}'_{\widetilde{\delta}})^L_\alpha = \inf_{\beta \geq \alpha} \left\{x : \mathbf{P}_{H_0^\beta}(T^+_\beta \leq x) = \delta\right\}, \ (\widetilde{C}'_{\widetilde{\delta}})^U_\alpha = \sup_{\beta \geq \alpha} \left\{x : \mathbf{P}_{H_0^\beta}(T^+_\beta \leq x) = \delta\right\},$$

where

$$H_0^\beta : (\widetilde{M}_{\widetilde{X}})_\beta = (\widetilde{M}_0)_\beta,\ T_\beta^+ = \sum_{i=1}^{n} r(|(\widetilde{X}_i)_\beta - (\widetilde{M}_0)_\beta|) I[(\widetilde{X}_i)_\beta > (\widetilde{M}_0)_\beta].$$

But from Definition 3, note that the sample distribution of T_β^+ under null hypothesis H_0^β is the same as the ordinary sample distribution of T^+ under H_0, i.e. for each $x \in \mathbb{R}$,

$$\mathbf{P}_{H_0^\beta}(T_\beta^+ \leq x) = \mathbf{P}_{H_0}(T^+ \leq x), \tag{18}$$

Therefore, the membership function of the fuzzy critical value reduces to

$$\mu_{\widetilde{C}'_\delta}(t) = I[t = C'_\delta],$$

which is the classical critical value. Therefore, at a given significance level $\delta \in (0, 1]$, the critical region is reminded as the classical case. Similar arguments can be stated for two cases (1) and (3) (Table 1).

5.4 Method of decision making

In the classical testing problem a decision rule is made to accept or to reject the null hypothesis, by investigating if the observed fuzzy test statistic belongs to the critical region or not. But, in the proposed method, the test statistics is defined as a fuzzy set. Therefore, we need a index to evaluate a degree that a discrete fuzzy set $\widetilde{A}$ belongs to the set $B \subseteq \mathbb{R}$. Here, we apply an index D for making decision to accept or reject the null fuzzy hypothesis $\widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0$.

Definition 10 Consider the problem of testing the sign test $\widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0$ versus an alternative hypothesis given in Table 1 based on the fuzzy sample $\widetilde{X}_1, \ldots, \widetilde{X}_n$. At significance level $\delta \in (0, 1]$, the fuzzy test function is defined as fuzzy set $\widetilde{\varphi}_\delta[\widetilde{X}_1, \ldots, \widetilde{X}_n]$, with the following membership function

$$\widetilde{\varphi}_\delta[\widetilde{X}_1, \ldots, \widetilde{X}_n](t) = \begin{cases} D(\widetilde{t}^+ \in R_\delta) & \text{t=rejet } \widetilde{H}_0, \\ D(\widetilde{t}^+ \notin R_\delta) & \text{t=accept } \widetilde{H}_0, \end{cases} \tag{19}$$

Therefore, based on the fuzzy random sample, the hypothesis $\widetilde{H}_0$ is accepted against the hypothesis $\widetilde{H}_1$ with credibility degree of acceptability $D(\widetilde{T}^+ \notin R_\delta)$, and it is rejected with credibility degree of $D(\widetilde{T}^+ \in R_\delta)$. With other words, in fuzzy test function $\widetilde{\varphi}_\delta[\widetilde{X}_1, \ldots, \widetilde{X}_n]$, $D(\widetilde{T}^+ \notin R_\delta)$ is called the credibility degree that $\widetilde{H}_0$ is accepted and $D(\widetilde{T}^+ \in R_\delta)$ is called the credibility degree that $\widetilde{H}_0$ is rejected.

Remark 9 At a nominal significance level δ, we can interpret degrees of accept or reject the fuzzy null hypothesis as follows:

1. if $D(\widetilde{T}^+ \notin R_\delta) > 0.5$, then the fuzzy random sample support $\widetilde{H}_0$ more than $\widetilde{H}_1$,

Table 2: Credbility degrees of rejection of the fuzzy null hypothesis for sign test

Case	$\widetilde{H}_1$:	Rejection degree of $\widetilde{H}_0$
$a)$	$\widetilde{M}_{\widetilde{X}} \succ \widetilde{M}_0$	$D(\widetilde{t}^+ \in \{C_\delta, C_\delta + 1, \ldots, n(n+1)/2\})$
$b)$	$\widetilde{M}_{\widetilde{X}} \prec \widetilde{M}_0$	$D(\widetilde{t}^+ \in \{0, 1, \ldots, C'_\delta\})$
$c)$	$\widetilde{M}_{\widetilde{X}} \neq \widetilde{M}_0$	$D\big(\widetilde{t}^+ \in (\{0, 1, \ldots, C'_{\delta/2}\} \cup \{n(n+1)/2 - C'_{\delta/2}, \ldots, n(n+1)/2\})\big)$

2. if $D(\widetilde{T}^+ \in R_\delta) > 0.5$, then the fuzzy random sample support $\widetilde{H}_1$ more than $\widetilde{H}_0$,
3. if $D(\widetilde{T}^+ \in R_\delta) = D(\widetilde{T}^+ \notin R_\delta) = 0.5$, then the fuzzy random sample equally support $\widetilde{H}_0$ and $\widetilde{H}_1$.

Remark 10 It is mentioned that Kahraman et al. [31], Grzegorzewski [24, 26, 27], and Hesamian and Taheri [47] considered the problem of sign test for fuzzy observations when the underlying hypotheses are crisp. But in this paper, we considered the fuzzy hypotheses about the median rather than crisp hypotheses. In addition, Hesamian and Taheri [29] also proposed a method to investigate imprecise hypothesis about the population median based on a concept of imprecise observation (i.e. a vague concept of a ordinary observations, for more see [14]). Using this idea, they proposed a fuzzy null hypothesis about the crisp population median M_X as "$\widetilde{H}_0 : M_X$ is about $\widetilde{M}_0$", where $\widetilde{M}_0$ is a fuzzy number. While the proposed method is relied on a concept of fuzzy random variable, i.e., we extend the concept of fuzzy median for fuzzy random variables. Then we proposed a notion of fuzzy hypothesis as "$\widetilde{H}_0 : \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0$". Introducing a notion of convergence w.p.1, we also discussed the large sample property between the fuzzy population median and fuzzy sample median.
To demonstrate the application of the proposed method, we provide a practical example using a real data set given in [54].
Example 4 A tire and rubber company is interested in the quality of a tire it has recently developed. Only 24 new tires were tested because the tests were destructive and took considerable time to complete. Six cars, all the same model and brand, were used to test the tires. Car model and brand were alike so that the car effects were not considered. The tire lifetimes are taken to be triangular fuzzy numbers as shown in Table 2.
Suppose that we wish to test the following fuzzy hypotheses

$$\begin{cases} \widetilde{H}_0 : & \widetilde{M}_{\widetilde{X}} = \widetilde{M}_0 = (30000, 32000, 34000)_T, \\ \widetilde{H}_1 : & \widetilde{M}_{\widetilde{X}} \succ \widetilde{M}_0. \end{cases}$$

To compute the observed fuzzy test statistic $\widetilde{t}^+$, we should calculate its α-cuts o for every

Table 2. Data set in Example 5.4.

$\widetilde{x}_1 = (33262, 33978, 34889)_T$	$\widetilde{x}_{13} = (32093, 32617, 33255)_T$
$\widetilde{x}_2 = (32585, 33052, 33787)_T$	$\widetilde{x}_{14} = (31720, 32611, 33497)_T$
$\widetilde{x}_3 = (32806, 33418, 33908)_T$	$\widetilde{x}_{15} = (31977, 32455, 33034)_T$
$\widetilde{x}_4 = (33065, 33463, 34131)_T$	$\widetilde{x}_{16} = (31943, 32466, 33212)_T$
$\widetilde{x}_5 = (30743, 31624, 32460)_T$	$\widetilde{x}_{17} = (32169, 33070, 33968)_T$
$\widetilde{x}_6 = (32415, 33127, 34072)_T$	$\widetilde{x}_{18} = (32900, 33543, 34335)_T$
$\widetilde{x}_7 = (32687, 33224, 33908)_T$	$\widetilde{x}_{19} = (30327, 30881, 31455)_T$
$\widetilde{x}_8 = (32185, 32597, 33186)_T$	$\widetilde{x}_{20} = (31187, 31565, 32237)_T$
$\widetilde{x}_9 = (33423, 34036, 34771)_T$	$\widetilde{x}_{21} = (33208, 34053, 34876)_T$
$\widetilde{x}_{10} = (31639, 32584, 33542)_T$	$\widetilde{x}_{22} = (30945, 31838, 32739)_T$
$\widetilde{x}_{11} = (31511, 32290, 33064)_T$	$\widetilde{x}_{23} = (31934, 32800, 33445)_T$
$\widetilde{x}_{12} = (33060, 33844, 34449)_T$	$\widetilde{x}_{24} = (33464, 34157, 34974)_T$

$\alpha \in (0, 1]$. For example, at level of $\alpha = 0.6$, from Equation (15), we obtain

$$(\widetilde{t}^{+})^{L}_{0.6} = \inf_{\beta \geq 0.6} \sum_{i=1}^{n} r(|(\widetilde{x}_i)_\beta - (\widetilde{M}_0)_\beta|) I[(\widetilde{x}_i)_\beta > (\widetilde{M}_0)_\beta] = 287,$$

$$(\widetilde{t}^{+})^{U}_{0.6} = \sup_{\beta \geq 0.6} \sum_{i=1}^{n} r(|(\widetilde{x}_i)_\beta - (\widetilde{M}_0)_\beta|) I[(\widetilde{x}_i)_\beta > (\widetilde{M}_0)_\beta] = 300.$$

So $(\widetilde{t}^{+})[0.5] = [274, 300]$. By continuing this procedure for other values of α, the fuzzy test statistics is obtained as "about 300" with the following membership function

$$\widetilde{t}^{+} = \Big\{ \frac{1}{300}, \frac{0.88}{299}, \frac{0.79}{298}, \frac{0.75}{296}, \frac{0.52}{32}, \frac{0.66}{295}, \frac{0.65}{294}, \frac{0.64}{293}, \frac{0.63}{292}, \frac{0.62}{291}, \frac{0.61}{289},$$
$$\frac{0.60}{287}, \frac{0.59}{286}, \frac{0.58}{284}, , \frac{0.57}{283}, \frac{0.56}{282}, \frac{0.52}{281}, \frac{0.51}{279}, \frac{0.5}{278}, \frac{0.49}{275}, \frac{0.49}{272}, \frac{0.47}{270},$$
$$\frac{0.46}{267}, \frac{0.41}{264}, \frac{0.44}{261}, \frac{0.43}{258}, \frac{0.41}{255}, \frac{0.40}{253}, \frac{0.39}{252}, \frac{0.38}{250}, \frac{0.37}{248}, \frac{0.36}{246}, \frac{0.35}{243}, \frac{0.34}{241},$$
$$\frac{0.33}{237}, \frac{0.32}{231}, \frac{0.31}{228}, \frac{0.30}{221}, \frac{0.29}{219}, \frac{0.28}{211}, \frac{0.27}{206}, \frac{0.26}{205}, \frac{0.25}{199}, \frac{0.24}{197}, \frac{023}{194}, \frac{0.22}{192},$$
$$\frac{0.21}{187}, \frac{0.20}{183}, \frac{0.19}{178}, \frac{0.18}{172}, \frac{0.17}{169}, \frac{0.16}{166}, \frac{0.15}{155}, \frac{0.14}{150}, \frac{0.13}{149}, \frac{0.12}{141}, \frac{0.11}{135}, \frac{0.10}{132},$$
$$\frac{0.09}{126}, \frac{0.08}{123}, \frac{0.07}{116}, \frac{0.05}{112}, \frac{0.04}{111}, \frac{0.03}{104}, \frac{0.02}{101}, \frac{0.01}{93} \Big\},$$

By considering the significance level $\delta = 0.05$, from Equation (19), finally the fuzzy test function is obtained as follows

$$\widetilde{\varphi}_\delta[\widetilde{X}_1, \ldots, \widetilde{X}_{24}](t) = \begin{cases} D(\widetilde{t}^{+} \in R_\delta) = 0.865 & \text{t=rejet } \widetilde{H}_0, \\ D(\widetilde{t}^{+} \notin R_\delta) = 0.135 & \text{t=accept } \widetilde{H}_0, \end{cases}$$

where, $R_\delta = \{208, 209, \ldots, 300\}$. Therefore, the fuzzy hypothesis of $\widetilde{H}_0$ is rejected with credibility degree of 0.865.

6 Conclusion

In this paper, we proposed a new method for sign test when data are observations of fuzzy random variables and underlying hypotheses about the population's median are imprecise quantities, rather than crisp. To do this, after introducing a new notion of fuzzy random variable, the concepts of fuzzy median and fuzzy sample median were extended for fuzzy random variables. Then, we stated and proved an essential large sample property of the fuzzy sample median. In addition, a sign test statistic was extended to fuzzy environment. Finally, for providing a fuzzy test function, the degree that the observed fuzzy test statistic belongs to the critical region were evaluated using an index called credibility degree. It is also worth noting that our approach could be applied for the generalization of other nonparametric statistical median-based tests for fuzzy random variables such as sign test for paired samples, two-sample median test, Kruskall-Wallis rank-sum test, Mann-Whitney-Wilcoxon rank-sum test, etc.
The study of developing the power of test and effect of vagueness on the results of the test is also a potential subject for further research.

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