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k-Total prime cordial labeling of graphs

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ABSTRACT

In this paper we introduce a new graph labeling method called k-Total prime cordial. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $gcd(f(u), f(v))$. f is called k-Total prime cordial labeling of G if $|t_f(i) - t_f(j)| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ where $t_f(x)$ denotes the total number of vertices and the edges labeled with x. We investigate k-total prime cordial labeling of some graphs and study the 4-total prime cordial labeling of path, cycle, complete graph etc.

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1 Introduction

Graphs in this paper are finite, simple and undirected. The join of two graphs G_1 and G_2 is the graph G_1+G_2 with $V(G_1+G_2) = V(G_1) \cup V(G_2)$, $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup \{uv :$

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$u \in V(G_1), v \in V(G_2)\}$ Cahit [1] notion of cordial labeling of graphs and A-cordial labeling was introduced in [4]. Ponraj [5] was introduced the concept of k-total product cordial graph and studied certain graphs for this labeling. Motivated by this, we introduced k-total prime cordial labeling of graphs. Also we investigate some general results on k-total prime cordial graphs and investigate 4-total prime cordial labeling of some standard graphs like path, cycle, star, bistar, complete graph, $K_2 + mK_1$.

2 k-Total prime cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $\gcd(f(u), f(v))$. f is called k-Total prime cordial labeling of G if $|t_f(i) - t_f(j)| \leq 1$, $i, j \in \{1, 2, \ldots, k\}$ where $t_f(x)$ denotes the total number of vertices and the edges labelled with x.
Remark. 2- total prime cordial graph is 2-total product cordial graph.
Theorem 2.1. Every graph is a subgraph of a connected k-total prime cordial graph.

Proof. Let G be a given (p, q) graph. Take k copies of the graph G and let G_i denote the i^{th} copy of k_G. Let $u_1^i, u_2^i, \ldots, u_p^i$ be the vertices of the i^{th} copy. Let $m = \begin{cases} \frac{k-1}{2}, & \text{if} \quad k \text{ is odd} \\ \frac{k-2}{2}, & \text{if} \quad k \text{ is even} \end{cases}$. Take $(k-1)$ copies of the star$K_{1,m}$. Let v_i' be the center of the i^{th} copy of the star. We construct the graph $\widehat{G}$ from k_G by joining the vertices u_1^i and u_1^{i+1} $(1 \leq i \leq k-1)$ and pasting the central vertex v_1^i with u_1^i $(2 \leq i \leq k-1)$. Clearly G is a subgraph of $\widehat{G}$. We now give a k-total prime cordial labeling of $\widehat{G}$. Assign the label i to all the vertices of G_i $(1 \leq i \leq k)$ and assign the label i to all the pendant vertices whose support received the label i. It is easy to verify that, $t_f(1) = p + \binom{p}{2} + k - 1$ and for $i = 2, 3, \cdots, k$. $t_f(i) = \begin{cases} p + \binom{p}{2} + k - 1 & \text{if} \quad k \text{ is odd} \\ p + \binom{p}{2} + k - 2, & \text{if} \quad k \text{ is even} \end{cases}$. Hence $\widehat{G}$ is a k-total prime cordial labeling. □

Theorem 2.2. If $m \equiv 0 \pmod{k}$, then mG is k-total prime cordial.

Proof. Let $m = kt$, $t \in \mathbb{N}$ and G be a (p, q) graph. Assign the label 1 to all the vertices of first $'t'$ copies of G. Next assign the label to all the vertices of $t+1, t+2, \ldots, 2t^{th}$ copies of G. We now assign 3 to all the vertices of $(2t+1)^{th}, \ldots, (3t)^{th}$ copies of G. Proceeding like this assign the label to the consecutive copies. That is in this process, the vertices of $((i-1)t+1)^{th}, ((i-1)t+2)^{th}, \ldots, (it)^{th}$ copies receive the label 3. It is easy to verify that $t_f(1) = t_f(2) = \cdots = t_f(k) = t$. □

Theorem 2.3. If G is a (p, q) graph, then $mG \cup (k-r)K_{1,p+q}$ is k-total prime cordial, where $m = kt + r$, $0 \leq r < k$.

Proof. Assign the label to the vertices of $(kt)G$ as in Theorem 2.2. We now move to the components $(k-r)K_{1,p+q}$. Consider the first star assign the label $r+1$ to the central vertex of the star and all the pendent vertices of the star. Next assign $r+2$ to all the vertices of the second star. Continue in this pattern until we reach the last star. Note that in this process, the vertices of the i^{th} star receive the label $r+i$. Obviously $t_f(1) = t_f(2) = \cdots = t_f(r) = p+q$ and $t_f(r+1) = t_f(r+2) = \cdots = t_f(k) = p+q+1$. □

Theorem 2.4. Let G_1 and G_2 be (p_1, q_1) and (p_2, q_2)-graph respectively with $(p_1+q_1) \equiv 0 \pmod k$ and $(p_2+q_2) \equiv 0 \pmod k$. If G_1 and G_2 are k-total prime cordial graph, then the graph $G_1 \sim G_2$ obtained from G_1 and G_2 by connecting an edge is also a k-total prime cordial.

Proof. Let f and g respectively be the k-total prime cordial labeling of G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_1)$ and $G_1 \sim G_2$ is obtained from G_1 and G_2 by connects an edge uv. Define $h : V(G_1 \sim G_2) \to \{1, 2, \dots, k\}$. $h(u) = \begin{cases} f(u), & \text{if} \quad u \in V(G_1) \\ g(u), & \text{if} \quad v \in V(G_1) \end{cases}$.
Case 1. $f(u) = f(v) = r$, $r \in \{1, 2, \dots, k\}$.
In this case $t_h(1) = t_h(2) = \cdots = t_h(r-1) = t_h(r+1) = \ldots = t_h(k) = \frac{p_1+p_2+q_1+q_2}{k}$ and $t_h(r) = \frac{p_1+p_2+q_1+q_2}{k} + 1$.
Case 2. $f(u) = s$, $f(v) = r$.
Subcase 1. $\gcd(s, r) = 1$.
Then $t_h(1) = \frac{p_1+p_2+q_1+q_2}{k} + 1$ and $t_h(2) = t_h(3) = \ldots = t_h(k) = \frac{p_1+p_2+q_1+q_2}{k}$.
Subcase 2. $\gcd(s, r) = m$, $m > 1$.
In this case, $t_h(1) = t_h(2) = \ldots = t_h(m-1) = t_h(m+1) = \ldots = t_h(k) = \frac{p_1+p_2+q_1+q_2}{k}$ and $t_h(m) = \frac{p_1+p_2+q_1+q_2}{k} + 1$. Hence h is k-total prime cordial labeling of $G_1 \sim G_2$. □

We now investigate the 4-total prime cordial labeling of some standard graphs.
Theorem 2.5. Let G be a (p, q)-k-total prime cordial graph with $(p+q) \equiv 0 \pmod k$. Then $G - e$ is also a k-total prime cordial graph.

Proof. Let $e = uv$ and f be a k-total prime cordial labeling of G.
Case 1. $f(u) = f(v) = r$.
Then $t_h(1) = t_h(2) = \cdots = t_h(r-1) = t_h(r+1) = \ldots = t_h(k) = \frac{p+q}{k}$ and $t_h(r) = \frac{p+q}{k} - 1$.
Case 2. $f(u) = s$, $f(v) = r$.
Subcase 1. $\gcd(s, r) = 1$.
Then $t_h(1) = \frac{p+q}{k} - 1$ and $t_h(2) = t_h(3) = \ldots = t_h(k) = \frac{p+q}{k}$.
Subcase 2. $\gcd(s, r) = m$, $m > 1$.
In this case, $t_h(1) = t_h(2) = \cdots = t_h(m-1) = t_h(m+1) = \ldots = t_h(k) = \frac{p+q}{k}$ and $t_h(m) = \frac{p+q}{k} - 1$. Hence h is k-total prime cordial labeling of $G - e$. □

Theorem 2.6. If G is a (p, q)-k-total prime cordial graph with $(p+q) \equiv 0 \pmod k$, then $G \cup K_2$ is also k-total prime cordial.

Proof. Let $V(K_2) = x, y$. Let f be a k-total prime cordial of G. Define $h : V(G) \cup K_2 \to \{1, 2, \dots, k\}$ by $h(u) = f(u)$, $u \in V_j$. $h(x) = r$, $h(y) = r+1$ which implies $t_h(1) = \frac{p+q}{k} + 1$. $t_h(2) = t_h(3) = \ldots = t_h(r-1) = t_h(r+2) = \cdots = t_h(k) = \frac{p+q}{k}$. Clearly h is a k-total prime cordial of $G \cup K_2$. □

Theorem 2.7. All paths are 4-total prime cordial.

Proof. Let P_n be the path $u_1u_2\dots u_n$.
Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4t$, $t > 1$. Assign the label 4 to all the path vertices $u_1, u_2, \dots, u_t$ and assign the label 2 to the path vertices $u_{t+1}, u_{t+2}, \dots, u_{2t}$. Next assign the label 3 to the path vertices $u_{2t+1}, u_{2t+2}, \dots, u_{3t}$ and u_n. Finally assign the label 1 to the remaining non-labeled vertices of the path. Here $t_f(1) = t_f(2) = t_f(3) = 2t$ and $t_f(4) = 2t - 1$.
Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4t+1$, $t > 1$. In this case assign the label to the path vertices u_i $(1 \le i \le n-1)$ as in case 1 and assign 4 to the vertex u_n. Here $t_f(1) = t_f(2) = t_f(3) = 2t$ and $t_f(4) = 2t-1$.
Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4t+2$, $t > 1$. In this case assign the label 4 to the path vertices $u_1, u_2, \dots, u_t, u_{t+1}$ and 2 to the vertices $u_{t+2}, u_{t+3}, \cdots, u_{2t+1}$. Next assign the label 3 to the path vertices $u_{2t+2}, u_{2t+3}, \dots, u_{3t+1}$ and assign the labels 3 and 2 to the vertices u_{n-1} and u_n respectively. Finally assign the label 1 to the remaining non-labelled vertices. It is easy to verify that $t_f(1) = t_f(2) = t_f(4) = 2t + 1$ and $t_f(3) = 2t$.
Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4t + 3$, $t > 1$. In this case assign the label to the path vertices u_i $(1 \le i \le n - 1)$ as in case 3, assign the label to the vertices $u_1, u_2, \cdots, u_{n-1}$. Next assign the label 3 to the pendent vertex u_n. Here $t_f(1) = 2t + 2$ and $t_f(2) = t_f(3) = 2t + 1$.
Case 5. $n = 4, 5, 6, 7$.
A 4-total prime cordial labeling of P_n, $n = 4, 5, 6, 7$ is given in Table 1.

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7
4	4	2	3	4			
5	4	3	2	4	3		
6	4	3	4	2	4	3	
7	4	1	4	2	4	3	3

Table 1:

□

Theorem 2.8. The cycle C_n is 4-total prime cordial iff $n \notin \{4, 6, 8\}$.

Proof. Let C_n be the path $u_1u_2\dots u_nu_1$.
Case 1. $n \equiv 0 \pmod 4$.
Assign the label to the vertices $u_1, u_2, \dots, u_{n-2}, u_n$ as in case 1 of Theorem 2.7. Next assign the label 4 to the vertex u_{n-1}.
Case 2. $n \equiv 1 \pmod 4$.
Clearly the vertex labeling in case 2 of Theorem 2.7 is also a 4-total prime cordial labeling of this case.
Case 3. $n \equiv 2 \pmod 4$.
As in case 3 of Theorem 2.7 the label to the vertices u_i $(1 \le i \le n - 4)$ and assign the labels 3, 1, 2, 3 to the vertices $u_{n-3}, u_{n-2}, u_{n-1}, u_n$ respectively.

Case 4. $n \equiv 3 \pmod 4$.
Assign the label to the vertices $u_1, u_2, \ldots, u_{n-3}, u_{n-2}$ as in case 3 of theorem 2.7. Next assign the labels 3 and 2 respectively to the vertices u_{n-1} and u_n. A 4-total prime cordial labeling of C_3, C_5 and C_7 a given below.
Case 5. $n = 4$.
Suppose f is a 4-total prime cordial labeling of C_3. Then $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 2$. To get the edge label 3 either 3 is label to adjacent vertices or non adjacent vertices. In the later case, $t_f(3) \geq 3$ and former case $t_f(1) \geq 4$, a contradiction.
Case 6. $n = 6$.
If there exist a 4-total prime cordial labeling of C_6, then $t_f(1) = t_f(2) = t_f(3) = t_f(4) = 3$. In this case 4 is labeled consecutive then either $t_f(3) < 3$ or $t_f(1) < 3$. In the case 4 is labeled alternatively, then $t_f(3) < 3$, a contradiction.
Case 7. $n = 8$.
Similar to case 6, we get a contradiction.
Case 8. $n = 3, 5, 7$.
A 4-total prime cordial labeling of C_n, $n = 3, 5, 7$ is given in Table 2.

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7
3	4	2	3				
5	4	2	4	3	3		
7	4	4	2	2	3	3	1

Table 2:

□

Theorem 2.9. If $n \equiv 0, 7 \pmod 8$, then the complete graph K_n is not 4-total prime cordial.

Proof. Suppose f is a 4-total prime cordial of K_n. Then $t_f(1) = t_f(2) = t_f(3) = t_f(4) = \frac{n(n-1)}{8}$. But $t_f(4) = m_1 + \binom{m_1}{2}$ and $t_f(3) = m_2 + \binom{m_2}{2}$, for some $m_1, m_2 \in \mathbb{N}$. This forces, $m_1 + \binom{m_1}{2} = m_2 + \binom{m_2}{2}$. That is, $m_1^2 + m_1 = m_2^2 + m_2$. This implies, $m_1 + m_2 = -1$, a contradiction. Therefore, K_n is not 4-total prime cordial if $n \equiv 0, 7 \pmod 8$. □

Theorem 2.10. The star $K_{1,n}$ is 4-total prime cordial for all n.

Proof. Let u be the center of the star and $u_1, u_2, \ldots, u_n$ be the pendant vertices adjacent to u. We divide the proof into four cases.
Case 1. $n = 4t$, $t \in \mathbb{N}$.
Assign the label 4 to the vertex u. Next consider the pendent vertices assign the label 4 to the vertices $u_1, u_2, \ldots, u_t$ and 2 to the vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$. Finally assign the label 3 to the remaining vertices $u_{3t+1}, u_{3t+2}, \ldots, u_{4t}$. Clearly $t_f(1) = t_f(2) = t_f(3) = 2t$ and $t_f(4) = 2t + 1$.
Case 2. $n = 4t + 1$, $t \in \mathbb{N}$.

As in case 1 assign the label to the vertices u, u_i $(1 \leq i \leq n-1)$ as in case 1 and assign 3 to the vertex u_n. It is easy to verify that $t_f(1) = t_f(3) = t_f(4) = 2t+1$ and $t_f(2) = 2t$.
Case 3. $n = 4t+2$, $t \in \mathbb{N}$.
In this case assign the label to the vertices u, u_i $(1 \leq i \leq n-1)$ as in case 2. Next assign the label 2 to the vertex u_n. Clearly $t_f(1) = t_f(3) = t_f(4) = 2t+1$ and $t_f(2) = 2t+2$.
Case 4. $n = 4t+3$, $t \in \mathbb{N}$.
Assign the label to the vertices u, u_i $(1 \leq i \leq n-1)$ as in case 3. Finally assign the label 3 to the vertex u_n. Obviously $t_f(1) = t_f(3) = t_f(2) = 2t+2$ and $t_f(4) = 2t+1$. □

Theorem 2.11. The bistar $B_{n,n}$ is 4-total prime cordial for all n.

Proof. Let u, v be the center vertices of the bistar $B_{n,n}$. Let u_i $(1 \leq i \leq n)$ be the pendent vertices adjacent to u and v_i $1 \leq i \leq n$ be the pendent vertices adjacent to v.
Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4t$, $t \in \mathbb{N}$. Assign the label 4, 3 to the vertices u and v respectively. We now move to the pendent vertices $u_1, u_2, \ldots, u_n$. Assign the label 4 to the vertices $u_1, u_2, \ldots, u_{2t}$ and 2 to the vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{4t}$. Now we consider the other side pendent vertices $v_1, v_2, \ldots, v_n$. Assign the label 3 to the vertices $v_1, v_2, \ldots, v_{2t}$ and 1 to the vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{4t}$. Clearly in the case $t_f(1) = t_f(3) = t_f(4) = 4t+1$ and $t_f(2) = 4t$.
Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4t+1$, $t \in \mathbb{N}$. In this case assign the label to the vertices u, v, u_i $(1 \leq i \leq n)$, v_i $(1 \leq i \leq n)$ as in case 1. Next assign the label 2 and 4 to the vertices u_n and v_n respectively. Obviously $t_f(1) = t_f(2) = t_f(4) = 4t+2$ and $t_f(3) = 4t$.
Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4t+2$, $t \in \mathbb{N}$. As in case 2 assign the label to the vertices u, v, u_i $(1 \leq i \leq n)$ and v_i $(1 \leq i \leq n)$. Next assign the label 1, 3, 2, 3 respectively to the vertices u_n, v_n, v_{n-2}, v_{n-1}. Clearly $t_f(1) = t_f(2) = t_f(3) = 4t+3$ and $t_f(4) = 4t+2$.
Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4t+3$, $t \in \mathbb{N}$. Assign the label to the vertices u, v, u_i $(1 \leq i \leq n)$, v_i $(1 \leq i \leq n)$ as in case 3. Finally assign the label 4, 2 respectively to the vertices u_n and v_n. It is easy to verify that $t_f(1) = t_f(2) = t_f(4) = 4t+4$ and $t_f(3) = 4t+3$. □

Theorem 2.12. The join of K_2 with mK_1, $K_2 + mK_1$ is 4-total prime cordial iff $m \leq 1$.

Proof. Let u, v be the vertex of degree $2m+1$ and $u_1, u_2, \ldots, u_m$ be the vertex of degree 2. Suppose f is a 4-total prime cordial labeling of $K_2 + mK_1$. Clearly $|V(K_2 + mK_1)| + |E(K_2 + mK_1)| = 3m+3$.
Case 1. $f(u) = f(v) = 4$.
They at least $[\frac{3m+3}{4}]$ vertices of $u_1, u_2, \cdots, u_m$ receive the label 3. But $t_f(2) < [\frac{3m+3}{4}]$, a contradiction.
Case 2. $f(u) = 4$, $f(v) = 2$.
In this case, $t_f(1) \geq [\frac{3m+3}{2}]$, a contradiction.
Case 3. $f(u) = 4$, $f(v) = 1$.
In this case, $t_f(1) \geq m + [\frac{3m+3}{4}]$, a contradiction.

Case 4. $f(u) = 4$, $f(v) = 3$.
In this case either $t_f(2) < [\frac{3m+3}{4}]$ or $t_f(4) < [\frac{3m+3}{4}]$, a contradiction.
Case 5. $f(u) = f(v) = 3$.
Then at least $[\frac{3m+3}{4}]$ vertices of $u_1, u_2, \ldots, u_n$ receive the label 4. This implies $t_f(2) < [\frac{3m+3}{4}]$, a contradiction.
Case 6. $f(u) = 3$, $f(v) = 2$.
In this case, $t_f(1) > [\frac{3m+3}{4}]$, a contradiction.
Case 7. $f(u) = 3$, $f(v) = 1$.
Clearly at least $[\frac{3m+3}{4}]$ vertices other than u and v receive the label 4. Therefore, $t_f(1) > [\frac{3m+3}{4}]$, a contradiction.
Case 8. $f(u) = f(v) = 2$.
Clearly, $t_f(2) > [\frac{3m+3}{4}]$, a contradiction.
Case 9. $f(u) = 2$, $f(v) = 1$.
Then at least $[\frac{3m+3}{4}]$ vertices of $u_1, u_2, \ldots, u_n$ receive the label 3. Then $t_f(1) \geq m + [\frac{3m+3}{4}]$, a contradiction.
Case 10. $f(u) = f(v) = 1$.
Clearly, $t_f(1) \geq 2m + 1$, a contradiction.
Case 11. $m = 1$.
$K_2 + K_1 \cong C_3$, follows from the Theorem 2.8. □

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