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Vulnerability in Networks - A Survey

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ABSTRACT

The analysis of vulnerability in networks generally involves some questions about how the underlying graph is connected. One is naturally interested in studying the types of disruption in the network that maybe caused by failures of certain links or nodes. In terms of a graph, the concept of connectedness is used in different forms to study many of the measures of vulnerability. When certain vertices or edges of a connected graph are deleted, one wants to know whether the remaining graph is still connected, and if so, what its vertex - or edge - connectivity is. If on the other hand, the graph is disconnected, the determination of the number of its components or their orders is useful. Our purpose here is to describe and analyses the current status of the vulnerability measures, identify its more interesting variants, and suggest a most suitable measure of vulnerability.

Keyword: integrity, connectivity, binding number, toughness and tenacity.

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PRELIMINARIES

Many graph theoretical parameters have been used to describe the vulnerability of communication networks, including connectivity, toughness, binding number, integrity and

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tenacity. Before we start to describe and analyse the current status of the vulnerability measures, we will give some basic definitions and notations.
We shall for the most part, use the terminlogy and notation of Bondy and Murty [3]; so a graph G has vertex set V(G), edge set E(G), $\nu(G) = n$ vertices, $\epsilon(G) = m$ edges. We use $\alpha(G)$ to denote the independence number of G. Let A be a subset of V(G). The neighborhood of A, N(A), consists of all vertices of G adjacent to at least one vertex of A. We define G-A to be the graph induced by the vertices of V-A. Also, for any graph G, $\tau(G)$ is the number of vertices in a largest component of G and $\omega(G)$ is the number of components of G. A cutset of a connected graph G is a collection of vertices whose removal results in a disconnected graph.

VULNERABILITY PARAMETERS

1. CONECTIVITY:
The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. There is a rich body of theorems concerning connectivity. Many of these are variations of a classical result of a Meneger, which involves the number of disjoint paths joining a given pair of vertices in a graph.

2. BINDING NUMBER:
In 1973 D. R. Woodall [47] introduced the concept of the binding number of a graph and studied some properties of binding number. The binding number of a graph G, denoted by bind(G), is defined to be $min\{\frac{|N(A)|}{|A|}; A \in F\}$, where $F = \{A \mid \phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)\}$.
Proposition 2.1: $bind(K_n) = n - 1 \qquad (n \geq 1)$
Proposition 2.2: $bind(K_{a,b}) = min(\frac{a}{b}, \frac{b}{a}) \qquad (a \geq 1, b \geq 1)$
Proposition 2.3: If $n \geq 3$, then $bind(C_n) = \begin{cases} 1 & \text{if } n \text{ } is \text{ } even, \\ \frac{n-1}{n-2} & \text{if } n \text{ } is \text{ } odd \end{cases}$
Proposition 2.4: (Fundemental Lemma). bind(G) is the largest number c such that $\mid N(A) \mid \geq \mid G \mid \frac{c-1}{c} + \frac{|A|}{c}$ for every $A \subseteq V(G), A \neq \phi$.
Proposition 2.5: If $\mid G \mid = n$ (≥ 1), and the connectivity of G is k (≥ 0) (so that G is k-cnnected but not (k+1)-connected), then $bind(G) \leq \frac{n+k}{n-k}$.

On the basis of these results in [47], Woodall gave a sufficient condition for the existence of a Hamiltonian circuit.

Theorem 2.1: Let G be a graph on n vertices such that $bind(G) \geq c$
a) If $c \geq \frac{3}{2}$, then G has a Hmiltonian circuit.
b) If $1 < c \leq \frac{3}{2}$, then G contains a circuit of length at least $\frac{3(n-1)(c-1)}{c}$, unless G consists either of two copies of K_4 with exactly one edge joining them, in which case the formula gives $4\frac{1}{2}$ and $4\frac{1}{5}$ respectively, and the longest circuit has length four.

Theorem 2.1 suggest the following conjectures:

Conjecture 2.1: If G is a graph on n vertices such that $bind(G) \geq c$, $(1 \leq c \leq \frac{3}{2})$, where n and c are sufficiently large (the precise conditions to be determined), and if G contains a circuit of length $m < \frac{3(n-1)(c-1)}{c}$, then G contains a circuit of length m+1.
Conjecture 2.2. If $bind(G) \geq \frac{3}{2}$, then G contains a triangle.
Conjecture 2.3: If $bind(G) \geq \frac{3}{2}$, then G is pancyclic (i.e., contains a circuit of every length m, $3 \leq m \leq \mid G \mid$).

The figure $\frac{3}{2}$ in Conjectures 2.2 and 2.3 is the least possible, in view of graphs of the following form: the vertices are spaced regualrly round the cicumference of a circle, and each vertex v is joined to all the vertices strictly within the arc of length $\frac{2}{3}\pi$ whose mid-point is diametrically opposite v. The conclusion of Conjecture 2 certainly follows if $bind(G) \geq \frac{1}{2}(1+\sqrt{5})$.

3. TOUGHNESS:
In 1972, Chvátal [6] introduced the concept of the toughness of a graph. It measures in a simple way how tightly various pieces of a graph hold together; therefore he called it toughness. Let G be a graph and t a real number such that the implication $\omega(G-A) > 1 \Rightarrow \mid A \mid \geq t \cdot \omega(G-A)$ holds for each set A of vertices of G. Then G will be said to be t-tough.

Proposition 3.1: $G \subset H \Rightarrow t(G) \leq t(H)$.

Thus toughness is a nondecreasing invaiant whose values range from zero to infinity. A graph G is disconnected if and only if $t(G) = 0$; G is complete if and only if $t(G) = +\infty$.

Proposition 3.2: $t \geq \frac{\kappa}{\alpha}$.
Proposition 3.3: If G is not complete, then $t \leq \frac{1}{2}\kappa$.
Proposition 3.4: If G is not complete, then $t \leq \frac{n-\alpha}{\alpha}$.
Proposition 3.5: $m \leq n \Rightarrow t(K_{m,n}) = \frac{m}{n}$.
Theorem 3.1: $t(K_m \times K_n) = \frac{1}{2}(m+n) - 1, (m, n \geq 2)$.

Proposition 3.2, 3.3 indicate a relationship between toughness and connectivity. Another indication of this relationship is given by:

Theorem 3.2: $t(G^2) \geq \kappa(G)$.
Corollary 3.1: If m is a positive integer and $n = 2^m$, then $t(G^n) \geq \frac{1}{2}n\kappa(G)$.
Proposition 3.6: Every hamiltonian graph is 1-tough.

Unfortunately, the converse of Proposition 3.6 holds for graphs with at most six vertices only. Eventhough its converse in general does not hold, one may wonder what additional conditions placed upon a 1-tough graph G would imply the existence of hamiltonian cycle in G. As in next conjecture, such conditions may have the flavour of Ramsey's theorem.

Conjecture 3.1: Every t-tough with $t > \frac{3}{2}$ is hamiltonian.

The toughness has been studied extensively; see for example [11,12,13,20,21]. Woodall in[47] proved the following proposition:

Proposition 3.7. $bind(G) \leq t(G) + 1$.

4. INTEGRITY:

The integrity of a graph G was introduced by Barefoot, Entringer and Swart in [2] as a useful measure of the vulnerability of G. The integrity of a graph G is given by $I(G) = min(\mid S \mid + \tau(G-S)$, where the minimum is taken over all vertex cutsets A of G, $\tau(G-S)$ is the maximum number of vertices in a component of G-S. Integrity has been studied in numerous papers including [1,8].

5.TENACITY:

The tenacity is a new invariant for graphs. It is another vulnerability measure, incorporating ideas of both toughness and integrity. The tenacity of a graph G, T(G) is defined by $T(G) = min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$, where the minimum is taken over all vertex cutset A of G, G-A is the graph induced by the vertices of V-A, $\tau(G-A)$ is the number of vertices in the largest component of the graph induced by G-A and $\omega(G-A)$ is the number of components of G-A. A connected graph G is called T-tenacious if $\mid A \mid + \tau(G-A) \geq T\omega(G-A)$ holds for any subset A of vertices of G with $\omega(G-A) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p $(p \geq 1)$. A set $A \subset V(G)$ is said to be a T-set of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$.

The tenacity has been studied extensively; see for example [20-33].

Without attempting to obtain the best possible result, we can prove quite easily the following relation between T(G) and t(G). This result gives us a number of corollaries.

Theorem 5.1: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.

Proof: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} + \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.

Proposition 5.1: If G is Hamiltonian-connected and $n \geq 3$, then $T(G) > 1$.

Proof: By Proposition 3.6, every Hamiltonian graph is 1-touph. Hence $t(G) \geq 1$. But if G is Hamiltonian-connected and $p \geq 3$ then G is Hamiltonian. Therefore $T(G) > 1$.

The following theorem, proved by Chvátal and Erdös [7], enable us to relate Proposition 5.1 to the connectivity and the independence number of a graph.

Theorem 5.2: (Chvátal and Erdös). If G is k-connected and $k > \alpha$, then G is Hamiltonian-connected.

Thus from Proposition 5.1 we have three possibilities for a graph G:

1) $1 < \frac{\kappa(G)}{\alpha(G)} < \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

2) $\frac{\kappa(G)+1}{\alpha(G)} \leq 1 \leq T(G)$

3) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1$

By Proposition 5.1, graphs satisfying the third inequality are not Hamiltonian-connected. By Theorem 5.2, graphs satisfying the first inequality are Hamiltonian-connected. The cycle C_p, $p \geq 6$, satisfies the second inequality but is not Hamiltonian-connected while the graph C_p^2, $p \geq 10$, satisfies the second inequality and is Hamiltonian-connected.

In [5] Chartrand, Kapoor and Lick considered some conditions necessay for a graph to be n-Hamiltonian. Let graph G be m-connected. By definition every Hamiltonian graph is 2-connected. Since the removal of any n vertices from an n-Hamiltonian graph G results in a Hamiltonian graph, it follows that G is (n+2)-connected.

Theorem 5.3: If G is n-Hamiltonian then $T(G) \geq 1 + \frac{n+1}{\alpha(G)}$.
Proof: Let A be a cutset of G. We know that $\mid A \mid \geq n+2$. Let A_n be an n-vertex subset of A. Since G is n-Hamiltonian, $G - A_n$ has a Hamiltonian cycle C. The components of C-A are disjoint paths $P_1, \cdots, P_r$. At least two vertices of A lie on C. Let v be one of these vertices. If we start at v and travel around C in a definite direction and return to v, we traverse each P_i exactly once. Let u_i be the next vertex of C encountered after having passed through P_i in the chosen direction. Then $u_1, \cdots, u_r$ are distinct vertices of A. Thus, $\mid A \mid \geq r + n$. Also the union of the P_i's includes all the vertices of G-A. Hence $\omega(G-A) \leq r$. Thus we have $\omega(G-A) \leq r \leq \mid A \mid -n$, and so $\frac{|A|+1}{\omega(G-A)} \geq 1 + \frac{n+1}{\omega(G-A)}$. Since $\omega(G-A) \leq \alpha(G)$ for any A, the theorem follows.

To relate Theorem 5.2 to the connectivity of G, we use a generalization of the following theorem of Chvátal and Erdös [7].

Theorem 5.4: (Chvátal and Erdös). If G is k-connected and $k \geq \alpha$, then G is Hamiltonian.
Theorem 5.5: (Molluzzo [34]). If G is k-connected and for any integer n, $k - n \geq \alpha$, then G is n-Hamiltonian.

For such k and n, we have the following three possiblities for a graph G:

1) $1 + \frac{n+1}{\alpha(G)} \leq \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

2) $\frac{\kappa(G)+1}{\alpha(G)} \leq 1 + \frac{n+1}{\alpha(G)} \leq T(G)$

3) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1 + \frac{n+1}{\alpha(G)}$

If G satisfies the third inequality it is not n-Hamiltonian by Theorem 5.3. If G satisfies the first inequlity then G is n-Hamiltonian by Theorem 5.5. Define the graph C_p^k for any positive k as follows: $V(C_p^k) = V(C_p) = \{0, 1, 2, \cdots, p-1\}$ and two vertices i and j are adjacent if and only if $\mid i - j \mid \leq k$. The graph C_p^{n+2}, for p sufficiently large, satisfies the second inequality and is n-Hamiltonian while the graph $G_{p,2}$, defined below, for p sufficiently large, satisfies the second inequality and is not n-Hamiltonian.

The graph $G_{p,m}$, with $1 \leq m \leq \frac{p-1}{2}$, has p vertices and vertex v which is adjacent to all vertices of the two complete subgraphs, copies of K_m and K_{p-m-1}, in other words we have $G_{p,m} \cong K_1 + (K_m \cup K_{p-m-1})$.

Now we can discuss about tenacity and its operation on graphs. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower by too much. In [9] we proved the following two theorems and corollaries:

Theorem 5.6: For any nontrivial, noncomplete graph G with n vertics and any vertex v, $T(G - v) \geq T(G) - \frac{1}{2}$.

The following theorem allow us to find the tenacity of several important classes of graphs.

Theorem 5.7: If G is a bipartite, r-regular, r-connected graph on n vertices, then $T(G) = \frac{n+2}{n}$.

This result gives several interesting corollaries.

Corollary 5.1: If G_1 is a bipartite, d-regular, d-connected graph with n_1 vertices and G_2 is a bipartite, q-regular, q-connected graph with n_2 vertices, then $T(G_1 \times G_2) = \frac{n_1 n_2 + 2}{n_1 n_2}$.
Corollary 5.2: For any integer n, $T(Q_n) = \frac{2^n+2}{2^n}$.
Corollary 5.3: For any integers n and m, $T(C_n \times C_m) = \frac{nm+2}{nm}$.
Corollary 5.4: For any even integer n, $T(C_n \times K_2) = \frac{n+1}{n}$.

We next obtain some bounds on the tenacity of products of graphs. Note that the first inequality in the following theorem is a corollary to Theorem 5.1

Theorem 5.8: If $n \geq m$, then $\frac{m^2+mn-2m+2}{2m} \leq T(K_m \times K_n) \leq \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.
Proof: By Theorem 3.1, $t(K_m \times K_n) = \frac{m+n-2}{2}$. It is easy to see that $\alpha(K_m \times K_n) = m$. Let $V(K_n) = \{1, 2, 3, \cdots, n\}$ and $V(K_m) = \{1, 2, 3, \cdots, m\}$. Then $V(K_m \times K_n) = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Also let $n = am + b$, for $0 \leq b < m$, so if $b = 0$ then

$a = \lceil \frac{n}{m} \rceil = \frac{n}{m}$ and otherwise $a+1 = \lceil \frac{n}{m} \rceil$. Now, if $b = 0$, then define the sets W_i as $W_i = \{(i, ia-a+1), \cdots, (i, ia)\}$ for $1 \leq i \leq m$, otherwise define the sets W_i as follows:

$$W_i = \begin{cases} \{(i, ia+i-a), \cdots, (i, ia+i)\} & 1 \leq i \leq b \\ \{(i, ia+b-a+1), \cdots, (i, ia+b)\}, & b+1 \leq i \leq m, \end{cases}$$

and let $W = \bigcup_{i=1}^{m} W_i$. Define $A = V(G) - W$ and so $\mid A \mid = mn - n$. It is easy to see that the W_i, $1 \leq i \leq m$, are the components of G-A and so $\tau(G-A) = \lceil \frac{n}{m} \rceil$ and $\omega(G-A) = n$. The result follows.

Corollary 5.5: For any integer n, $T(K_n \times K_n) = n - 1 + \frac{1}{n}$.

Recently proved the following:

Corollary 5.6: If $n \geq m \geq 2$ then $T(K_m \times K_n) = \frac{mn-n+\lceil \frac{n}{m} \rceil}{m}$.

CONCLUDING REMARKS:

The vulnerability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes there is a loss in it's effectiveness. Normally new nodes or links are added so that the network is reconstructed in an attempt to regain it's effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

Since such a network can be represented by a graph, G, with a vertex set, V(G), and an edge set, E(G), many graph theoretical parameters have been used to describe the vulnerability of communication networks. Most notably, the parameter called connectivity have been frequently used. The difficulty with this parameter is that they do not take into account what remains after the graph is disconnected. That is, two graphs with the same number of vertices and the same connectivity may result in entirely different forms after a minimum disconnected set of vertices is removed. One maybe totally disconnected while the other may consist of a few very stable components, and thus be much easier to reconstruct. Consequently, a number of other parameters have been introduced to cope with this difficulty. The parameters considered in this paper which were introduced in different references, in order to deal with this problem.

In [18], we compared integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.

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