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Linear programming on SS-fuzzy inequality constrained problems

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ABSTRACT

In this paper, a linear optimization problem is investigated whose constraints are defined with fuzzy relational inequality. These constraints are formed as the intersection of two inequality fuzzy systems and Schweizer-Sklar family of t-norms. Schweizer-Sklar family of t-norms is a parametric family of continuous t-norms, which covers the whole spectrum of t-norms when the parameter is changed from zero to infinity. Firstly, we investigate the resolution of the feasible region of the problem and studysome theoretical results. A necessary and sufficient condition and three other necessary conditions are derived for determining the feasibility. Moreover, in order to simplify the problem, some procedures are presented. It is proved that the optimal solution of the problem is always resulted from the unique maximum solution and a minimal solution of the feasible region. A method

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1 Abstract continued

is proposed to generate random feasible max-Schweizer-Sklar fuzzy relational inequalities and an algorithm is presented to solve the problem. Finally, an example is described to illustrate these algorithms.

2 Introduction

In this paper, we study the following linear problem:

$$\begin{aligned} \min \quad & Z = c^T x \\ & A\varphi x \leq b^1 \\ & D\varphi x \geq b^2 \\ & x \in [0,1]^n \end{aligned} \tag{1}$$

where $I_1 = \{1, 2, \ldots, m_1\}$, $I_2 = \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$ and $J = \{1, 2, \ldots, n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ fuzzy matrices such that $0 \leq a_{ij} \leq 1$ ($\forall i \in I_1$ and $\forall j \in J$) and $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an $m_1$dimensional fuzzy vectorin $[0,1]^{m_1}$ (i.e., $0 \leq b_i^1 \leq 1$, $\forall i \in I_1$), $b^2 = (b_i^2)_{m_2 \times 1}$ is an $m_2$dimensional fuzzy vectorin $[0,1]^{m_2}$ (i.e., $0 \leq b_i^2 \leq 1$, $\forall i \in I_2$), and c is a vector in $\Box^n$. Moreover, II is the max-Schweizer-Sklar composition, that is, $\varphi(x,y) = T_{SS}^p(x,y) = (\max\{x^p + y^p - 1, 0\})^{\frac{1}{p}}$ in which $p > 0$. By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned} \min \quad & Z = c^T x \\ & \max_{j \in J}\{T_{SS}^p(a_{ij}, x_j)\} \leq d_i^1, i \in I_1 \\ & \max_{j \in J}\{T_{SS}^p(d_{ij}, x_j)\} \leq d_i^2, i \in I_2 \\ & \qquad x \in [0,1]^n \end{aligned} \tag{2}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max-Schweizer-Sklar fuzzy relational equations. The family $\{T_{SS}^p\}$ is increasing in the parameter p. It can be easily shown that Schweizer-Sklar t-norm $T_{SS}^p(x,y)$ converges to the basic fuzzy intersection $\{x,y\}$ when $p \longrightarrow +\infty$ [7].

Sanchez was first who developed the theory of fuzzy relational equations (FRE) [44]. Nowadays, it has been shown that many issues associated with a body knowledge can be formulated as FRE problems [39]. FRE theory has been also applied in many fields, including fuzzy control, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [38].

The finding of solutions set is the primary subject concerning with FRE problems [2, 3, 5, 34, 33, 37]. Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [40, 41, 43, 45, 48, 47, 50, 53, 56].

Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI) [11, 16, 21, 22, 32, 55].
The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1, 8, 9, 13, 23, 28, 30, 35, 42, 46, 51, 49, 55]. The topic of the linear optimization problem was also investigated with max-product operation [14, 26, 36]. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [27, 49], max-star composition [17] and max-t-norm composition [20, 23, 26, 30, 46].
Recently, many interesting generalizations of the linear programming have been introduced and developed, that are subjected to a system of fuzzy relations [4, 6, 10, 12, 22, 31, 35, 52].
The optimization problem subjected to various versions of FRI could be found in the literature as well [19, 15, 16, 21, 22, 25, 54, 55]. Yang [54] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Xiao et al. [55] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model of wireless communication emission base stations.Ghodousian et al. [19, 15] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition. They used this fuzzy system to convincingly optimize the educational quality of a school (with minimum cost) to be selected by parents.
The remainder of the paper is organized as follows. In section 3, some preliminary notions and definitions and three necessary conditions for the feasibility of problem (1) are presented. In section **??**, the feasible region of problem (1) is determined as a union of the finite number of closed convex intervals. Two simplification operations are introduced to accelerate the resolution of the problem. Moreover, a necessary and sufficient condition based on the simplification operations is presented to realize the feasibility of the problem. Problem (1) is resolved by optimization of the linear objective function considered in section 5. In addition, the existence of an optimal solution is proved if problem (1) is not empty. The preceding results are summarized as an algorithm and, finally in section **??** an example is described to illustrate.Additionally, in section **??**, a method is proposed to generate feasible test problems for problem (1).

3 Basic properties of max-Schweizer-Sklar FRI

This section describes the basic definitions and structural properties concerning problem (1) that are used throughout the paper. For the sake of simplicity, let $S_{T^p_{SS}}(A, b^1)$ and $S_{T^p_{SS}}(D, b^2)$ denote the feasible solutions sets of inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, respectively, that is, $S_{T^p_{SS}}(A, b^1) = \{x \in [0, 1]^n : A\varphi x \leq b^1\}$ and $S_{T^p_{SS}}(D, b^2) = \{x \in [0, 1]^n : D\varphi x \leq b^2\}$.Also, let $S_{T^p_{SS}}(A, D, b^1, b^2)$ denote the feasible solutions set of problem

(1). Based on the foregoing notations, it is clear that $S_{T^p_{SS}}(A, D, b^1, b^2) = S_{T^p_{SS}}(A, b^1) \cap S_{T^p_{SS}}(D, b^2)$.

Definition 1. *For each $i \in I$ and each $j \in J$, we define $S_{T^p_{SS}}(a_{ij}, b^1_i) = \{x \in [0,1] : T^p_{SS}(a_{ij}, x) \leq b^1_i\}$. Similarly, for each $i \in I_2$ and each $j \in J$, $S_{T^p_{SS}}(d_{ij}, b^2_i) = \{x \in [0,1] : T^p_{SS}(d_{ij}, x) \leq b^2_i\}$. Furthermore, the notations $J^1_i = \{j \in J : S_{T^p_{SS}}(a_{ij}, b^1_i) \neq \varnothing\}$, $\forall i \in I_1$, and $J^2_i = \{j \in J : S_{T^p_{SS}}(d_{ij}, b^2_i) \neq \varnothing\}$, $\forall i \in I_2$, are used in the text.*

From the least-upper-bound property of $\square$, it is clear that $\inf\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}$ and $\sup\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}$ exist, if $S_{T^p_{SS}}(a_{ij}, b^1_i) \neq \varnothing$. Moreover, since T^p_{SS} is a t-norm, its monotonicity property implies that $S_{T^p_{SS}}(a_{ij}, b^1_i)$ is actually a connected subset of $[0,1]$. Additionally, due to the continuity of T^p_{SS}, we must have $\inf\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\} = \min\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}$ and $\sup\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\} = \max\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}$. Therefore,
$S_{T^p_{SS}}(a_{ij}, b^1_i) = [\min\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}, \max\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}]$, i.e., $S_{T^p_{SS}}(a_{ij}, b^1_i)$ is a closed subinterval of $[0,1]$. By the similar argument, if $S_{T^p_{SS}}(a_{ij}, b^2_i) \neq \varnothing$, then we have $S_{T^p_{SS}}(a_{ij}, b^2_i) = [\min\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^2_i)\}, \max\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^2_i)\}] \subseteq [0,1]$. From Definition 1 and the above statements, the following two corollaries are easily resulted.

Corollary 1. *For each $i \in I_1$ and each $j \in J$, $S_{T^p_{SS}}(a_{ij}, b^1_i) \neq \varnothing$.Also, $S_{T^p_{SS}}(a_{ij}, b^1_i) = [0, \max\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}]$.*

Corollary 2. *If $S_{T^p_{SS}}(a_{ij}, b^2_i) \neq \varnothing$ for some $i \in I_2$ and $j \in J$, then $S_{T^p_{SS}}(a_{ij}, b^2_i) = [\min\limits_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^2_i)\}, 1]$.*

Remark 1. *Corollary 1 together with Definition 1 implies $J^1_i = J$, $\forall i \in I_1$.*

Definition 2. *For each $i \in I_1$ and each $j \in J$, we define*

$$U_{ij} = \begin{cases} 1 & a_{ij} < b^1_i \\ \left((b^1_i)^p + 1 - a^p_{ij}\right)^{\frac{1}{p}} & a_{ij} \geq b^1_i \end{cases}$$

Also, for each $i \in I_2$ and each $j \in J$, we set

$$L_{ij} = \begin{cases} +\infty & d_{ij} < b^2_i \\ 0 & b^2_i = 0, d_{ij} \geq b^2_i \\ \left((b^1_i)^p + 1 - d^p_{ij}\right)^{\frac{1}{p}} & b^2_i \neq 0, d_{ij} \geq b^2_i \end{cases}$$

From Definition 2, if $a_{ij} = b^1_i$, then $U_{ij} = 1$. Also, we have $L_{ij} = 1$, if $d_{ij} = b^2_i$ and $b^2_i \neq 0$.

Lemma 1 below shows that U_{ij} and L_{ij} stated in Definition 2, determine the maximum and minimum solutions of sets $S_{T^p_{SS}}(a_{ij}, b^1_i)(i \in I_1)$ and $S_{T^p_{SS}}(a_{ij}, b^2_i)(i \in I_2)$, respectively.

Lemma 1. *(a)* $U_{ij} = \max_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^1_i)\}, \forall i \in I_1,$ *and* $\forall j \in J$. *(b) If* $S_{T^p_{SS}}(a_{ij}, b^2_i) \neq \varnothing$ *for some* $i \in I_2$ *and* $j \in J$, *then* $L_{ij} = \min_{x\in[0,1]}\{S_{T^p_{SS}}(a_{ij}, b^2_i)\}$.

Proof. The proof is similar to the proof of Lemma 1 in [15]. □

Corollary 3. *(a) For each* $i \in I_1$ *and* $j \in J$, $S_{T^p_{SS}}(a_{ij}, b^1_i) = [0, U_{ij}]$. *(b) If* $S_{T^p_{SS}}(a_{ij}, b^2_i) \neq \varnothing$ *for some* $i \in I_2$ *and* $j \in J$, *then* $S_{T^p_{SS}}(a_{ij}, b^2_i) = [L_{ij}, 1]$.

Definition 3. *For each* $i \in I_1$, *let* $S_{T^p_{SS}}(a_{ij}, b^1_i) = \{x \in [0,1]^n : \max_{j=1}^{n}\{T^p_{SS}(a_{ij}, x_j)\} \leq b^1_i\}$. *Similarly, for each* $i \in I_2$,*we define* $S_{T^p_{SS}}(d_{ij}, b^2_i) = \{x \in [0,1]^n : \max_{j=1}^{n}\{T^p_{SS}(d_{ij}, x_j)\} \leq b^2_i\}$.

According to Definition 3 and the constraints stated in (2), sets $S_{T^p_{SS}}(a_i, b^1_i)$ and $S_{T^p_{SS}}(d_i, b^2_i)$ actually denote the feasible solutions sets of the ith inequality $\max_{j\in J}\{T^p_{SS}(a_{ij}, x_j)\} \leq b^1_i($ $i \in I_1)$ and $\max_{j\in J}\{T^p_{SS}(d_{ij}, x_j)\} \geq b^2_i($ $i \in I_2)$ of problem (1), respectively. Based on (2) and Definitions 1 and 3, it can be easily concluded that for a fixed $i \in I_1$, $S_{T^p_{SS}}(a_i, b^1_i) \neq \varnothing$ iff $S_{T^p_{SS}}(a_{ij}, b^1_i) \neq \varnothing$, $\forall j \in J$. On the other hand, by Corollary 1 we know that $S_{T^p_{SS}}(a_{ij}, b^1_i) \neq \varnothing$, $i \in I_1$ and $\forall j \in J$. As a result, $S_{T^p_{SS}}(a_i, b^1_i) \neq \varnothing$ for each $i \in I_2$. However, in contrast to $S_{T^p_{SS}}(a_i, b^1_i)$, set $S_{T^p_{SS}}(d_i, b^2_i)$ may be empty. Actually, for a fixed $i \in I_2$, $S_{T^p_{SS}}(d_i, b^2_i)$ is nonempty if and only if $S_{T^p_{SS}}(d_{ij}, b^2_i)$ is nonempty for at least some $j \in J$. Additionally, for each $i \in I_2$ and $j \in J$ we have $S_{T^p_{SS}}(d_{ij}, b^2_i) \neq \varnothing$ if and only if $d_{ij} \geq b^2_i$. These results have been summarized in the following lemma1 Part (b) of the lemma gives a necessary and sufficient condition for the feasibility of set $S_{T^p_{SS}}(d_i, b^2_i)($ $\forall i \in I_2)$. It is to be noted that the lemma 2 (part (b))also provides a necessary condition for problem (1).

Lemma 2. *(a)* $S_{T^p_{SS}}(a_i, b^1_i) \neq \varnothing$, $\forall i \in I_1$. *(b) For a fixed* $i \in I_2$, $S_{T^p_{SS}}(d_{ij}, b^2_i) \neq \varnothing$ *iff* $\bigcup_{j=1}^{n} S_{T^p_{SS}}(d_{ij}, b^2_i) \neq \varnothing$. *Also, for each* $i \in I_2$ *and* $j \in J$, $S_{T^p_{SS}}(d_{ij}, b^2_i) \neq \varnothing$ *iff* $d_{ij} \geq b^2_i$.

Definition 4. *For each* $i \in I_2$ *and* $j \in J^2_i$, *we define* $S_{T^p_{SS}}(d_i, b^2_i, j) = [0,1] \times \cdots \times [0,1] \times [L_{ij}, 1] \times [0,1] \times \cdots \times [0,1]$, *where* $[L_{ij}, 1]$ *is in the* j*th position.*

Lemma 3. *(a)* $S_{T^p_{SS}}(a_i, b^2_i) = [0, U_{i,1}] \times [0, U_{i,2}] \times \cdots \times [0, U_{i,n}]$, $\forall i \in I_1$. *(b)* $S_{T^p_{SS}}(d_i, b^2_i) = \bigcup_{j\in J^2_i} S_{T^p_{SS}}(d_i, b^2_i, j)$, $\forall i \in I_2$.

Proof. For a more general case, see Lemma 2.3 in [16]. □

Definition 5. *Let* $\overline{X}(i) = [U_{i1}, U_{i2}, \ldots, U_{i,n}]$, $\forall i \in I_1$. *Also, let* $\underline{X}(i,j) = [\underline{X}(i,j)_1, \underline{X}(i,j)_2, \ldots, \underline{X}(i,j)_n]$, $\forall i \in I_2$ *and* $\forall i \in J^2_i$, *where*

$$\underline{X}(i,j)_k \begin{cases} L_{ij} & k = j \\ 0 & k \neq j \end{cases}$$

Lemma 3 together with Definitions 4 and 5, results in Theorem 1, which completely determines the feasible region for the ith relational inequality.

Theorem 1. *(a)* $S_{T_{SS}^p}(a_i, b_i^1) = [0, \overline{X}(i)]$, $\forall i \in I_1$. *(b)* $S_{T_{SS}^p}(d_i, b_i^2) = \bigcup_{j \in J_i^2} [\underline{X}(i,j), 1]$, $\forall i \in I_2$, *where* 0 *and* 1 *are ndimensional vectors with each component equal to zero and one, respectively.*

Theorem 1 gives the upper and lower bounds for the feasible solutions set of the ith relational inequality. Actually, for each $i \in I_1$, vectors 0 and $\overline{X}(i)$ are the unique minimum and the unique maximum of set $S_{T_{SS}^p}(a_i, b_i^1)$. In addition, for each $i \in I_2$, set $S_{T_{SS}^p}(d_i, b_i^2)$ has the unique maximum (i.e., vector 1), but the finite number of minimal solutions $\underline{X}(i,j)(\forall j \in J_i^2)$. Furthermore, part (b) of Theorem 1 presents another feasible necessary condition for problem (1) as stated in the following corollary.

Corollary 4. *If* $S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$, *then* $1 \in S_{T_{SS}^p}(d_i, b_i^2)$, $\forall i \in I_2$*(i.e.,* $1 \in \bigcap_{j \in I_2} S_{T_{SS}^p}(d_i, b_i^2) = S_{T_{SS}^p}(D, b^2)$*).*

Proof. Let $S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$. Then, $S_{T_{SS}^p}(D, b^2) \neq \varnothing$, and therefore, $S_{T_{SS}^p}(d_i, b_i^2) \neq \varnothing$, $\forall i \in I_2$. Now, Theorem 1 (part (b)) implies $1 \in S_{T_{SS}^p}(d_i, b_i^2)$, $\forall i \in I_2$. □

Lemma 4 describes the shape of the feasible solutions set for the fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \leq b^2$, separately.

Lemma 4. *(a)* $S_{T_{SS}^p}(A, b^1) = \bigcap_{i \in I_1} [0, U_{i1}] \times \bigcap_{i \in I_1} [0, U_{i2}] \times \cdots \times \bigcap_{i \in I_1} [0, U_{in}]$.
(b) $S_{T_{SS}^p}(D, b^2) = \bigcap_{i \in I_2} \bigcup_{j \in J_i^2} S_{T_{SS}^p}(d_i, b_i^2, j)$.

Proof. The proof is obtained from Lemma 3 and equations $S_{T_{SS}^p}(A, b^1) = \bigcap_{i \in I_1} S_{T_{SS}^p}(a_i, b_i^1)$ and
$S_{T_{SS}^p}(D, b^2) = \bigcap_{i \in I_2} S_{T_{SS}^p}(d_i, b_i^2)$. □

Definition 6. *Let* $e : I_2 \longrightarrow J_i^2$ *so that* $e(i) = j \in J_i^2$, $\forall i \in I_2$, *and let* E_D *be the set of all vectors* e*. For the sake of convenience, we represent each* $e \in E_D$ *as an* m_2*dimensional vector* $e = [j_1, j_2, \ldots, j_{m_2}]$ *in which* $j_k = e(k)$, $k = 1, 2, \ldots, m_2$.

Definition 7. *Let* $e = [j_1, j_2, \ldots, j_{m_2}] \in E_D$. *We define* $\overline{X} = \min_{i \in I_1}\{\overline{X}(i)\}$, *that is,* $\overline{X}_j = \min_{i \in I_1}\{\overline{X}(i)_j\}$, $\forall j \in J$*.Moreover, let* $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, \ldots, \underline{X}(e)_n]$, *where* $\underline{X}(e)_j = \max_{i \in I_2}\{\underline{X}(i, e(i))_j\} = \max_{i \in I_2}\{\underline{X}(i, j_i)_j\}$, $\forall j \in J$.

Based on Theorem 1 and the above definition, we have the following theorem characterizing the feasible regions of the general inequalities $A\varphi x \gneqq b^1$ and $D\varphi x \geq b^2$ in the most familiar way.

Theorem 2. *(a)* $S_{T^p_{SS}}(A, b^1) = [0, \overline{X}], \forall i \in I_1$. *(b)* $S_{T^p_{SS}}(D, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), 1]$.

Proof. See Theorem 2.2 and Remark 2.5 in [16]. □

Corollary 5. *Assume that* $S_{T^p_{SS}}(A, D, b^1, b^2) \neq \varnothing$*.Then, there exists some* $e \in E_D$ *such that* $[0, \overline{X}] \cap [\underline{X}(e), 1] \neq \varnothing$*.*

Corollary 6. *Assume that* $S_{T^p_{SS}}(A, D, b^1, b^2) \neq \varnothing$*. Then,* $\overline{X} \in S_{T^p_{SS}}(D, b^2)$ *.*

Proof. Let $S_{T^p_{SS}}(A, D, b^1, b^2) \neq \varnothing$. By Corollary 5, $[0, \overline{X}] \cap [\underline{X}(\acute{e}), 1] \neq \varnothing$ for some $\acute{e} \in E_D$. Thus, $\overline{X} \in [\underline{X}(\acute{e}), 1]$ that means $\overline{X} \in \bigcup_{e \in E_D} [\underline{X}(e), 1]$. Therefore, from Theorem 2(part(b)), $\overline{X} \in S_{T^p_{SS}}(D, b^2)$. □

4 Feasible solutions set and simplification operations

In this section, two operations are presented to simplify the matrices A and D, and a necessary and sufficient condition is derived to determine the feasibility of the main problem. At first, we give a theorem in which the bounds of the feasible solutions set of problem (1) are attained. As is shown in the following theorem, by using these bounds, the feasible region is completely found.

Theorem 3. *Suppose that* $S_{T^p_{SS}}(A, D, b^1, b^2) \neq \varnothing$*. Then*

$$S_{T^p_{SS}}(A, D, b^1, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}]$$

.

Proof. $S_{T^p_{SS}}(A, D, b^1, b^2) = S_{T^p_{SS}}(A, b^1) \cap S_{T^p_{SS}}(D, b^2)$, then by Theorem 2, $S_{T^p_{SS}}(A, D, b^1, b^2) = [0, \overline{X}] \cap \left(\bigcup_{e \in E_D} [\underline{X}(e), 1] \right)$ and the statement is established. □

In practice, there are often some components of matrices A and D, which have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. We refer the interesting reader to [16] where a brief review of such these processes is given. Here, we present two simplification techniques based on the Schweizer-Sklarfamily of t-norms.

Definition 8. *If a value changing in an element, say* a_{ij}*, of a given fuzzy relation matrix* A *has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.*

Corollary 7. *Suppose that* $i \in I_1$ *and* $T^p_{SS}(a_{ij_0}, x_{j_0})$*,* $\forall x \in S_{T^p_{SS}}(A, b^1)$*. In this case, it is obvious that* $\max_{j=1}^{n} \{T^p_{SS}(a_{ij}, x_j)\} \leq b^1_i$ *is equivalent to* $\max_{\substack{j=1 \\ j \neq j_0}}^{n} \{T^p_{SS}(a_{ij}, x_j)\} \leq b^1_i$*,that* *is, resetting a_{ij} to zero has no effect on the solutions of problem (1) (since component a_{ij_0} only appears in the ith constraint of problem (1)). Therefore, if $T^p_{SS}(a_{ij_0}, x_{j_0}) < b^1_i$, $\forall x \in S_{T^p_{SS}}(A, b^1)$, then resetting a_{ij_0} to zero is an equivalence operation.*

Lemma 5 (simplification of matrix A). *Suppose that matrix $\tilde{A} = (\tilde{a}_{ij})_{m_1 \times n}$ is resulted from matrix A as follows:*

$$\tilde{a}_{ij} = \begin{cases} 0 & a_{ij} < b^1_i \\ a_{ij} & a_{ij} \geq b^1_i \end{cases}$$

for each $i \in I_1$ and $j \in J$. Then, $S_{T^p_{SS}}(A, b^1) = S_{T^p_{SS}}(\tilde{A}, b^1)$.

Proof. From corollary 7, it is sufficient to show that $T^p_{SS}(a_{ij_0}, x_{j_0}) < b^1_i$, $\forall x \in S_{T^p_{SS}}(A, b^1)$. But, from the monotonicity and identity laws of T^p_{SS}, we have
$T^p_{SS}(a_{ij_0}, x_{j_0}) \leq T^p_{SS}(a_{ij_0}, 1) = a_{ij_0} < b^1_i$, $\forall x_{j_0} \in [0, 1]$. Thus, $T^p_{SS}(a_{ij_0}, x_{j_0}) < b^1_i$, $\forall x \in S_{T^p_{SS}}(A, b^1)$. □

Lemma 5 gives a condition to reduce the matrix A. In this lemma, $\tilde{A}$ denote the simplified matrix resulted from A after applying the simplification process. Based on this notation, we define $\tilde{J}^1_i = \{j \in J : S_{T^p_{SS}}(\tilde{a}_{ij}, b^1_i) \neq \varnothing\}$($\forall i \in I_1$) where $\tilde{a}_{ij}$ denotes (i, j)th component of matrix $\tilde{A}$. So, from Corollary 1 and Remark 1, it is clear that $\tilde{J}^1_i = J^1_i = J$. Moreover, since $S_{T^p_{SS}}(A, D, b^1, b^2) = S_{T^p_{SS}}(A, b^1) \cap S_{T^p_{SS}}(D, b^2)$, from Lemma 5 we can also conclude that $S_{T^p_{SS}}(A, D, b^1, b^2) = S_{T^p_{SS}}(\tilde{A}, D, b^1, b^2)$.
By considering a fixed vector $e \in E_D$ in Theorem 3, interval $[\underline{X}(e), \overline{X}]$ is meaningful iff $\underline{X}(e) \leq \underline{X}$. Therefore, by deletinginfeasible intervals $[\underline{X}(e), \overline{X}]$ in which $\underline{X}(e) \nleq \underline{X}$, the feasible solutions set of problem (1) stays unchanged. In order to remove such infeasible intervals from the feasible region, it is sufficient to neglect vectors e generating infeasible solutions $\underline{X}(e)$(i.e., solutions $\underline{X}(e)$ such that $\underline{X}(e) \nleq \underline{X}$). These considerations lead us to introduce a new set $\acute{E}_D = \{e \in E_D : \underline{X}(e) \leq \underline{X}\}$ to strengthen Theorem 3. By this new set, Theorem 3 can be written as $S_{T^p_{SS}}(A, D, b^1, b^2) = \bigcup_{e \in \acute{E}_D} [\underline{X}(e), \overline{X}]$, if $S_{T^p_{SS}}(A, D, b^1, b^2) \neq \varnothing$.

Lemma 6. *Let $I_j(e) = \{i \in I_2 : e(i) = j\}$ and $J(e) = \{j \in J : I_j(e) \neq \varnothing\}$, $\forall e \in E_D$. Then,*

$$\underline{X}(e)_j = \begin{cases} \max\limits_{i \in I_j(e)} \{L_{ie(i)}\} & j \in J(e) \\ 0 & j \notin J(e) \end{cases}$$

Proof. From Definition 7, $\underline{X}(e)_j = \max\limits_{i \in I_j(e)} \{\underline{X}(i, e(i))_j\}$, $\forall j \in J$. On the other hand, by Definition 5, we have

$$\underline{X}(i, e(i))_j = \begin{cases} L_{ie(i)} & j = e(i) \\ 0 & j \neq e(i) \end{cases}$$

Now, the result follows by combining these two equations. □

Corollary 8. *$e \in \acute{E}_D$ if and only if $L_{ie(i)} \le \overline{X}_{e(i)}$, $\forall i \in I_2$.*

Proof. Firstly, from the definition of set $\acute{E}_D$, we note that $e \in \acute{E}_D$ if and only if $\underline{X}(e)_j \le \overline{X}_j$, $\forall j \in J$. Now, let $e \in \acute{E}_D$ and by contradiction, suppose that $L_{i_0e(i_0)} > \overline{X}_{e(i_0)}$ for some $i_0 \in I_2$. So, by setting $e(i_0) = j_0$, we have $j_0 \in J(e)$, and therefore lemma 6 implies $\underline{X}(e)_{j_0} = \max_{i \in I_{j_0}(e)} \{L_{ie(i)}\} \ge L_{i_0e(i_0)} > \overline{X}_{e(i_0)}$. Thus, $\underline{X}(e)_{j_0} > \overline{X}_{e(i_0)}$ that contradicts $e \in \acute{E}_D$. The converse statement is easily proved by Lemma 6. □

As mentioned before, to accelerate identification of the meaningful solutions $\underline{X}(e)$, we reduce our search to set $\acute{E}_D$ instead of set E_D.As a result from Corollary 8, we can confine set J_i^2 by removing each $j \in J_i^2$ such that $L_{ij} > \overline{X}_j$ before selecting the vectors e to construct solutions $\underline{X}(e)$. However, lemma 7 below shows that this purpose can be accomplished by resetting some components of matrix D to zeros. Before formally presenting the lemma, some useful notations are introduced.

Definition 9 (simplification of matrix D)**.** *Let $\tilde{D} = (\tilde{d}_{ij})_{m_2 \times n}$ denote a matrix resulted from as follows:*

$$\tilde{d}_{ij} = \begin{cases} 0 & j \in J_i^2 \; and \, L_{ij} > \overline{X}_j \\ d_{ij} & otherwise \end{cases}$$

Also, similar to Definition 1, assume that $\tilde{J}_i^2 = \{i \in J : S_{T_{SS}^p}(\tilde{d}_{ij}, b_i^2) \ne \varnothing\}$($\forall i \in I_2$) where $\tilde{d}_{ij}$ denotes (i,j)th components of matrix $\tilde{D}$.

According to the above definition, it is easy to verify that $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$. Furthermore, the following lemma demonstrates that the infeasible solutions $\underline{X}(e)$ are not generated, if we only consider those vectors generated by the components of the matrix $\tilde{D}$, or equivalently vectors generated based on the set $\tilde{J}_i^2$ instead of J_i^2.

Lemma 7. *$E_{\tilde{D}} = \acute{E}_{\tilde{D}}$, where $E_{\tilde{D}}$ is the set of all functions $e : I_2 \longrightarrow \tilde{J}_i^2$ so that $e(i) = j \in \tilde{J}_i^2$, $\forall i \in I_2$.*

Proof. Let $e \in \acute{E}_D$. Then, by Corollary 8, $L_{ie(i)} \le \overline{X}_{e(i)}$, $\forall i \in I_2$. Therefore, we have $\tilde{d}_{ie(i)} = d_{ie(i)}$, $\forall i \in I_2$, that necessitates $\tilde{J}_i^2 = J_i^2$, $\forall i \in I_2$. Hence, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$, and then $e \in E_{\tilde{D}}$. Conversely, let $e \in E_{\tilde{D}}$. Therefore, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$. Since $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$, then $e(i) \in J_i^2$, $\forall i \in I_2$, and therefore $e \in E_D$. By contradiction, suppose that $e \notin \acute{E}_D$. So, by Corollary 8, there is some $i_0 \in I_2$ such that $L_{i_0e(i_0)} > \overline{X}_{e(i_0)}$. Hence, $\tilde{d}_{i_0e(i_0)} = 0$ (since $e(i_0) \in J_{i_0}^2$ and $L_{i_0e(i_0)} > \overline{X}_{e(i_0)}$) and $L_{i_0e(i_0)} > 0$.The latter inequality together with Definition 2 implies $b_{i_0}^2 > 0$. But in this case, $T_{SS}^p(\tilde{d}_{i_0e(i_0)}) = T_{SS}^p(0, x) = 0 < b_{i_0}^2$, $\forall x \in [0,1]$, that contradicts $e(i_0) \in J_{i_0}^2$. □

By Lemma 7, we always have $\underline{X}(e) \le \overline{X}$ for each vector e, which is selected based on the components of matrix $\tilde{D}$. Actually, matrix $\tilde{D}$ as a reduced version of matrix D, removes all the infeasible intervals from the feasible region by neglecting those vectors generating the infeasible solutions $\underline{X}(e)$. Also, similar to Lemma 5 we have $S_{T_{SS}^p}(A, D, b^1, b^2) =$

$S_{T_{SS}^p}(A, \tilde{D}, b^1, b^2)$. This result and Lemma 5 can be summarized by $S_{T_{SS}^p}(A, D, b^1, b^2) = S_{T_{SS}^p}(\tilde{A}, \tilde{D}, b^1, b^2)$.

Definition 10. *Let $L = (L_{ij})_{m_2 \times n}$ be a matrix whose (i, j)th component is equal to L_{ij}. We define the modified matrix $L^* = (L^*_{ij})_{m_2 \times n}$ from the matrix L as follows:*

$$L^*_{ij} = \begin{cases} +\infty & L_{ij} > \overline{X}_j \\ L_{ij} & \textit{otherwise} \end{cases}$$

As will be shown in the following theorem, matrix L^* is useful for deriving a necessary and sufficient condition for the feasibility of problem (1)and accelerating identification of the set $S_{T_{SS}^p}(A, D, b^1, b^2)$.

Theorem 4. *$S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$ iff there exists at least some $j \in J_i^2$ such that $L^*_{ij} \neq +\infty$, $\forall i \in I_2$.*

Proof. Let $x \in S_{T_{SS}^p}(A, D, b^1, b^2)$. Then, from Corollary 5, there exists some $\acute{e} \in E_D$ such that $[\underline{X}(\acute{e}), \overline{X}] \neq \varnothing$. Therefore, $\underline{X}(\acute{e}) \leq \overline{X}$ that implies $\acute{e} \in \acute{E}_D$. Now, by Corollary 8, we have $L_{i\acute{e}(i)} \leq \overline{X}_{\acute{e}(i)}$, $\forall i \in I_2$. Hence, by considering Definition 10, $L^*_{i\acute{e}(i)} \neq +\infty$, $\forall i \in I_2$. Conversely, suppose that $L^*_{ij_i} \neq +\infty$ for some $j_i \in J_i^2$, $\forall i \in I_2$. Then, from Definition 10 we have

$$L_{ij_i} \leq \overline{X}_{j_i}, \quad \forall i \in I_2 \tag{3}$$

Consider vector $\acute{a} = [j_1, j_2, \ldots, j_m] \in E_D$. So, by noting Lemma 6, $\underline{X}(\acute{e})_{j_i} = \max\limits_{i \in I_j(\acute{e})}\{L_{i\acute{e}(i)}\} = \max\limits_{i \in I_j(\acute{e})}\{L_{ij_i}\}$, $\forall i \in I_2$, and $\underline{X}(\acute{e})_j = 0$ for each $j \in J - \{j_1, j_2, \ldots, j_m\}$. These equations together with (3) imply $\underline{X}(\acute{e}) \leq \overline{X}$ that means $[\underline{X}(\acute{e}), \overline{X}] \neq \varnothing$. Now, the result follows from Corollary 5.

□

5 Optimization of the problem

According to the well-known schemes used for optimization of linear problems such as (1) [9, 16, 21, 30], problem (1) is converted to the following two sub-problems:

$$\begin{array}{ll} \min & Z_1 = \sum_{j=1}^n c_j^+ x_j \\ & A\varphi x \leq b^1 \\ & D\varphi x \geq b^2 \\ & x \in [0,1]^n \end{array} \tag{4}$$

$$\begin{array}{ll} \min & Z_1 = \sum_{j=1}^n c_j^- x_j \\ & A\varphi x \leq b^1 \\ & D\varphi x \geq b^2 \\ & x \in [0,1]^n \end{array} \tag{5}$$

Where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \max\{c_j, 0\}$ for $j = 1, 2, \ldots, n$. It is easy to prove that $\overline{X}$ is the optimal solution of (5), and the optimal solution of (4) is $\underline{X}(\acute{e})$ for some $\acute{e} \in \acute{E}_D$.

Theorem 5. *Suppose that $S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems* (5) *and* (4)*, respectively. Then $c^T x^*$ is the lower bound of the optimal objective function in* (1)*, where $x^+ = [x_1^*, x_2^*, \ldots, x_n^*]$ is defined as follows:*

$$x_j^* = \begin{cases} \overline{X}_j & c_j < 0 \\ \underline{X}(e^*)_j & c_j \geq 0 \end{cases} \tag{6}$$

for $j = 1, 2, \ldots, n$

Proof. See Colrollary 4.1 in [16]. □

Corollary 9. *Suppose that $S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$. Then, $x^+ = [x_1^*, x_2^*, \ldots, x_n^*]$ as defined in* (6)*,is the optimal solution of problem (1).*

Proof. As in the poof of Theorem 5, $c^T x^*$ is the lower bound of the optimal objective function. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \leq x_j^* \leq \overline{X}_j$, $\forall j \in J$, which implies

$$x^* \in \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}] = S_{T_{SS}^p}(A, D, b^1, b^2)$$

. □

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))

Given problem (1):

1. Compute $U_{ij}(\ \forall i \in I_1$ and $\forall j \in J)$ and $L_{ij}(\ \forall i \in I_2$ and $\forall j \in J)$ by Definition 2.

2. If $1 \in S_{T_{SS}^p}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 4).

3. Compute vectors $\overline{X}(i)(\ \forall i \in I_1)$ from Definition 5, and then vector $\overline{X}$ from Definition 7.

4. If $\overline{X} \in S_{T_{SS}^p}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 6).

5. Compute simplified matrices $\tilde{A}$ and $\tilde{D}$ from Lemma 5 and Definition 9, respectively.

6. Compute modified matrix L^* from Definition 10.

7. For each $i \in I_2$, if there exists at least some $j \in J_i^2$ such that $L_{ij}^* \neq +\infty$, then continue; otherwise, stop, the problem is infeasible (Theorem 4).

8. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem (4) by considering vectors $e \in E_{\tilde{D}}$ and set $\tilde{J}_i^2$, $\forall i \in I_2$(Lemma 7).

9. Find the optimal solution $x^* = [x_1^*, x_2^*, \ldots, x_n^*]$ for the problem (1) by (6) (Corollary 9).

It should be noted that there is no polynomial time algorithm for complete solution of FRIs with the expectation $N \neq NP$. Hence, the problem of solving FRIs is an NP-hard problem in terms of computational complexity [2].

6 Construction of test problems and numerical example

In this section, we present a method to generate random feasible regions formed as the intersection of two fuzzy inequalities with Schweizer-Sklar family of t-norms. In section 6.1, we prove that the max-Schweizer-Sklar fuzzy relational inequalities constructed by the introduced method are actually feasible. In section 6.2, the method is used to generate a random test problem for problem (1), and then the test problem is solved by Algorithm 1 presented in section 5.

6.1 Construction of test problems

There are several ways to generate a feasible FRI defined with max-Schweizer-Sklar composition. In what follows, we present a procedure to generate random feasible max-Schweizer-Sklar fuzzy relational inequalities:

Algorithm2 (construction of feasible Max-Schweizer-Sklar FRI)

1. Generate randon scalars $a_{ij} \in [0,1]$, $i = 1, 2, \ldots, m_1$ and $b_i^1 \in [0,1]$, $i = 1, 2, \ldots, m_1$.

2. Compute $\overline{X}$ by Definition 7.

2. Randomly select m_2 columns $\{j_1, j_2, \ldots, j_{m_2}\}$ from $J = \{1, 2, \ldots, n\}$.

2. For $i \in \{1, 2, \ldots, m_2\}$, assign a random number from $[0, \underline{X}_{j_i}]$ to b_i^2.

3. For $i \in \{1, 2, \ldots, m_2\}$, if $b_i^2 \neq 0$, then

Assign a random number from the following interval to d_{ij_i}:

$$\left[\max\left\{b_i^2, \left((b_i^2)^p + 1 - \overline{X}_{j_i}^p\right)^{\frac{1}{p}}\right\}, 1\right]$$

End

4. For $i \in \{1, 2, \ldots, m_2\}$

 For each $k \in \{1, 2, \ldots, m_2\} - \{1\}$

 Assign a random number from $[0, 1]$ to d_{kj_i}.

 End

 End

5. For each $i \in \{1, 2, \ldots, m_2\}$ and each $j \notin \{j_1, j_2, \ldots, j_{m_2}\}$

 Assign a random number for $[0, 1]$ to d_{ij}.

 End

By the following theorem, it is proved that Algorithm 2 always generates random feasible max-Schweizer-Sklar fuzzy relational inequalities.

Theorem 6. *Problem (1) with feasible region constructed by Algorithm (2) has the nonempty feasible solutions set (i.e., $S_{T_{SS}^p}(A, D, b^1, b^2) \neq \varnothing$).*

Proof. By considering the columns $\{j_1, j_2, \ldots, j_{m_2}\}$ selected by Algorithm 2, let $\acute{e} = [j_1, j_2, \ldots, j_{m_2}]$. We show that $\acute{e} \in E_D$ and $\underline{X}(\acute{e}) \leq \overline{X}$. Then, the result follows from Corollary 5. From Algorithm 2, the following inequalities are resulted for each $i \in I_2$: (I) $b_i^2 \leq \overline{X}_{j_i}$; (II) $b_i^2 \leq d_{ij_i}$; (III) $\left((b_i^2)^p + 1 - \overline{X}_{j_i}^p\right)^{\frac{1}{p}} \leq d_{ij_i}$. By (I), we have $\left((b_i^2)^p + 1 - \overline{X}_{j_i}^p\right)^{\frac{1}{p}} \leq 1$. This inequality together with $b_i^2 \in [0, 1]$, $\forall i \in I_2$, implies that the interval $\left[\max\left\{b_i^2, \left((b_i^2)^p + 1 - \overline{X}_{j_i}^p\right)^{\frac{1}{p}}\right\}, 1\right]$ is meaningful. Also, by (II), $\acute{e}(i) = j_i \in J_i^2$, $\forall i \in I_2$. Therefore, $\acute{e} \in E_D$. Moreover, since the columns $\{j_1, j_2, \ldots, j_{m_2}\}$ are distinct, sets $I_{j_i}(\acute{e})($ $i \in I_2)$are all singleton, i.e.,

$$I_{j_i}(\acute{e}) = \{i\}, \quad \forall i \in I_2 \tag{7}$$

As a result, we also have $J(\acute{e}) = \{j_1, j_2, \ldots, j_{m_2}\}$ and $I_j(\acute{e}) = \varnothing$ for each $j \notin \{j_1, j_2, \ldots, j_{m_2}\}$. On the other hand, from Definition 5, we have $\underline{X}(i, \acute{e})_{\acute{e}(i)} = \underline{X}(i, j_i)_{j_i} = L_{ij_i}$ and $\underline{X}(i, \acute{e}(i))_j = 0$ for each $j \notin J - \{j_i\}$. This fact together with (7) and Lemma 6 implies $\underline{X}(\acute{e})_{j_i} = L_{ij_i}$, $\forall i \in I_2$, and $\underline{X}(\acute{e})_j = 0$ for $j \notin \{j_1, j_2, \ldots, j_{m_2}\}$. So, in order to prove $\underline{X}(\acute{e}) \leq \overline{X}$, it is sufficient to show that $\underline{X}(\acute{e})_{j_i} \leq \overline{X}_{j_i}$, $\forall i \in I_2$. But, from Definition 2,

$$\underline{X}(\acute{e})_{j_i} = L_{ij_i} = \begin{cases} 0 & b_i^2 = 0 \\ \left((b_i^2)^p + 1 - d_{ij_i}^p\right)^{\frac{1}{p}} & b_i^2 \notin 0 \end{cases} \tag{8}$$

Now, inequality (III) implies

$$\left((b_i^2)^p + 1 - d_{ij_i}^p\right)^{\frac{1}{p}} \leq \overline{X}_{j_i} \tag{9}$$

Therefore, by relations (8) and (9), we have $\underline{X}(\acute{e})_{j_i} \leq \overline{X}_{j_i}$, $\forall i \in I_2$. This completes the proof. $\square$

6.2 Numerical example

Consider the following linear optimization problem (1) in which the feasible region has been randomly generated by Algorithm 2 presented in section 6.1.

$$\begin{aligned}
\min\ Z = {} & 6.2945x_1 + 8.1158x_2 - 7.4603x_3 + 8.2675x_4 + 26472x_5 - 8.0492x_6 - 4.4300x_7 0.9376x_8 \\
& - 9.1501x_9 + 9.2978x_{10}
\end{aligned}$$

$$\begin{bmatrix}
0.1576 & 0.6557 & 0.7060 & 0.4387 & 0.2713 & 0.8407 & 0.3517 & 0.3517 & 0.0759 & 0.1622 \\
0.9706 & 0.0357 & 0.0318 & 0.3816 & 0.6797 & 0.2551 & 0.2543 & 0.8308 & 0.0540 & 0.7943 \\
0.9572 & 0.8491 & 0.2769 & 0.7655 & 0.6551 & 0.5060 & 0.8143 & 0.5853 & 0.5308 & 0.3112 \\
0.4854 & 0.9340 & 0.0462 & 0.7952 & 0.1626 & 0.6991 & 0.2435 & 0.5497 & 0.7792 & 0.5285 \\
0.8003 & 0.6787 & 0.0971 & 0.1869 & 0.1190 & 0.8909 & 0.9293 & 0.9172 & 0.9340 & 0.1656 \\
0.1419 & 0.7577 & 0.8235 & 0.4898 & 0.4984 & 0.9593 & 0.3500 & 0.2858 & 0.1299 & 0.6020 \\
0.4218 & 0.7431 & 0.6948 & 0.4456 & 0.9597 & 0.5472 & 0.1966 & 0.7572 & 0.5688 & 0.2630 \\
0.9157 & 0.3922 & 0.3171 & 0.6463 & 0.3404 & 0.1386 & 0.2511 & 0.7537 & 0.4694 & 0.6541 \\
0.7922 & 0.6555 & 0.9502 & 0.7094 & 0.5853 & 0.1493 & 0.6160 & 0.3804 & 0.0119 & 0.6892 \\
0.9595 & 0.1712 & 0.0344 & 0.7547 & 0.2238 & 0.2575 & 0.4733 & 0.5678 & 0.3371 & 0.7482
\end{bmatrix} \varphi x \leq \begin{bmatrix}
0.4505 \\ 0.0838 \\ 0.2290 \\ 0.9133 \\ 0.1524 \\ 0.8258 \\ 0.5383 \\ 0.9961 \\ 0.0782 \\ 0.4427
\end{bmatrix}$$

$$\begin{bmatrix}
0.3674 & 0.4899 & 0.9730 & 0.6099 & 0.3909 & 0.4588 & 0.6797 & 0.9934 & 0.8909 & 0.0714 \\
0.9880 & 0.1679 & 0.6490 & 0.6177 & 0.8314 & 0.9198 & 0.1366 & 0.0855 & 0.3342 & 0.5216 \\
0.9999 & 0.9787 & 0.8003 & 0.8594 & 0.8034 & 0.9631 & 0.7212 & 0.2625 & 0.6987 & 0.0967 \\
0.0377 & 0.7127 & 0.4538 & 0.8055 & 0.0605 & 0.5468 & 0.9407 & 0.8010 & 0.1978 & 0.8181 \\
0.8852 & 0.5005 & 0.4324 & 0.5767 & 0.3993 & 0.5211 & 0.1068 & 0.0292 & 0.9568 & 0.8175 \\
0.9133 & 0.4711 & 0.8253 & 0.8789 & 0.5269 & 0.2316 & 0.6538 & 0.9289 & 0.0305 & 0.7224 \\
0.7962 & 0.9677 & 0.0835 & 0.1829 & 0.4168 & 0.4889 & 0.4942 & 0.7303 & 0.7441 & 0.1499 \\
0.0987 & 0.0596 & 0.1332 & 0.2399 & 0.6569 & 0.6241 & 0.7791 & 0.4886 & 0.5000 & 0.8248 \\
0.2619 & 0.6820 & 0.9672 & 0.8865 & 0.6280 & 0.6791 & 0.7150 & 0.5785 & 0.4799 & 0.6596 \\
0.3354 & 0.0424 & 0.1734 & 0.0287 & 0.9189 & 0.3955 & 0.9037 & 0.2373 & 0.9047 & 0.5186
\end{bmatrix} \varphi x \geq \begin{bmatrix}
0.1327 \\ 0.2061 \\ 0.2307 \\ 0.1754 \\ 0.1002 \\ 0.4063 \\ 0.3470 \\ 0.1360 \\ 0.0953 \\ 0.2576
\end{bmatrix}$$

$$x \in [0,1]^n$$

where $|I_1| = |I_2| = |J| = 10$ and $\varphi(x,y) = T_{SS}^p(x,y) = \sqrt{\max\{x^2 + y^2 - 10\}}$(i.e., $p = 2$). Moreover, $Z_1 = 6.2945x_1 + 8.1158x_2 + 8.2675x_4 + 2.6472x_5 + 0.9376x_8 + 9.1501x^9 + 9.2978x_{10}$ the objective function of sub-problem (4) and $Z_2 = -7.4603x_3 - 8.0492x_6 - 4.4300x_7$ is that of sub-problem (5). By Definition 2, matrices $U = (U_{ij})_{10\times 10}$ and $L = (L_{ij})_{10\times 10}$ are as follows:

$$U = \begin{bmatrix}
1.0000 & 0.8792 & 0.8393 & 1.0000 & 1.0000 & 0.7991 & 0.7044 & 1.0000 & 1.0000 & 1.0000 \\
0.2549 & 1.0000 & 1.0000 & 0.9281 & 0.7383 & 0.9705 & 0.9708 & 0.5628 & 1.0000 & 0.6133 \\
0.3691 & 0.5757 & 0.9878 & 0.6829 & 0.7895 & 0.8924 & 0.6240 & 0.8426 & 0.8779 & 0.9775 \\
1.0000 & 0.9807 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
0.6187 & 0.7500 & 1.0000 & 0.9941 & 1.0000 & 0.4791 & 0.3996 & 0.4266 & 0.3884 & 0.9979 \\
1.0000 & 0.9807 & 1.0000 & 1.0000 & 1.0000 & 0.8728 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
1.0000 & 0.8588 & 0.8983 & 1.0000 & 0.6072 & 0.9952 & 1.0000 & 0.8464 & 0.9830 & 1.0000 \\
1.0000 & 0.9807 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
0.6152 & 0.7592 & 0.3212 & 0.7092 & 0.8146 & 0.9919 & 0.7916 & 0.9281 & 1.0000 & 0.7288 \\
0.5247 & 1.0000 & 1.0000 & 0.7915 & 1.0000 & 1.0000 & 0.9859 & 0.9346 & 1.0000 & 0.7976
\end{bmatrix}$$

$$L = \begin{bmatrix} 0.9395 & 0.8818 & 0.2663 & 0.8035 & 0.9299 & 0.8984 & 0.7454 & 0.1753 & 0.4731 & \infty \\ 0.2576 & \infty & 0.7882 & 0.8130 & 0.5927 & 0.4432 & \infty & \infty & 0.9648 & 0.8777 \\ 0.2312 & 0.3088 & 0.6424 & 0.5609 & 0.6386 & 0.3545 & 0.7301 & 0.9921 & 0.7516 & \infty \\ \infty & 0.7231 & 0.9082 & 0.6180 & \infty & 0.8554 & 0.3818 & 0.6238 & 0.9958 & 0.6012 \\ 0.4759 & 0.8715 & 0.9072 & 0.8231 & 0.9223 & 0.8593 & 0.9993 & \infty & 0.3075 & 0.5845 \\ 0.5753 & 0.9712 & 0.6956 & 0.6266 & 0.9421 & \infty & 0.8589 & 0.5498 & \infty & 0.8020 \\ 0.6975 & 0.4289 & \infty & \infty & 0.9730 & 0.9388 & 0.9361 & 0.7662 & 0.7529 & \infty \\ \infty & \infty & \infty & 0.9803 & 0.7662 & 0.7931 & 0.6415 & 0.8812 & 0.8825 & 0.7576 \\ 0.9767 & \infty & \infty & \infty & 0.4711 & 0.9539 & 0.4996 & \infty & 0.4978 & 0.8930 \end{bmatrix}$$

Therefore, by Corollary 3 we have, for example:

$$S_{T^2_{SS}}(a_{32}, b^1_3) = [0, U_{32}] = [0, 0.5757] \text{and} S_{T^2_{SS}}(a_{57}, b^1_5) = [0, U_{57}] = [0, 0.3996].$$
$$S_{T^2_{SS}}(d_{12}, b^2_1) = [L_{12}, 1] = [0.8818, 1] \text{and} S_{T^2_{SS}}(d_{64}, b^2_6) = [L_{64}, 1] = [0.6266, 1].$$

Also, from Definition 1, $J^2_1 = J^2_3 = J - \{10\}$, $J^2_2 = J - \{2, 7, 8\}$, $J^2_4 = J - \{1, 5\}$, $J^2_5 = J - \{8\}$,
$J^2_6 = J - \{6, 9\}$, $J^2_7 = J - \{3, 4, 10\}$, $J^2_8 = J - \{1, 2, 3\}$, $J^2_9 = J = \{1, 2, \ldots, 10\}$ and $J^2_{10} = J - \{2, 3, 4, 8\}$.Moreover, the only components of matrix D such that $d_{ij} < b^2_i$ are as follows: $d_{1,10}$(in the first row), d_{22}, d_{27}, d_28(in the second row), $d_{3,10}$(in the third row), d_{41}, d_{45}(in the fourth row), d_{58}(in the fifth row), d_{66}, d_{69}(in the sixth row), d_{73}, $_{74}$, $d_{7,10}$(in the seventh row), d_{81}, d_{82}, d_{83}(in the eighth row) and $d_{10,2}$, $d_{10,3}$, $d_{10,4}$, $d_{10,8}$(in the tenth row).Therefore, by Lemma 2(part (b)), $S_{T^2_{SS}}(d_i, b^2_i) = \bigcup_{j=1}^{10} S_{T^2_{SS}}(d_{ij}, b^2_i) \neq \varnothing$, $\forall i \in I_2$.

By Definition 5, we have

$$\overline{X}(1) = \begin{bmatrix} 1 & 0.8792 & 0.8393 & 1 & 1 & 0.7991 & 0.7044 & 1 & 1 & 1 \end{bmatrix}$$
$$\overline{X}(2) = \begin{bmatrix} 0.2549 & 1 & 1 & 0.9281 & 0.7383 & 0.9705 & 0.9708 & 0.5628 & 1 & 0.6133 \end{bmatrix}$$
$$\overline{X}(3) = \begin{bmatrix} 0.3691 & 0.5757 & 0.9878 & 0.6829 & 0.7895 & 0.8924 & 0.6240 & 0.8426 & 0.8779 & 0.9775 \end{bmatrix}$$
$$\overline{X}(4) = \begin{bmatrix} 1 & 0.9807 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\overline{X}(5) = \begin{bmatrix} 0.6187 & 0.7500 & 1 & 0.9941 & 1 & 0.4791 & 0.3996 & 0.4266 & 0.3884 & 0.9979 \end{bmatrix}$$
$$\overline{X}(6) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0.8728 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\overline{X}(7) = \begin{bmatrix} 1 & 0.8588 & 0.8983 & 1 & 0.6072 & 0.9952 & 1 & 0.8464 & 0.9830 & 1 \end{bmatrix}$$
$$\overline{X}(8) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$\overline{X}(9) = \begin{bmatrix} 0.6152 & 0.7592 & 0.3212 & 0.7092 & 0.8146 & 0.9919 & 0.7916 & 0.9281 & 1 & 0.7288 \end{bmatrix}$$
$$\overline{X}(10) = \begin{bmatrix} 0.5247 & 1 & 1 & 0.7915 & 1 & 1 & 0.9859 & 0.9346 & 1 & 0.7976 \end{bmatrix}$$

Also, for example

$$\underline{X}(10,1) = \begin{bmatrix} 0.9767 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \underline{X}(10,5) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.4711 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\underline{X}(10,6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.9539 & 0 & 0 & 0 & 0 \end{bmatrix}, \underline{X}(10,7) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.4996 & 0 & 0 & 0 \end{bmatrix},$$
$$\underline{X}(10,9) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4978 & 0 \end{bmatrix}, \underline{X}(10,10) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8930 \end{bmatrix}.$$

Therefore, by Theorem 1, $S_{T_{SS}^2}(a_i, b_i^1) = [0, \overline{X}(i)]$, $\forall i \in I_1$, and for example $S_{T_{SS}^2}(d_{10}, b_{10}^2) = \bigcup[\underline{X}(10,1),1] \bigcup_{j=5}^{7} [\underline{X}(10,j),1] \bigcup[\underline{X}(10,9),1] \bigcup[\underline{X}(10,10),1]$, forthe tenth row of matrix D(i.e., $i = 10 \in I_2$).
From Corollary 4, the necessary condition holds for the feasibility of the problem. More precisely, we have

$$\begin{aligned} D\varphi 1 &= \begin{bmatrix} 0.9934 & 0.9880 & 0.9999 & 0.9407 & 0.9289 & 0.9677 & 0.8248 & 0.9672 & 0.9189 \end{bmatrix} \\ &\geq \begin{bmatrix} 0.1327 & 0.2061 & 0.2307 & 0.1754 & 0.1002 & 0.4063 & 0.3470 & 0.1360 & 0.0953 & 0.2576 \end{bmatrix} = b^2 \end{aligned}$$

that means $1 \in S_{T_{SS}^2}(D, b^2)$.
From Definition 7,

$$\overline{X} = \begin{bmatrix} 0.2549 & 0.57568 & 0.32123 & 0.68295 & 0.60721 & 0.47907 & 0.39961 & 0.42659 & 0.38839 & 0.6133 \end{bmatrix}$$

which determines the feasible region of the first inequalities, i.e., $S_{T_{SS}^2}(A, b^1) = [0, \overline{X}]$(Theorem 2, part (a)).Also,

$$\begin{aligned} D\varphi\overline{X} &= \begin{bmatrix} 0.4109 & 0.2749 & 0.5378 & 0.3395 & 0.2575 & 0.4887 & 0.5176 & 0.2375 & 0.5023 & 0.4616 \end{bmatrix} \\ &\geq \begin{bmatrix} 0.1327 & 0.2061 & 0.2307 & 0.1754 & 0.1002 & 0.4063 & 0.3470 & 0.1360 & 0.0953 & 0.2576 \end{bmatrix} = b^2 \end{aligned}$$

Therefore, we have $\overline{X} \in S_{T_{SS}^2}(D, b^2)$, which satisfies the necessary feasibility condition stated in Corollary 6. On the other hand, from Definition 6, we have $|E_D| = 960180480$. Therefore, the number of all vectors $e \in E_D$ is equal to 960180480. However, each solution $\underline{X}(e)$ generated by vectors $e \in E_D$ is not necessary a feasible solution. For example, for $\acute{e} = [8,3,1,4,10,2,6,6,6,9]$, we attain from Definition 7

$$\begin{aligned} \underline{X}(\acute{e}) &= \max_{i \in I_2} \{\underline{X}(i, \acute{e}(i))\} \\ &= \max\{\underline{X}(1,8), \underline{X}(2,3), \underline{X}(3,1), \underline{X}(4,4), \underline{X}(5,10), \underline{X}(6,2), \underline{X}(7,6), \underline{X}(8,6), \underline{X}(9,6), \underline{X}(10,9)\} \end{aligned}$$

where

$$\begin{aligned} \underline{X}(1,8) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1753 & 0 & 0 \end{bmatrix}, \underline{X}(2,3) = \begin{bmatrix} 0 & 0 & 0.7882 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \underline{X}(3,1) &= \begin{bmatrix} 0.2312 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \underline{X}(4,4) = \begin{bmatrix} 0 & 0 & 0 & 0.6180 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \underline{X}(5,10) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5845 \end{bmatrix}, \underline{X}(6,2) = \begin{bmatrix} 0 & 0.9712 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \underline{X}(7,6) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.9388 & 0 & 0 & 0 & 0 \end{bmatrix}, \underline{X}(8,6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.7931 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \underline{X}(9,6) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.7402 & 0 & 0 & 0 & 0 \end{bmatrix}, \underline{X}(10,6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 00.4978 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $\underline{X}(\acute{e}) = \begin{bmatrix} 0.2312 & 0.9712 & 0.7882 & 0.6180 & 0 & 0.9388 & 0 & 0.1753 & 0.4978 & 0.5845 \end{bmatrix}$
It is obvious that $\underline{X}(\acute{e}) \nleq \overline{X}$ (actually, $\underline{X}(\acute{e})_2 > overline X_2$, $\underline{X}(\acute{e})_3 > \overline{X}_3$, $\underline{X}(\acute{e})_6 > \overline{X}_6$ and $\underline{X}(\acute{e})_9 \nleq \overline{X}_9$) which means $\underline{X}(\acute{e}) \notin S_{T_{SS}^2}(A, D, b^1, b^2)$ from Theorem 3.

From the first simplification (Lemma 5), resetting the following components a_{ij} to zeros are equivalence operations: a_{11}, a_{14}, a_{15}, a_{18}, a_{19}, $a_{1,10}$; a_{22}, a_{23}, a_{29}; a_{41}, a_{4j} ($j = 3, 4, \ldots, 10$); a_{53}, a_{55}; a_{6j} ($j \in J - \{6\}$); a_{71}, a_{74}, a_{77}, $a_{7,10}$; a_{8j} ($j \in J$); a_{99}; $a_{10,2}$, $a_{10,3}$, $a_{10,5}$, $a_{10,6}$, $a_{10,9}$. So, matrix $\tilde{A}$ is resulted as follows:

$$\tilde{A} = \begin{bmatrix} 0 & 0.6557 & 0.7060 & 0 & 0 & 0.7513 & 0.8407 & 0 & 0 & 0 \\ 0.9706 & 0 & 0 & 0.3816 & 0.6797 & 0.2551 & 0.2543 & 0.8308 & 0 & 0.7943 \\ 0.9572 & 0.8491 & 0.2767 & 0.7655 & 0.6551 & 0.5060 & 0.8143 & 0.5853 & 0.5308 & 0.3112 \\ 0 & 0.9340 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8003 & 0.6787 & 0 & 0.1869 & 0 & 0.8909 & 0.9293 & 0.9172 & 0.9340 & 0.1656 \\ 0 & 0 & 0 & 0 & 0 & 0.9593 & 0 & 0 & 0 & 0 \\ 0 & 0.7431 & 0.6948 & 0 & 0.9597 & 0.5472 & 0 & 0.7572 & 0.5688 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7922 & 0.6555 & 0.9502 & 0.5853 & 0.1460 & 0.6160 & 0.3804 & 0 & 0.6892 & \\ 0.9595 & 0 & 0 & 0.7547 & 0 & 0 & 0.4733 & 0.5678 & 0 & 0.7482 \end{bmatrix}$$

Also, by Definition 9, we can change the value of components d_{1j} ($j \in J - \{3, 8, 10\}$), d_{2j} ($j \in J - \{2, 5, 6, 7, 8\}$), d_{3j} ($j \in J - \{1, 2, 4, 6, 10\}$), d_{4j} ($j \in J - \{1, 4, 5, 7, 10\}$), d_{5j} ($j \in J - \{8, 9, 10\}$), d_{6j} ($j \in J - \{4, 6, 9\}$), d_{7j} ($j \in J - \{2, 3, 4, 10\}$), d_{8j} ($j \in J - \{1, 2, 3, 10\}$), d_{9j} ($j \in J - \{3, 4\}$) and $d_{10,j}$ ($j \in J - \{2, 3, 4, 5, 8\}$)to zeros. For example, since $7 \in J_1^2$ and $L_{17} = 0.7454 > 0.39961 = \overline{X}$, then $\tilde{d}_{17} = 0$. Simplified matrix $\tilde{D}$ is obtained as follows:

$$\tilde{D} = \begin{bmatrix} 0 & 0 & 0.9730 & 0 & 0 & 0 & 0 & 0.9934 & 0 & 0.0714 \\ 0 & 0.1679 & 0 & 0 & 0.8314 & 0.9198 & 0.1366 & 0.0855 & 0 & 0 \\ 0.9999 & 0.9787 & 0 & 0.8594 & 0 & 0.9631 & 0 & 0 & 0 & 0.0967 \\ 0.0377 & 0 & 0 & 0.8055 & 0.0605 & 0 & 0.9407 & 0 & 0 & 0.8181 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02922 & 0.9568 & 0.8175 \\ 0 & 0 & 0 & 0.8789 & 0 & 0.2316 & 0 & 0 & .03050 & 0 \\ 0 & 0.9677 & 0.0835 & 0 & 0.1829 & 0 & 0 & 0 & 0 & 0.1499 \\ 0.0987 & 0.0596 & 0.1332 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8248 \\ 0 & 0 & 0.9672 & 0.8865 & 0 & 0 & & 0 & 0 & \\ 0 & 0.0424 & 0.1734 & 0.0287 & 0.9189 & 0 & 0 & 0.2373 & 0 & 0 \end{bmatrix}$$

Additionally, $\tilde{J}_1^2 = \{3, 8\}$, $\tilde{J}_2^2 = \{5, 6\}$, $\tilde{J}_3^2 = \{1, 2, 4, 6\}$, $\tilde{J}_4^2 = \{4, 7, 10\}$, $\tilde{J}_5^2 = \{9, 10\}$, $\tilde{J}_6^2 = \{4\}$, $\tilde{J}_7^2 = \{2\}$, $\tilde{J}_8^2 = \{10\}$, $\tilde{J}_9^2 = \{3, 4\}$ and $\tilde{J}_{10}^2 = \{5\}$. Based on these results and Lemma 7, we have $|E_{\tilde{D}}| = |\acute{E}_D| = 192$. Therefore, the simplification processes reduced the number of the minimal candidate solutions from 960180480 to 192, by removing 960180288 infeasible points $\underline{X}(e)$. Consequently, the feasible region has 192 minimal candidate solutions, which are feasible. In other words, for each $e \in E_{\tilde{D}}$, we have $\underline{X}(e) \in S_{T_{SS}^2}(A, D, b^1, b^2)$. However, each feasible solution $\underline{X}(e)$ ($e \in E_{\tilde{D}}$) may not be a minimal solution for the problem. For example, by selecting $\acute{e} = [3, 6, 1, 7, 9, 4, 2, 10, 3, 5]$, the corresponding solution is obtained as

$$\underline{X}(\acute{e}) = \begin{bmatrix} 0.2311 & 0.4289 & 0.2713 & 0.6266 & 0.4711 & 0.4432 & 0.3819 & 0 & 0.3075 & 0.5816 \end{bmatrix}.$$

Although $\underline{X}(\acute{e})$ is feasible (because of the inequality $\underline{X}(\acute{e}) \leq \overline{X}$) but it is not actually a minimal solution. To see this, let $e'' = [3, 6, 2, 4, 9, 4, 2, 10, 4, 5]$. Then,

$$\underline{X}(e'') = \begin{bmatrix} 0 & 0.4289 & 0.2662 & 0.6266 & 0.4711 & 0.4432 & 0 & 0 & 0.3075 & 0.5816 \end{bmatrix}.$$

Obviously, $\underline{X}(e'') \leq \underline{X}(e')$ which shows that $\underline{X}(e')$ is not a minimal solution.
Now, we obtainthe modified matrix L^* according to Definition 10:

$$L^* = \begin{bmatrix} \infty & \infty & 0.2662 & \infty & \infty & \infty & \infty & 0.1754 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.5927 & 0.4432 & \infty & \infty & \infty & \infty \\ 0.2311 & 0.3088 & \infty & 0.5609 & \infty & 0.3545 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0.6180 & \infty & \infty & 0.3819 & \infty & \infty & 0.6012 \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0.3075 & 0.5846 \\ \infty & \infty & \infty & 0.6266 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0.4289 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0.5816 \\ \infty & \infty & 0.2713 & 0.4724 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.4711 & \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

As is shown in matrix L^*, for each $i \in I_2$ there exists at least some $j \in J_i^2$ such that $L^* \neq +\infty$. Thus, by Theorem 4 we have $S_{T_{SS}^2}(A, D, b^1, b^2) \neq \varnothing$.
Finally, vector $\overline{X}$ is optimal solution of sub-problem (5). For this solution, $Z_2 = -7.4603\overline{X}_3 - 4.4300\overline{X}_7 = -80232$. Also, $Z = c^T\overline{X} = 15.1638$. In order to find the optimal solution $\underline{X}(e^*)$ of sub-problems (4), we firstly compute all minimal solutions by making pairwise comparisons between all solutions $\underline{X}(e)$ ($\forall e \in E_{\tilde{D}}$), and then we find $\underline{X}(e^*)$ among the resulted minimal solutions. Actually, the feasible region has eight minimal solutions as follows:

$$\begin{aligned} e_1 =& [3, 6, 2, 4, 10, 4, 2, 10, 4, 5] \\ \underline{X}(e_1) =& \begin{bmatrix} 0 & 0.4289 & 0.2662 & 0.6266 & 0.4711 & 0.4432 & 0 & 0 & 0 & 0.5846 \end{bmatrix} \\ e_2 =& [8, 6, 2, 4, 10, 4, 2, 10, 4, 5] \\ \underline{X}(e_2) =& \begin{bmatrix} 0 & 0.4289 & 0 & 0.6266 & 0.4711 & 0.4432 & 0 & 0.1754 & 0 & 0.5846 \end{bmatrix} \\ e_3 =& [3, 5, 2, 4, 10, 4, 2, 10, 4, 5] \\ \underline{X}(e_3) =& \begin{bmatrix} 0 & 0.4289 & 0.2662 & 0.6266 & 0.5927 & 0 & 0 & 0 & 0 & 0.5846 \end{bmatrix} \\ e_4 =& [8, 5, 2, 4, 10, 4, 2, 10, 4, 5] \\ \underline{X}(e_4) =& \begin{bmatrix} 0 & 0.4289 & 0 & 0.6266 & 0.5927 & 0 & 0 & 0.1754 & 0 & 0.5846 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
e_5 =& [3, 6, 2, 4, 9, 4, 2, 10, 4, 5] \\
\underline{X}(e_5) =& \begin{bmatrix} 0 & 0.4289 & 0.2662 & 0.6266 & 0.4711 & 0.4432 & 0 & 0 & 0.3075 & 0.5816 \end{bmatrix} \\
e_6 =& [8, 6, 2, 4, 9, 4, 2, 10, 4, 5] \\
\underline{X}(e_6) =& \begin{bmatrix} 0 & 0.4289 & 0 & 0.6266 & 0.4711 & 0.4432 & 0 & 0.1754 & 0.3075 & 0.5816 \end{bmatrix} \\
e_7 =& [3, 5, 2, 4, 9, 4, 2, 10, 4, 5] \\
\underline{X}(e_7) =& \begin{bmatrix} 0 & 0.4289 & 0.2662 & 0.6266 & 0.5927 & 0 & 0 & 0 & 0.3075 & 0.5816 \end{bmatrix} \\
e_8 =& [8, 5, 2, 4, 9, 4, 2, 10, 4, 5] \\
\underline{X}(e_8) =& \begin{bmatrix} 0 & 0.4289 & 0 & 0.6266 & 0.5927 & 0 & 0 & 0.1754 & 0.3075 & 0.5816 \end{bmatrix}
\end{aligned}$$

By comparison of the values of the objective function for the minimal solutions, $\underline{X}(e_1)$ is optimal in (4) (i.e., $e^* = e_1$). For this solution,

$$\begin{aligned}
Z_1 =& \sum_{j=1}^{n} c_j^+ \underline{X}(e_1)_j \\
=& 6.2945\underline{X}(e_1)_1 + 8.1158\underline{X}(e_1)_2 + 8.2675\underline{X}(e_1)_4 + 2.6472\underline{X}(e_1)_5 0.9376\underline{X}(e_1)_8 + 9.1501\underline{X}(e_1)_9 \\
&+ 9.2978\underline{X}(e_1)_{10} = 15.3438
\end{aligned}$$

Also, $Z = c^T \underline{X}(e_1) = 9.7901$. Thus, from Corollary 9,

$$x^* = \begin{bmatrix} 0 & 0.4289 & 0.3213 & 0.6266 & 0.4711 & 0.4791 & 0.3995 & 0 & 0 & 0.5846 \end{bmatrix}$$

and then $Z^* = c^T x^* = 7.3206$.

Conclusion

In this paper, an algorithm was proposed for finding the optimal solution of linear problems subjected to two fuzzy relational inequalities with Schweizer-Sklar family of t-norms. The feasible solutions set of the problem is completely resolved and a necessary and sufficient condition and three necessary conditions were presented to determine the feasibility of the problem. Moreover, depending on the max-Schweizer-Sklar composition, two simplification operations were proposed to accelerate the solution of the problem. Additionally, a method was introduced for generating feasible random max-Schweizer-Sklar inequalities. This method was used to generate a test problem for our algorithm. The resulted test problem was then solved by the proposed algorithm. As future works, we aim at testing our algorithm in other type of linear optimization problems whose constraints are defined as FRI with other well-known t-norms.

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