Civil Engineering Infrastructures Journal, 51(1): 169 – 198, June 2018
Print ISSN: 2322-2093; Online ISSN: 2423-6691
DOI: 10.7508/ceij.2018.01.010

Analytical D'Alembert Series Solution for Multi-Layered One-Dimensional Elastic Wave Propagation with the Use of General Dirichlet Series

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Received: 08 Oct. 2017; Revised: 27 Feb. 2018; Accepted: 13 Mar. 2018

ABSTRACT: A general initial-boundary value problem of one-dimensional transient wave propagation in a multi-layered elastic medium due to arbitrary boundary or interface excitations (either prescribed tractions or displacements) is considered. Laplace transformation technique is utilised and the Laplace transform inversion is facilitated via an unconventional method, where the expansion of complex-valued functions in the Laplace domain in the form of general Dirichlet series is used. The final solutions are presented in the form of finite series involving forward and backward travelling wave functions of the d'Alembert type for a finite time interval. This elegant method of Laplace transform inversion used for the special class of problems at hand eliminates the need for finding singularities of the complex-valued functions in the Laplace domain and it does not need utilising the tedious calculations of the more conventional methods which use complex integration on the Bromwich contour and the techniques of residue calculus. Justification for the solutions is then considered. Some illustrations of the exact solutions as time-histories of stress or displacement of different points in the medium due to excitations of arbitrary form or of impulsive nature are presented to further investigate and interpret the mathematical solutions. It is shown via illustrations that the one-dimensional wave motions in multi-layered elastic media are generally of complicated forms and are affected significantly by the changes in the geometrical and mechanical properties of the layers as well as the nature of the excitation functions. The method presented here can readily be extended for three-dimensional problems. It is also particularly useful in seismology and earthquake engineering since the exact time-histories of response in a multi-layered medium due to arbitrary excitations can be obtained as finite sums.

Keywords: Analytical D'alembert Solution, General Dirichlet Series, Inverse Laplace Transform, Multi-Layered, One-Dimensional, Wave Propagation.

INTRODUCTION

One-dimensional elastic wave propagation can be used as a simple, yet sufficiently accurate, mathematical model for many Initial-Boundary Value Problems (IBVPs)

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that arise in the elastodynamic theory (Achenbach, 1975). Longitudinal wave propagation in rods and piles or transverse and longitudinal wave propagation in soil and rock are some examples with applications in mechanical engineering, civil engineering, geophysics and seismology; see for instance (Achenbach, 1975; Achenbach et al., 1968). Moreover, deep understanding of one-dimensional elastic wave propagation in periodic materials in any size is the basis for dynamics of composite materials. Furthermore, studying one-dimensional elastic wave propagation can be used for gaining a better insight into this phenomenon and developing methods; for solving such one-dimensional IBVPs can be a starting point for more complicated three-dimensional problems. Propagation of short time waves in time domain in heterogeneous media are also of significant interest in mechanics and other branches of physics.

D'Alembert's general solution of the one-dimensional wave equation (Weinberger, 1995) is arguably the most helpful solution for the above mentioned engineering problems. Beddoe (1966) considered a general problem of wave propagation in a sectionally uniform string when subjected to arbitrary initial conditions. This author gave a comparison between the two methods of solution of such problems: the Fourier's method of separation of variables (the method of eigenfunction expansion) and the method of wave reflection (based on d'Alembert's solution). The method of wave reflection makes use of reflection and transmission coefficients to derive recurrence relations for d'Alembert solutions, which are then solved by Laplace transformation methods. It was shown in this study that one could fully circumvent the difficulties of obtaining the natural frequencies (eigenvalues) of the complicated systems, often involving transcendental equations that cannot be solved analytically, by the application of wave reflection method instead of Fourier's method (Beddoe, 1966).

In addition, as it was discussed in (Kobayashi and Genest, 2014), because of the inaccuracies that stem from Gibbs phenomenon in Fourier's method, this method does not yield satisfactory results for problems of wave propagation which involve discontinuities in the solutions. The d'Alembert's method and its various extensions are, therefore, more suitable for this class of problems. The authors in this study (Kobayashi and Genest, 2014) have investigated, from a fundamentally theoretical aspect, an extension of d'Alembert's method for the wave-propagation in multi-layered domains.

Generally, the study of transient wave motions of structures or mechanical systems are of great significance in earthquake and structural engineering (Thambiratnam, 1986; Cheung et al., 1995; Yang and Yin, 2015). Yang and Yin (2015), for example, investigated the transient response of girder bridges subjected to base excitations with the use of eigenfunction expansion approach. Yang investigated a unified solution for one-dimensional longitudinal wave propagation in an elastic rod that takes into the account the contributions from non-zero initial conditions, inhomogeneous boundary conditions and external excitations altogether (Yang, 2008). The method of solution was based on Laplace transform and its inversion. With these techniques, the author found formulae, particularly suitable for transient problems of wave propagation in finite homogenous (single-layered) media, that capture the essence of forward and backward travelling wave functions which are typically found in d'Alembert's method (Yang, 2008).

The Laplace integral transform is a very powerful mathematical tool for wave propagation problems studied in the time domain, which allows one to capture the solution in the transformed domain (Cohen, 2007). The immediate problem after finding the solution in the transformed domain is to obtain the inverse Laplace transform of the solution to capture the real solution in time domain. However, there does not exist a general method for determining the analytical form of the inverse Laplace transform of any function, and because of this the numerical methods have been improved very much over the years (Durbin, 1974; Duffy, 1993; Cohen, 2007). Because of the engineering applications of transient wave propagation and the mathematical challenge of the inverse Laplace transformation, the subject is interesting for both engineers and mathematicians.

The importance of developing a firm conception of one-dimensional wave propagation in layered media causes the researchers to study the phenomena in many different points of view. As mentioned by Berlyand and Burridge, a pulse propagates normally through a stratified medium (Berlyand and Burridge, 1995). From a probabilistic aspect, it is known that the shape and travel time of the forward propagating pulse as it travels over large distances depends on the autocorrelation function of the medium properties (O'Doherty and Anstey, 1971). This phenomenon was recognized by O'Doherty and Anstey (1971) during their study of reflecting waves in a multi-layered one-dimensional elastic media and is, nowadays, known as O'Doherty and Anstey approximation.

Burridge studied the propagation of pulses in a multi-layered medium with an exact theory, where no probabilistic approach was used (Burridge, 1988). Then, Berlyand and Burridge (1995) gave a theoretical estimate of the error in making this O'Doherty and Anstey approximation. Their analyses were done by presenting a closed-form equation governing the evolution of the propagation mode of interest, where Berlyand and Burridge recognized that there is an interplay of frequency-domain and time-domain considerations.

Chiu and Erdogan (1999) have studied the one-dimensional wave propagation in a functionally graded slab, where the density and the mechanical properties of the slab vary in thickness. Bruck (2000) by studying the wave propagation in functionally graded one-dimensional materials, proposed a model for controlling the stress level in the medium. Cheshmehkani and Eskandari-Ghadi (2017) with the use of two-dimensional wave propagation in a functionally graded material proposed models for the passive control of displacement and stress level in the medium. Lin and Daraio (2016) with the use of noncontact optical technique, have experimentally and numerically generated and measured stress wave propagation in one-dimensional microscopic granular materials and have shown that the wave propagation in granular materials are highly nonlinear. Ponge and Croënne (2016) have investigated the wave propagation in a piezo-magnetic phononic crystal and proposed a model to control the propagation of waves. Among other recent contributions that investigated various aspects of wave propagation concepts, either displacement induced or force induced waves, the papers (Ardeshir-Behrestaghi et al., 2013; Eskandari-Ghadi et al., 2014; Shafiei and Khaji, 2015; Raoofian Naeeni and Eskandari-Ghadi, 2016) could be mentioned.

As mentioned, the Laplace integral transform is a very powerful tool for solving time-dependent ordinary and partial differential equations. As an example of far-reaching applications of the Laplace transforms in such initial-boundary value problems, the reader is referred to (Eskandari-Ghadi et al., 2013). Nevertheless, the main issue after finding a solution in Laplace transformed domain is the inverse transformation procedure. In many IBVPs encountered in elastodynamics, the solution in the Laplace transformed domain is a meromorphic function of the Laplace transform parameter (usually denoted by the complex variable s). By definition, a meromorphic function is a complex-valued function that is analytic in some region Ω in the complex plane (of the transform parameter in this context) except for some isolated singularities which are poles (Ahlfors, 1966). If the solution of a problem in Laplace transformed domain is a meromorphic function that has infinitely many poles, then under certain conditions, the inverse Laplace transform can be expressed in the form of an infinite series (Churchill, 1937; 1958).

Another interesting method for Laplace transform inversion of the meromorphic functions is to express the meromorphic function as a Mittag-Leffler expansion, which is a generalisation of partial fraction decomposition of rational functions (Ahlfors, 1966). For a discussion of the Mittag-Leffler series expansion for meromorphic functions involving simple poles only, the reader is referred to (Hassani, 2013). After expanding the Laplace domain solution in the form of a Mittag-Leffler series, the inverse Laplace transform can then be taken term-by-term and the solution will be expressed, again, as an infinite series. Churchill successfully used this method to find the solution of one-dimensional heat conduction problem in a two-layered slab (Churchill, 1936). This method is also well applicable to the problems of wave propagation.

In the context of one-dimensional transient wave propagation problems in cylindrical and spherical system of coordinates (cylindrically and spherically symmetric waves) via multiple integral transformations (usually Fourier or Hankel transforms joined with the Laplace transform), De Hoop has used a modified version of Cagniard's method. This method is essentially about deforming the contour of integration in the complex plane in an appropriate manner so as to obtain, directly and elegantly, the inverse of the multiple transform and the solution of the problem (De Hoop, 1960). The Cagniard-De Hoop method is a tool that has been extensively used for the solution of IBVPs involving transient wave motions from one to three space dimensions in isotropic and anisotropic elastic media (see for instance: Achenbach, 1975; Van Der Hijden, 2016; Eskandari-Ghadi and Sattar, 2009; Raoofian Naeeni and Eskandari-Ghadi, 2016).

In this paper, an analytical method is presented to derive the solution in the form of infinite series of linear combinations of d'Alembert solutions for transient wave propagation in a general multi-layered one-dimensional domain. The layers can be of finite thickness or of infinite extent (half-space), which is of interest in IBVPs in seismology. It should be emphasised that the formulation of one-dimensional problems of propagation of transverse or longitudinal transient elastic waves in layers and rods are mathematically analogous and thus can be treated by a unified approach, as pursued in this study. For instance, the methods utilised in this study can be used to study transient pulse propagation in uniform or sectionally uniform rods and piles, which is of interest in many engineering and industrial applications. The one-dimensional axially symmetric wave propagation theory in rods, however, involves certain considerations in terms of being physically reliable. More refined versions of the simple one-dimensional wave equation might be needed in some situations, as was thoroughly studied by Boström (2000).

The source of wave motion in this study is the excitation in the form of either a prescribed traction or displacement applied at the interface of any two adjacent layers and/or at the boundary of the very top or the very bottom layer. In addition, the excitation can be in the form of the combination of prescribed traction or displacement, however applied at different interfaces. Obviously, the prescribed traction or displacement can induce many different types of waves (including impulsive waves) in the layered domain.

The solution is determined via Laplace integral transform, after which a systematic and simple procedure is presented for inverse Laplace transformation. The inversion procedure is based on the expansion of the complex-valued functions in the Laplace domain in the form of general Dirichlet series (Hardy, 1915). As a result, we can obtain the final solutions elegantly in the form of infinite d'Alembert series for each layer, which is valid for any time. Such series, however, become finite sums for a finite time interval. By d'Alembert series, we shall mean in this paper a series that is composed of an infinite linear combination of solutions that take the simple form of general solution of the one-dimensional wave equation first found in 1747 by the prominent French mathematician, physicist and philosopher, *Jean-Baptist Le Rond d'Alembert*. The interplay between the general Dirichlet series in the Laplace domain and the d'Alembert series in the time domain is manifested and emphasised on in this study.

The methods and the results obtained in this paper are particularly useful in seismology and earthquake engineering since, as will be illustrated in later sections, the exact form of time-histories of response (displacement and stress) in a multi-layered medium due to arbitrary excitations can be constructed from the excitation functions as finite sums.

Because of the different elastic properties of different layers, the wavefronts travelling in the domain are split up upon collision with layer interfaces. Each wavefront in partly reflected and partly transmitted to the adjacent layer. This succession of reflections and transmissions may create complicated and peculiar wave motions and some unexpected effects (Achenbach, 1975; Achenbach et al., 1968), which can be observed in the d'Alembert series solutions presented in this paper. The validity of the solutions presented here is shown via both intuitively logical properties of the mathematical formulae, which satisfy the governing equations, boundary and initial conditions and also by illustrations for the traveling waves. Moreover, physical phenomena such as energy loss due to one-dimensional mechanical radiation of waves through semi-infinite layers are examined in this study, which were not considered in the previous studies.

As it will become apparent later on, these exact d'Alembert series solutions are valid for the whole (semi-infinite) time interval $0 < t < \infty$. One major advantage of representing solutions as d'Alembert series is that the infinite series becomes a *finite* sum for a finite time interval. This fact greatly reduces the computational efforts needed to evaluate and plot the time-history of stress or displacement at any point in the multi-layered elastic medium caused by wave motions. This important assertion and its physical significance will be further discussed and elaborated on in this study.

DESCRIPTION OF THE MULTI-LAYERED IBVP

Let us consider a multi-layered elastic medium that consists of N layers, each having a finite thickness H_m, mass density ρ_m and wave propagation speed c_m ($m = 1, 2, \dots, N$). As depicted in Figure 1, the coordinate system is set on the top of the first layer, having the x-axis point into the medium. We divide the x-axis from zero to x_N into N intervals with the boundaries set as $0 = x_0 < x_1 < \cdots < x_m < \cdots < x_N$, then the open interval $I_m = (x_{m-1}, x_m)$ is associated with the m'th layer. As we will see in the next section, the case in which a finite number of layers are rested on an elastic half-space can also be considered as a special case of this configuration, in which both x_N and H_N approach infinity.

We assume that either motions (displacements) or surface tractions are prescribed at boundary and/or interface planes $x = x_m$ $(m = 0, 1, 2, \dots, N)$ in such a manner that they produce either longitudinal or transverse waves travelling in the direction of x-axis in the whole domain. Thus, we have at each layer the one-dimensional, homogeneous wave equation:

$$\frac{\partial^2 u_m}{\partial x^2} - \frac{1}{c_m^2}\frac{\partial^2 u_m}{\partial t^2} = 0, \qquad x \in I_m, \quad t > 0, \tag{1}$$

for $m = 1, 2, \dots, N$, where $u_m = u_m(x, t)$ denotes the elastic displacement in the m'th layer. We further assume that the layers are fully connected via their interfaces, i.e.

$$u_m(x_m, t) = u_{m+1}(x_m, t), \quad m = 1, 2, \dots, N-1. \tag{2}$$

Hence, for a more systematic treatment of the problem we define functions $q_m(t)$ for all $m = 0, 1, 2, \dots, N$ that denote the displacement of the boundary or interface planes. At each of these planes, either a prescribed motion $q_m(t)$ or a surface traction $f_m(t)$ is applied. This means that either

$$\begin{aligned} &u_m(x_m, t) = u_{m+1}(x_m, t) = q_m(t), \quad m = 1,2, \dots, N-1, \\ &u_1(x_0, t) = q_0(t), \\ &u_N(x_N, t) = q_N(t), \end{aligned} \tag{3}$$

or alternatively

$$\begin{aligned} &\alpha_{m+1} c_{m+1} \frac{\partial u_{m+1}}{\partial x}(x_m, t) - \alpha_m c_m \frac{\partial u_m}{\partial x}(x_m, t) = -f_m(t), \quad m = 1, 2, \dots, N-1, \\ &\alpha_1 c_1 \frac{\partial u_1}{\partial x}(x_0, t) = -f_0(t), \\ &\alpha_N c_N \frac{\partial u_N}{\partial x}(x_N, t) = f_N(t), \end{aligned} \tag{4}$$

are given, in which $\alpha_m = \rho_m c_m$ is the mechanical impedance (Achenbach, 1975) of the m'th layer. We allow also a combination of the conditions (3) and (4), which means that one can prescribe displacements at some boundaries or interface planes and apply tractions at the others. Finally, the whole media is assumed to be of quiescent past, i.e. we assume zero initial conditions for the whole domain

$$u_m(x, 0) = \frac{\partial u_m}{\partial t}(x, 0) = 0, \qquad x \in I_m, \tag{5}$$

for all $m = 1, 2, \dots, N$. The system of partial differential equations (1) with the continuity conditions written as Eq. (2), alternate interface or boundary conditions given by Eq. (3) or (4) or a combination of them and the initial conditions written as Eq. (5) defines an IBVP that we are considering in this paper.

GENERAL SOLUTION OF THE IBVP IN THE LAPLACE DOMAIN

In this section with the aid of Laplace integral transform with respect to time variable t, we attack the initial-boundary value problem that we described in the preceding section. Thus the solutions will be found in the *Laplace domain* in this section, i.e. we will find the Laplace transforms of functions $u_m(x, t)$ $(m = 1, 2, \dots, N)$ defined by Sneddon (1995) as follows:

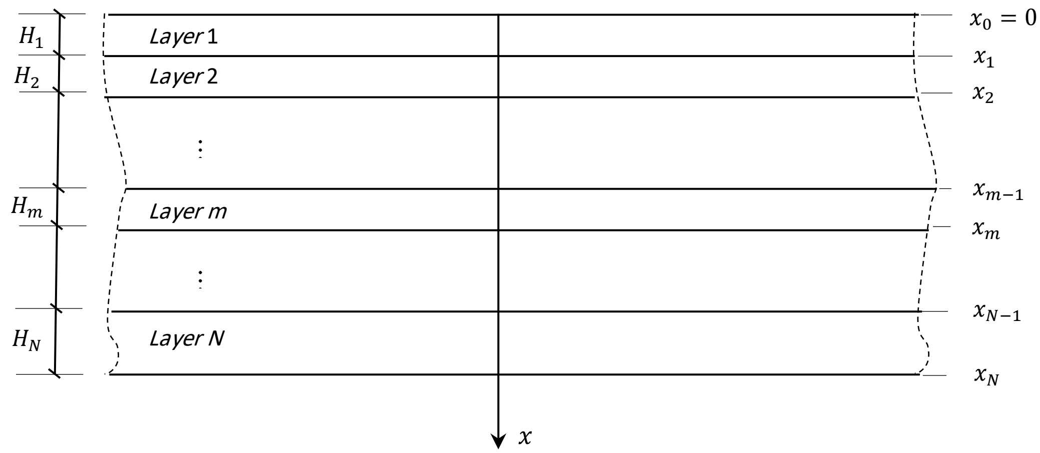

Fig. 1. A multi-layered one-dimensional elastic medium

$$U_m(x,s) = \int_0^{\infty} u_m(x,t)e^{-st}dt, \quad (6)$$

provided that the integral on the right hand side exists. Here s is assumed to be a complex variable in some region Ω of the complex s-plane. The final solutions $u_m(x,t)$ of the IBVP will then need to be determined by a Laplace transform inversion of functions $U_m(x,s)$ given by

$$u_m(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U_m(x,s)e^{st}ds, \quad (7)$$

in which γ is an arbitrary real number such that $U_m(x,s)$ is an analytic function of s in the half-plane $\Re(s) > \gamma$. The inversion formula in (7) is the Bromwich integral (also known as Mellin's inversion theorem) and is valid under certain conditions involving the functions $U_m(x,s)$, (Sneddon, 1995). Other inversion formulae for Laplace transforms do exist. For example the Post-Widder inversion formula (Cohen, 2007), which involves a limiting process instead of complex integration. The inversion process will be discussed in the following sections.

The first step of the solution is to take the Laplace transforms of both sides of Eqs. (1) while the initial conditions written as Eq. (5) are applied. Doing this, one obtains an ordinary differential equation for each layer as:

$$\frac{d^2}{dx^2}U_m(x,s) - \frac{s^2}{c_m^2}U_m(x,s) = 0, \quad x \in I_m, \quad (8)$$

that has general solution in the form of:

$$U_m(x,s) = A_m(s)\cosh\left(\frac{x}{c_m}s\right) + B_m(s)\sinh\left(\frac{x}{c_m}s\right), \quad x \in I_m. \quad (9)$$

where $A_m(s)$ and $B_m(s)$ are unknown functions which have to be determined using the boundary conditions for each layer. If one takes the Laplace transform of Eq. (3), the following equation is obtained:

$$Q_m(s) = U_m(x_m,s) = U_{m+1}(x_m,s), \quad m = 1,2,\dots,N-1, \quad (10)$$

where $Q_m(s)$ denotes the Laplace transform of the function $q_m(t)$. By combining (9) with (10), one can write

$$Q_m(s) = A_m(s)\cosh\left(\frac{x_m}{c_m}s\right) + B_m(s)\sinh\left(\frac{x_m}{c_m}s\right),$$
$$Q_m(s) = A_{m+1}(s)\cosh\left(\frac{x_m}{c_{m+1}}s\right) + B_{m+1}(s)\sinh\left(\frac{x_m}{c_{m+1}}s\right). \quad (11)$$

Upon shifting the indices of the Eq. $(11)_2$ one obtains

$$Q_{m-1}(s) = A_m(s)\cosh\left(\frac{x_{m-1}}{c_m}s\right) + B_m(s)\sinh\left(\frac{x_{m-1}}{c_m}s\right). \tag{12}$$

The Eq. $(11)_1$ and Eq. (12) can be simultaneously solved for $A_m(s)$ and $B_m(s)$ in terms of $Q_m(s)$ and $Q_{m-1}(s)$, which after some algebraic manipulations and using $H_m = x_m - x_{m-1}$ result in:

$$\begin{aligned} A_m(s) &= \operatorname{csch}\left(\frac{H_m}{c_m}s\right)\left[-\sinh\left(\frac{x_{m-1}}{c_m}s\right)Q_m(s) + \sinh\left(\frac{x_m}{c_m}s\right)Q_{m-1}(s)\right], \\ B_m(s) &= \operatorname{csch}\left(\frac{H_m}{c_m}s\right)\left[\cosh\left(\frac{x_{m-1}}{c_m}s\right)Q_m(s) - \cosh\left(\frac{x_m}{c_m}s\right)Q_{m-1}(s)\right]. \end{aligned} \tag{13}$$

Alternatively, one may take the Laplace transforms from both sides of Eq. $(4)_1$ to obtain

$$\alpha_{m+1}c_{m+1}\frac{dU_{m+1}}{dx}(x_m, s) - \alpha_m c_m \frac{dU_m}{dx}(x_m, s) = -F_m(s), \quad m = 1, 2, \dots, N-1, \tag{14}$$

in which $F_m(s)$ denotes the Laplace transform of the function $f_m(t)$. Using $U_m(x,s)$, $A_m(s)$ and $B_m(s)$, as given in (9) and (13), in Eq. (14) and after some rather lengthy algebraic manipulations of hyperbolic functions, one can derive the important relation

$$F_m(s) = [X_m(s) + X_{m+1}(s)]Q_m(s) + Y_m(s)Q_{m-1}(s) + Y_{m+1}(s)Q_{m+1}(s), \tag{15}$$

in which $X_m(s)$ and $Y_m(s)$ are defined as:

$$\begin{aligned} X_m(s) &= \alpha_m s \coth\left(\frac{H_m}{c_m}s\right), \\ Y_m(s) &= -\alpha_m s \operatorname{csch}\left(\frac{H_m}{c_m}s\right). \end{aligned} \tag{16}$$

Relation (15) is valid only for $m = 1, 2, \dots, N-1$. To obtain similar relations for $F_0(s)$ and $F_N(s)$ one has to repeat these steps for the second and the third equations in (4). Eliminating the details, they are found as:

$$\begin{aligned} F_0(s) &= X_1(s)Q_0(s) + Y_1(s)Q_1(s), \\ F_N(s) &= X_N(s)Q_N(s) + Y_N(s)Q_{N-1}(s). \end{aligned} \tag{17}$$

Writing out the Eqs. (15) and (17) for a few indices, we see that:

$$\begin{aligned} F_0(s) &= X_1(s)Q_0(s) + Y_1(s)Q_1(s), \\ F_1(s) &= [X_1(s) + X_2(s)]Q_1(s) + Y_1(s)Q_0(s) + Y_2(s)Q_2(s), \\ F_2(s) &= [X_2(s) + X_3(s)]Q_2(s) + Y_2(s)Q_1(s) + Y_3(s)Q_3(s), \\ &\vdots \\ F_{N-1} &= [X_{N-1}(s) + X_N(s)]Q_{N-1}(s) + Y_{N-1}(s)Q_{N-2}(s) + Y_N(s)Q_N(s) \\ F_N(s) &= X_N(s)Q_N(s) + Y_N(s)Q_{N-1}(s). \end{aligned} \tag{18}$$

Arranging the above system of $N+1$ equations in the matrix form, the Eq. (18) can be written as:

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 & 0 & \dots & 0 & 0 \\ Y_1 & X_1 + X_2 & Y_2 & \cdots & 0 & 0 \\ 0 & Y_2 & X_2 + X_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_{N-1} + X_N & Y_N \\ 0 & 0 & 0 & \dots & Y_N & X_N \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_{N-1} \\ Q_N \end{bmatrix}, \tag{19}$$

where we have omitted the argument s from the functions for brevity. We, thus, see that in the Laplace domain, the tractions $f_m(t)$ and displacements $q_m(t)$ of the interface or boundary planes are related by a linear equation of the matrix form:

$$\mathbf{F}(s) = \mathbf{K}(s)\mathbf{Q}(s). \tag{20}$$

where the column matrices $\mathbf{F}(s) = [F_0(s) \quad F_1(s) \quad \cdots \quad F_N(s)]^{\mathrm{T}}$, $\mathbf{Q}(s) = [Q_0(s) \quad Q_1(s) \quad \cdots \quad Q_N(s)]^{\mathrm{T}}$ and the square symmetric and band (tridiagonal) matrix $\mathbf{K}(s)$ as given in Eq. (19). As it has been mentioned earlier, at each interface or boundary plane, either the displacement or the traction is given. Hence, some of the functions F_m and the other functions Q_m in Eq. (19) or (20) are known. Thus, one may need to interchange the place of some of F_m and Q_m to move all the unknown functions to one side and all the known functions to the other side, then solve it for the unknown functions. Moreover, substituting $A_m(s)$ and $B_m(s)$ from (13) in Eq. (9), one may obtain the relation

$$\begin{aligned} U_m(x,s) &= \operatorname{csch}\left(\frac{H_m}{c_m}s\right)\left[\sinh\left(\frac{x_m - x}{c_m}s\right)Q_{m-1}(s) \right. \\ &\left. + \sinh\left(\frac{x - x_{m-1}}{c_m}s\right)Q_m(s)\right], \end{aligned} \tag{21}$$

for all $x \in I_m$ $(m = 1, 2, \ldots, N)$. This relation gives the functions $U_m(x,s)$ in terms of the previously obtained functions $Q_m(s)$. Accordingly, the Laplace transforms of the solutions of the multi-layered system are at hand.

The last step would be to find the inverse Laplace transforms of $U_m(x,s)$, i.e. the functions $u_m(x,t)$, which are the desired wave functions in each layer. They give us the precise pattern of transient motions at each arbitrary point of the multi-layered system. It is noteworthy that the motions of boundary or interfaces planes can be directly determined by applying the inversion process to $Q_m(s)$ functions in order to find $q_m(t)$. We are going to show the inversion procedure from some simple to moderately complicated problems. It will be seen that by solving each case with an analytical procedure, the inverse determined as the sum of some d'Alembert type wave functions.

A THREE-LAYERED PROBLEM

A sample problem is considered in this section to demonstrate the Laplace transform inversion technique used in this paper. We formulate and solve this sample problem in the Laplace domain using the methods just discussed in the previous section. A few degenerated cases of this problem will also be considered for clarifying the procedure.

An elastic multi-layered medium that consists of three different layers ($N = 3$) of finite thickness is considered (see Figure 2). A time-dependent uniform traction $f(t)$ is exerted at the plane $x = x_1$, which is the interface of the first and the second layers. The upper boundary of the medium ($x = x_0 = 0$) is free of traction and the lower boundary ($x = x_3$) is assumed to be fixed on a rigid base.

It is clear that in this problem the conditions at the boundaries and interfaces are simply given by $f_0(t) = 0$, $f_1(t) = f(t)$, $f_2(t) = 0$ and $q_3(t) = 0$. Thus, one can immediately express the linear relation (20) in the form of

$$\begin{bmatrix} 0 \\ F(s) \\ 0 \\ F_3(s) \end{bmatrix} = \begin{bmatrix} X_1(s) & Y_1(s) & 0 & 0 \\ Y_1(s) & X_1(s) + X_2(s) & Y_2(s) & 0 \\ 0 & Y_2(s) & X_2(s) + X_3(s) & Y_3(s) \\ 0 & 0 & Y_3(s) & X_3(s) \end{bmatrix} \begin{bmatrix} Q_0(s) \\ Q_1(s) \\ Q_2(s) \\ 0 \end{bmatrix}. \tag{22}$$

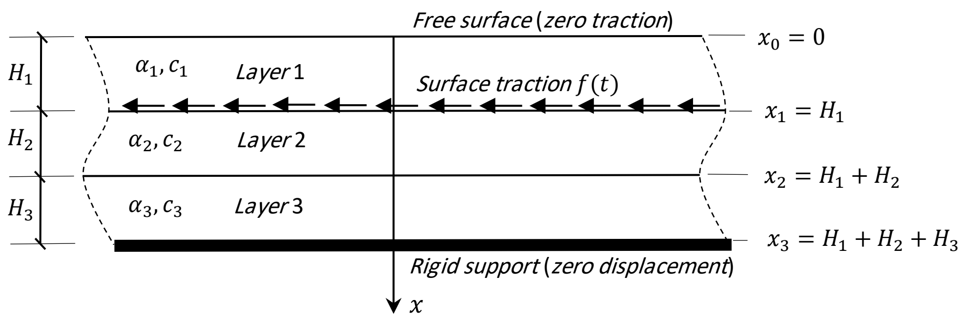

Fig. 2. A three-layered elastic medium subjected to an arbitrary time-dependent traction

By partitioning the upper three rows of the above system of equations and solving them for $Q_m(s)$, $m = 0, 1, 2$, one obtains

$$
\begin{aligned}
Q_0(s) &= -\frac{Y_1(s)[X_2(s) + X_3(s)]}{Z(s)} F(s), \\
Q_1(s) &= \frac{X_1(s)[X_2(s) + X_3(s)]}{Z(s)} F(s), \\
Q_2(s) &= \frac{-X_1(s)Y_2(s)}{Z(s)} F(s),
\end{aligned} \tag{23}
$$

in which

$$
\begin{aligned}
Z(s) = X_1^2(s)X_2(s) &+ X_1^2(s)X_3(s) \\
&+ X_1(s)X_2^2(s) \\
&+ X_1(s)X_2(s)X_3(s) \\
&- X_1(s)Y_2^2(s) \\
-X_2 \; s \; Y_1^2 \; s &- X_3 \; s \; Y_1^2(s)
\end{aligned} \tag{24}
$$

Recalling the definitions (16), one can now expand and simplify the above equations to obtain:

$$
\begin{aligned}
Q_0(s) &= \frac{E(s)}{H(s)} \frac{F(s)}{s}, \\
Q_1(s) &= \cosh(\beta_1 s)\, Q_0(s), \\
Q_2(s) &= \frac{\alpha_2 \cosh(\beta_1 s) \sinh(\beta_3 s)}{E(s)} Q_0(s),
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
&E(s) \\
&= \alpha_2 \cosh(\beta_2 s) \sinh(\beta_3 s) \\
&+ \alpha_3 \sinh(\beta_2 s) \cosh(\beta_3 s),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&H(s) \\
&= \alpha_1 \alpha_2 \sinh(\beta_1 s) \cosh(\beta_2 s) \sinh(\beta_3 s) \\
&+ \alpha_1 \alpha_3 \sinh(\beta_1 s) \sinh(\beta_2 s) \cosh(\beta_3 s) \\
&+ \alpha_2 \alpha_3 \cosh(\beta_1 s) \cosh(\beta_2 s) \cosh(\beta_3 s) \\
&+ \alpha_2^2 \cosh(\beta_1 s) \sinh(\beta_2 s) \sinh(\beta_3 s),
\end{aligned}
$$

are functions of s and

$$
\beta_m = \frac{H_m}{c_m}, \qquad m = 1, 2, 3. \tag{27}
$$

are constants. Finally, one can derive the expressions for $U_m(x, s)$ from formula (21), where we need to use $x_0 = 0$, $x_1 = H_1$, $x_2 = H_1 + H_2$ and $x_3 = H_1 + H_2 + H_3$ in this case. Thus, the displacements at each layer in the Laplace domain are given as

$$
\begin{aligned}
U_1(x, s) &= \operatorname{csch}(\beta_1 s) \left[\sinh\left(\frac{H_1 - x}{c_1} s\right) Q_0(s) \right. \\
&\qquad \left. + \sinh\left(\frac{x}{c_1} s\right) Q_1(s) \right], \\
&\text{for } 0 < x < x_1, \\
U_2(x, s) &= \operatorname{csch}(\beta_2 s) \left[\sinh\left(\frac{H_1 + H_2 - x}{c_2} s\right) Q_1(s) \right. \\
&\qquad \left. + \sinh\left(\frac{x - H_1}{c_2} s\right) Q_2(s) \right], \\
&\text{for } x_1 < x < x_2 \text{ and} \\
U_3(x, s) & \\
&= \operatorname{csch}(\beta_3 s) \left[\sinh\left(\frac{H_1 + H_2 + H_3 - x}{c_3} s\right) Q_2(s) \right], \\
&\text{for } x_2 < x < x_3.
\end{aligned} \tag{28}
$$

in order to determine the inverse Laplace transform of $U_m(x, s)$ analytically, it is more helpful to have some simpler and degenerated cases of this problem via exploiting the available results. The cases that are considered here are as follows. Case 1: A single layer subjected to an arbitrary traction at the bottom and traction-free at the top. Case 2: A single layer rested on the top of a half-space subjected to an arbitrary traction at the interface. Case 3: A two-layered medium fixed at the bottom and free at surface subjected to an arbitrary traction at the interface, and Case 4: Two elastic layers resting on an elastic half-space subjected to an arbitrary traction at the bottom of the first layer.

Case 1: A single elastic layer illustrated in Figure 3 is subjected to an arbitrary traction $f(t)$ at the bottom, while it is free of tractions on the top boundary. The results for this case can be easily derived by exploiting the results (25) and (28) in the case of $\alpha_1 = \alpha$, $H_1 = H$ and $\alpha_2 \to 0$. By setting these values, one may obtain

$$Q_0(s) = \frac{1}{\alpha}\operatorname{csch}\left(\frac{H}{c}s\right)\frac{F(s)}{s}, \quad Q_1(s) = \frac{1}{\alpha}\coth\left(\frac{H}{c}s\right)\frac{F(s)}{s}, \quad Q_2(s) = 0, \tag{29}$$

and consequently,

$$U(x,s) = U_1(x,s) = \frac{\cosh\left(\frac{x}{c}s\right)}{\alpha\sinh\left(\frac{H}{c}s\right)}\frac{F(s)}{s}, \quad 0 < x < H. \tag{30}$$

Case 2: An elastic layer is resting on an elastic half-space. The top boundary is free of traction and the interface of the layer and the half-space is subjected to an arbitrary traction $f(t)$, as depicted in Fig. 4. For this problem, we set $H_1 = H$ and take the limit of results (25) and (28) as $H_2 \to \infty$, which implies $\beta_2 \to \infty$. Eventually, after some algebraic manipulations, one may find

$$Q_0(s) = \frac{1}{\alpha_1\sinh\left(\frac{H}{c_1}s\right) + \alpha_2\cosh\left(\frac{H}{c_1}s\right)}\frac{F(s)}{s}, \quad Q_1(s) = \frac{\cosh\left(\frac{H}{c_1}s\right)}{\alpha_1\sinh\left(\frac{H}{c_1}s\right) + \alpha_2\cosh\left(\frac{H}{c_1}s\right)}\frac{F(s)}{s}, \quad Q_2(s) = 0, \tag{31}$$

which results in

$$U_1(x,s) = \frac{\cosh\left(\frac{x}{c_1}s\right)}{\alpha_1\sinh\left(\frac{H}{c_1}s\right) + \alpha_2\cosh\left(\frac{H}{c_1}s\right)}\frac{F(s)}{s}, \quad 0 < x < H, \quad U_2(x,s) = \exp\left(\frac{H-x}{c_2}s\right)U_1(H,s). \quad x > H. \tag{32}$$

Case 3: Two elastic layers resting on a rigid base, free of tractions at the top and subject to arbitrary traction $f(t)$ at the interface plane $x = x_1$ (see Figure 5). In this case, by exploiting the results given in Eqs. (25) and (28) for the limiting case of $\alpha_3 \to \infty$, one may obtain

$$Q_0(s) = \frac{\sinh\left(\frac{H_2}{c_2}s\right)}{\alpha_1\sinh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right) + \alpha_2\cosh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right)}\frac{F(s)}{s}, \quad Q_1(s) = \frac{\cosh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right)}{\alpha_1\sinh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right) + \alpha_2\cosh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right)}\frac{F(s)}{s}, \quad Q_2(s) = 0. \tag{33}$$

Moreover,

$$U_1(x,s) = \frac{\cosh\left(\frac{x}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right)}{\alpha_1 \sinh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right) + \alpha_2 \cosh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right)}\frac{F(s)}{s}, \tag{34}$$

for $0 < x < H_1$ and

$$U_2(x,s) = \frac{\cosh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_1+H_2-x}{c_2}s\right)}{\alpha_1 \sinh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right) + \alpha_2 \cosh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right)}\frac{F(s)}{s}, \tag{35}$$

for $H_1 < x < H_1 + H_2$.

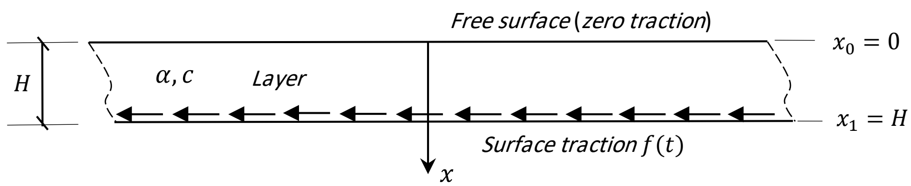

Fig. 3. A single layer subjected to an arbitrary traction at the bottom

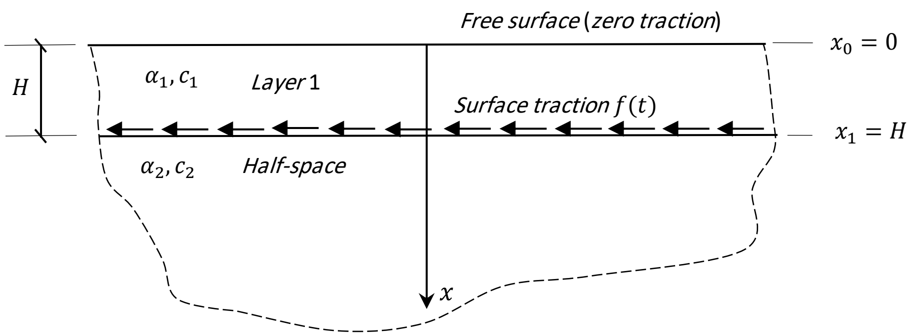

Fig. 4. A single layer rested on the top of a half-space subjected to an arbitrary traction at the interface

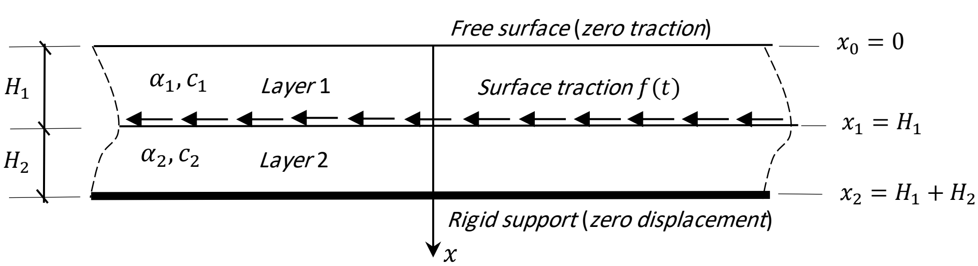

Fig. 5. A two-layered medium fixed at the bottom and subjected to an arbitrary traction at the interface

Case 4: Two elastic layers resting on an elastic half-space. The top boundary plane ($x = x_0 = 0$) is free of tractions and an arbitrary uniform traction $f(t)$ is applied at the interface plane $x = x_1$, as illustrated in Figure 6. To obtain the results for this case it is sufficient to use the results (25) and (28) and taking their limit as $H_3 \to \infty$, or equivalently $\beta_3 \to \infty$, which after some rather tedious manipulations result in

$$Q_0(s) = \frac{\alpha_2 \cosh\left(\frac{H_2}{c_2}s\right) + \alpha_3 \sinh\left(\frac{H_2}{c_2}s\right)}{\Delta(s)} \frac{F(s)}{s},$$
$$Q_1(s) = \frac{\cosh\left(\frac{H_1}{c_1}s\right)\left[\alpha_2 \cosh\left(\frac{H_2}{c_2}s\right) + \alpha_3 \sinh\left(\frac{H_2}{c_2}s\right)\right]}{\Delta(s)} \frac{F(s)}{s}, \tag{36}$$
$$Q_2(s) = \frac{\alpha_2 \cosh\left(\frac{H_1}{c_1}s\right)}{\Delta(s)} \frac{F(s)}{s},$$
$$Q_3(s) = 0.$$

in addition, one obtains

$$U_1(x,s) = \frac{\cosh\left(\frac{x}{c_1}s\right)\left[\alpha_2 \cosh\left(\frac{H_2}{c_2}s\right) + \alpha_3 \sinh\left(\frac{H_2}{c_2}s\right)\right]}{\Delta(s)} \frac{F(s)}{s}, \tag{37}$$

for $0 < x < H_1$,

$$U_2(x,s) = \frac{\cosh\left(\frac{H_1}{c_1}s\right)\left[\alpha_2 \cosh\left(\frac{H_1+H_2-x}{c_2}s\right) + \alpha_3 \sinh\left(\frac{H_1+H_2-x}{c_2}s\right)\right]}{\Delta(s)} \frac{F(s)}{s}, \tag{38}$$

for $H_1 < x < H_1 + H_2$ and

$$U_3(x,s) = \exp\left(\frac{H_1 + H_2 - x}{c_3}s\right) U_2(H_1 + H_2, s), \tag{39}$$

for $x > H_1 + H_2$. Here, $\Delta(s)$ is defined as:

$$\Delta(s) = \alpha_1\alpha_2 \sinh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right) + \alpha_1\alpha_3 \sinh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right)$$
$$+\alpha_2\alpha_3 \cosh\left(\frac{H_1}{c_1}s\right)\cosh\left(\frac{H_2}{c_2}s\right) + \alpha_2^2 \cosh\left(\frac{H_1}{c_1}s\right)\sinh\left(\frac{H_2}{c_2}s\right). \tag{40}$$

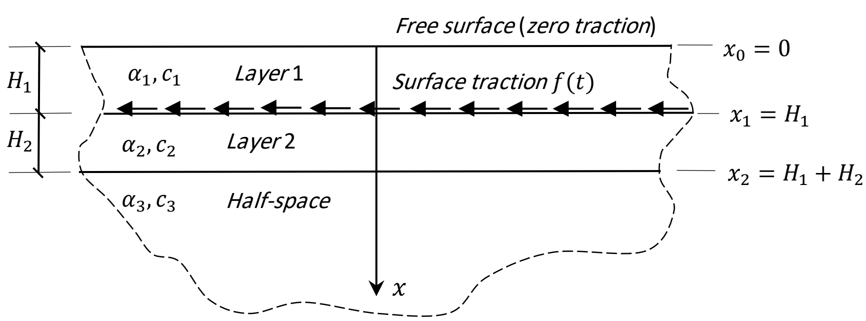

Fig. 6. Two elastic layers resting on an elastic half-space subjected to an arbitrary traction at the bottom of the first layer

Obviously, the special cases studied above, could be stated and solved directly as independent problems by the matrix method developed in the previous section.

INVERSE LAPLACE TRANSFORMATION WITH THE USE OF GENERAL DIRICHLET SERIES

The method of solution of one-dimensional wave propagation problems in multi-layered elastic media in the Laplace domain has been discussed in the preceding sections. The last step is to find the inverse Laplace transform for these solutions in order to derive the solutions in time domain for the IBVP. This section is devoted to a discussion of the idea of Laplace transform inversion with the use of general Dirichlet series.

In this study, the functions $u_m(x,t)$ need to be obtained by applying the inverse Laplace transform to the functions $U_m(x,t)$, ($m = 1, 2, \dots, N$). These two functions are related to each other via Eqs. (6) and (7), which are the Laplace transform and inverse pairs. The classical and frequently used method of Laplace transform inversion is the contour integration of the integral involved in Eq. (7). This is often done by the methods of residue calculus (see Sneddon, 1995; Weinberger, 1995). The residue integration methods involve finding singular points of the functions $U_m(x,s)$. However, as we have seen in the previous section, the complex-valued functions arising as the Laplace transform of the solutions of the one-dimensional transient wave propagation problems in multi-layered elastic media are, in fact, meromorphic functions (Ahlfors, 1966) having infinitely many poles. The position of these poles in the complex s-plane depend on both the geometry and the mechanical properties of the layers.

The methods of Laplace transform inversion hitherto mentioned in the introduction and the preceding paragraph depend upon the ability of finding the singularities of the function $U_m(x,s)$ in the complex s-plane. However, as we have seen, the meromorphic function that one has to deal with in our IBPVs are of rather complicated forms and their poles cannot be analytically found in many cases. In addition, finding the residues at the poles and simplifying the final results often need some computational efforts. Hence, here a more direct method is presented for finding the inverse Laplace transform in which it is not required to deal with the poles of the complex-valued functions. Detailed investigation of the general form of the function $U_m(x,s)$ in section 4 results in the fact that they are expressible as a product of two other complex-valued functions in the form of:

$$U_m(x,s) = \widehat{\Psi}_m(x;s)\Phi(s), \tag{41}$$

for all s in some domain Ω in the complex s-plane. $\widehat{\Psi}_m(x;s)$ are functions that appear as quotients of combinations of hyperbolic functions, while $\Phi(s)$ is defined to be $F(s)/s$. In problems where instead of a given traction, a prescribed displacement at an interface or boundary plane is given, one shall define $\Phi(s) = Q(s)$, where $Q(s)$ is the Laplace transform of the prescribed displacement $q(t)$ at that specific plane. Here, we can treat the real variable x as a real parameter that appears in complex-valued functions $\widehat{\Psi}_m(x;s)$. Thus for a more convenient notation we can denote $\widehat{\Psi}_m(x;s)$ by $\Psi_m(s)$. It is proposed that the function $\Psi_m(s)$ is expressible in the form of a general Dirichlet series (Dirichlet series of type λ_n); (see Hardy, 1915) for the definition. If that is true, one is able to write

$$\Psi_m(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{42}$$

where $\{\lambda_n\}$ is some strictly increasing sequence of nonnegative real numbers. The numbers a_n can generally be complex, yet in applications that we are concerned with in this paper they are assumed to be real numbers. The real numbers λ_n here are themselves functions of the real variable x. Moreover, it is seen that the inverse Laplace transform of $\Phi(s)$ can be easily found. In the case $\Phi(s) = F(s)/s$, we have

$$\phi(t) = \mathfrak{L}^{-1}[\Phi(s)] = \mathfrak{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(\tau)d\tau = \tilde{f}(t), \quad t > 0, \tag{43}$$

where $f(t) = \mathfrak{L}^{-1}[F(s)]$, as before. If on the other hand $\Phi(s) = Q(s)$, we simply have $\phi(t) = \mathfrak{L}^{-1}[Q(s)] = q(t)$. Now, making use of the well-known *shifting* property of Laplace transform results in

$$\mathfrak{L}^{-1}\left[\Phi(s)e^{-\lambda s}\right] = \phi(t-\lambda)\Theta(t-\lambda), \quad \lambda \geq 0, \tag{44}$$

where Θ represents the Heaviside unit step function (Kreyszig, 2011). The above formula can be proven easily using the definition of Laplace transform or, equivalently, the use of convolution (Faltung) theorem for the Laplace transform, as discussed in (Kreyszig, 2011) and (Sneddon, 1995). Now, by combining Eqs. (41) and (42), one may infer that:

$$U_m(x,s) = \sum_{n=1}^{\infty} a_n \Phi(s) e^{-\lambda_n s}. \tag{45}$$

Then, upon using the shifting Eq. (44) term-by-term, the inverse Laplace transform of the function $U_m(x,s)$ is obtained formally as:

$$u_m(x,t) = \mathfrak{L}^{-1}[U_m(x,s)] = \sum_{n=1}^{\infty} a_n\, \phi(t-\lambda_n)\Theta(t-\lambda_n), \quad t > 0, \quad x \in I_m. \tag{46}$$

Accordingly, if the term-by-term inversion is justified then one is able to express the inverse Laplace transform of $U_m(x,t)$ in the form of an infinite series. Moreover, the Eq. (46) can be expressed more concisely in the form of:

$$u_m(x,t) = \mathfrak{L}^{-1}[U_m(x,s)] = \sum_{n=1}^{\infty} a_n\, \phi\langle t-\lambda_n \rangle, \quad t > 0, \quad x \in I_m. \tag{47}$$

where the symbols $\langle \cdot \rangle$ denotes the Macaulay's brackets, also known as singularity brackets (Macaulay, 1919), defined by:

$$\phi\langle \xi \rangle = \phi(\xi)\Theta(\xi) = \begin{cases} \phi(\xi), & \text{if } \xi > 0 \\ 0, & \text{if } \xi < 0 \end{cases}, \tag{48}$$

for any typical function ϕ. As suggested by the above discussion, for Laplace transform inversions used in this paper, we only need to express the Laplace transforms of solutions in the form of (41), where the inverse Laplace transform of the function Φ is known. Thus, if we are able to develop the complex-valued function $\Psi_m(s) = \hat{\Psi}_m(x,s)$ in the form of a general Dirichlet series, then the Laplace transform inversion process will be greatly facilitated by the use of the shifting Eq. (44). The final solution will then be in the form of an infinite series given in Eq. (47). The justification of this final solution can then be considered.

Now, we consider the special case 1 from the sample problem defined in the previous section. It is clear from Eq. (30) that:

$$\Psi(s) = \frac{\cosh(as)}{\alpha \sinh(bs)} = \frac{1}{\alpha}\frac{e^{as}+e^{-as}}{e^{bs}-e^{-bs}}, \quad 0 < a < b. \tag{49}$$

where the real positive parameters a and b are defined as $a = x/c$ and $b = H/c$. Now, by factoring the exponential term e^{bs} in the denominator of (49) which has the greatest real coefficient of s, one may write:

$$\Psi(s) = \frac{1}{\alpha}\frac{e^{as}+e^{-as}}{e^{bs}(1-e^{-2bs})} = \frac{1}{\alpha}\frac{e^{-(b-a)s}+e^{-(b+a)s}}{1-e^{-2bs}}, \quad 0 < a < b. \tag{50}$$

The expression $1/(1-e^{-2bs})$, upon using the substitution $\zeta = e^{-2bs}$, can be represented by the geometric series $1/(1-\zeta) = 1+\zeta+\zeta^2+\cdots = \sum_{n=0}^{\infty}\zeta^n$ for $|\zeta| = |e^{-2bs}| < 1$. Thus, $\Psi(s)$ can be written in the form:

$$\Psi(s) = \frac{1}{\alpha}\left[e^{-(b-a)s} + e^{-(b+a)s}\right]\sum_{n=0}^{\infty} e^{-2nbs} = \frac{1}{\alpha}\sum_{n=0}^{\infty}\left[e^{-(b-a+2nb)s} + e^{-(b+a+2nb)s}\right], \tag{51}$$

The above series is indeed in the form of a general Dirichlet series (Hardy, 1915) if its terms are rearranged in an appropriate manner. The geometric series used here, converge absolutely throughout the open disc $|\zeta| < 1$ in the complex ζ-plane, thus we must have $|e^{-2bs}| < 1$ or more explicitly, $\Re(s) > 0$ on the complex s-plane, which is because $b = H/c > 0$. Thus, the series representation (51) for $\Psi(s)$ is valid in the right half-plane $\Re(s) > 0$. The geometric series expansion discussed here for inverse Laplace transformation has been suggested in different sources; see for instance (Churchill, 1958).

The inverse Laplace transform of the function $U(x,s)$ in (30) can now be written by the help of Eqs. (47) and (51) in the form of an infinite series as

$$u(x,t) = \mathfrak{L}^{-1}[U(x,s)] = \frac{1}{\alpha}\sum_{n=0}^{\infty}\left[\tilde{f}\left\langle t + \frac{x}{c} - \frac{(2n+1)H}{c}\right\rangle + \tilde{f}\left\langle t - \frac{x}{c} - \frac{(2n+1)H}{c}\right\rangle\right], \tag{52}$$

for all $0 < x < H$ and $t > 0$, where the definitions $a = x/c$ and $b = H/c$ have been utilised. The function $\tilde{f}$ has been defined in (43) as a definite integral of f. The above equation represents, at least formally, the final solution of the special case 1 defined in the sample problem in the form of an infinite series. It is not irrelevant to name the series in (52) a d'Alembert series, for it is but a linear combination of d'Alembert's solutions to one-dimensional wave equation. Physically, it can be interpreted as an infinite succession of wavefronts that travel back and forth in the elastic layer.

In addition, one can find the corresponding stress wave function $T(x,t)$ in the medium by the stress-displacement relation $T(x,t) = \alpha c\,\partial u(x,t)/\partial x$. Hence, from (52), $T(x,t)$ is derived as:

$$T(x,t) = \sum_{n=0}^{\infty}\left[f\left\langle t + \frac{x}{c} - \frac{(2n+1)H}{c}\right\rangle - f\left\langle t - \frac{x}{c} - \frac{(2n+1)H}{c}\right\rangle\right], \tag{53}$$

for all $0 < x < H$ and $t > 0$. As mentioned earlier, the term-by-term Laplace transform inversion of the infinite series in (45) needs some justification to prove the validity of the solution. Primarily, we notice from Eq. (53) that $T(0,t)$ vanishes identically, which is required by the boundary conditions at $x = 0$. Moreover, $T(H,t)$ can be verified to be a telescoping series (Apostol, 1974) that reduces to a single term $f\langle t\rangle = f(t)$ since $f(t)$ is zero for all $t < 0$. This means that the traction on the plane $x = H$ is equal to $f(t)$ as it is required by the boundary condition. Therefore, all the boundary conditions are satisfied exactly. The initial conditions also hold trivially because of the presence of Heaviside step functions in all terms of the series (52). In addition, it should be noted that all the terms in the series in (52) are in the form of $\chi(t \pm x/c + \nu)$ where χ denotes a typical real-valued function and ν is an arbitrary real constant. The function χ can be readily verified to satisfy the one-dimensional homogeneous wave equation of the form (1). Due to linearity and homogeneity of Eq. (1), an arbitrary linear combination of such functions also satisfy (1). Thus, the final solution of the special case 1 of the sample problem is correctly represented in (52).

The series solution (52) is valid for the whole (semi-infinite) interval $0 < t < \infty$. It was mentioned earlier in the introduction that this d'Alembert series reduces to a finite sum if one is interested in a solution that is valid only in a finite time interval $0 < t < t_0$. Indeed, from (52) it can be verified that if $n \geq (x + ct - H)/2H$ then the terms of series in (52) vanish. Now, if $t \leq t_0$ then all the terms having the index $n \geq n_0 = (x + ct_0 - H)/2H$ in the series vanish. This means that only a finite number of the terms in the series are non-zero in this case and the series is, in fact, reduced to a finite sum.

More generally and from a physical aspect, the wavefronts generated due collisions of existing waves to the boundaries or interfaces that take place in $t > t_0$, are effectively removed from the solution. This is indeed a reflection of the principle of causality in physics, which asserts that no phenomenon can be affected in the present time by causes that are going to occur in later times. In this context, this simply asserts that the wave motions of the medium are completely unaffected by the future interactions of waves with the boundaries and interfaces.

The reduction of the infinite series to a finite sum is a significant advantage of d'Alembert series representations of the solutions when one is interested in numerical solution for any arbitrary point in the elastic medium for a finite time interval. This means that the exact, and not a mere approximation, of the time-history can be plotted with minimal computational efforts.

An interesting special case arises when $f(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta distribution. In this case, we have

$$\begin{aligned} u^*(x,t) &= \frac{1}{\alpha}\sum_{n=0}^{\infty}\left[\Theta\left(t + \frac{x}{c} - \frac{(2n+1)H}{c}\right) + \Theta\left(t - \frac{x}{c} - \frac{(2n+1)H}{c}\right)\right], \\ T^*(x,t) &= \sum_{n=0}^{\infty}\left[\delta\left(t + \frac{x}{c} - \frac{(2n+1)H}{c}\right) - \delta\left(t - \frac{x}{c} - \frac{(2n+1)H}{c}\right)\right], \end{aligned} \tag{54}$$

for all $0 < x < H$ and $t > 0$. Here u^* and T^* are the influence functions, also known as one-sided Green's functions, see (Weinberger, 1995), of displacement and stress in the elastic layer. The second equation in (54) formally shows an infinite train of impulsive stress waves that travel back and forth in the elastic medium.

We can now focus on the special case 2 of our sample problem. One may notice from the first Eq. (32) that

$$\begin{aligned} \Psi_1(s) &= \frac{\cosh(as)}{\alpha_1 \sinh(bs) + \alpha_2 \cosh(bs)} \\ &= \frac{1}{\alpha_1 + \alpha_2}\frac{e^{-(b-a)s} + e^{-(b+a)s}}{1 - \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)e^{-2bs}}, \\ & 0 < a < b, \end{aligned} \tag{55}$$

where $a = x/c_1$ and $b = H/c_1$. Substituting $\kappa = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)$ and $\zeta = \kappa e^{-2bs}$, one can expand the expression $1/(1 - \zeta)$ as a geometric series $1/(1 - \zeta) = 1 + \zeta + \zeta^2 + \cdots = \sum_{n=0}^{\infty} \zeta^n$, which results in the general Dirichlet series representation for $\Psi_1(s)$ as:

$$\Psi_1(s) = \frac{1}{\alpha_1+\alpha_2}\sum_{n=0}^{\infty}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^n \left[e^{-(b-a+2nb)s} + e^{-(b+a+2nb)s}\right], \quad 0<a<b. \tag{56}$$

As before, the geometric series that is used here converge absolutely for $|\zeta|<1$ or, equivalently, $\Re(s) > (1/2b)\log|\kappa|$. Thus, the series representation (56) is valid for a right half-plane in the complex s-plane. Using Eqs. (47) and (56), the solution $u_1(x,t)$ for the case 2 is now expressible by the d'Alembert series:

$$u_1(x,t) = \frac{1}{\alpha_1+\alpha_2}\sum_{n=0}^{\infty}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^n \left[\tilde{f}\left\langle t + \frac{x}{c_1} - \frac{(2n+1)H}{c_1}\right\rangle + \tilde{f}\left\langle t - \frac{x}{c_1} - \frac{(2n+1)H}{c_1}\right\rangle\right], \tag{57}$$

for all $0<x<H$ and $t>0$. This physically shows an infinite succession of forward and backward travelling wavefronts in the upper layer. Returning to (32), one notes that the solution $u_2(x,t)$ can be easily obtained from $u_1(x,t)$ and the use of shifting Eq. (44) in the form of:

$$u_2(x,t) = u_1\left(H, t - \frac{x}{c_2} + \frac{H}{c_2}\right)\Theta\left(t - \frac{x}{c_2} + \frac{H}{c_2}\right). \tag{58}$$

Hence, by substituting Eq. (57) into (58), we have the d'Alembert series representation

$$u_2(x,t) = \frac{1}{\alpha_1+\alpha_2}\sum_{n=0}^{\infty}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^n \left[\tilde{f}\left\langle t - \frac{x}{c_2} - \frac{2nH}{c_1} + \frac{H}{c_2}\right\rangle + \tilde{f}\left\langle t - \frac{x}{c_2} - \frac{2(n+1)H}{c_1} + \frac{H}{c_2}\right\rangle\right], \tag{59}$$

for all $x>H$ and $t>0$. Physically, this solution indicates an infinite succession of wavefronts travelling only forward, i.e. in the positive direction of the x-axis, in the lower half-space. The stress wave functions can also be derived from Eqs. (57) and (59) by the stress-displacement relation $T_m(x,t) = \alpha_m c_m\, \partial u_m(x,t)/\partial x$ as

$$T_1(x,t) = \frac{\alpha_1}{\alpha_1+\alpha_2}\sum_{n=0}^{\infty}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^n \left[f\left\langle t + \frac{x}{c_1} - \frac{(2n+1)H}{c_1}\right\rangle - f\left\langle t - \frac{x}{c_1} - \frac{(2n+1)H}{c_1}\right\rangle\right], \tag{60}$$

for $0<x<H$ and $t>0$ and

$$T_2(x,t) = \frac{-\alpha_2}{\alpha_1+\alpha_2}\sum_{n=0}^{\infty}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^n \left[f\left\langle t - \frac{x}{c_2} - \frac{2nH}{c_1} + \frac{H}{c_2}\right\rangle + f\left\langle t - \frac{x}{c_2} - \frac{2(n+1)H}{c_1} + \frac{H}{c_2}\right\rangle\right], \tag{61}$$

for $x>H$ and $t>0$.

Our formal derivation of solutions via term-by-term Laplace transform inversion is in need of justification, as before. From Eq. (60), we note that $T_1(0,t)$ vanishes. In addition, from Eqs. (57) and (59) one can readily verify that $u_1(H,t) = u_2(H,t)$. Moreover, some manipulation of terms shows that $T_2(H,t) - T_1(H,t) = -f(t)$, for it is a telescoping series (Apostol, 1974) in such a way that all its other terms are cancelled out. It is readily seen that the initial conditions are satisfied trivially, as well. Eqs. (57) and (59) also satisfy one-dimensional wave equations for their respective layers. The reasoning is identical to that used for the special case 1.

Accordingly, the validity of the solutions (57) and (59) is now established.

Interestingly, the constant $\kappa = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)$ that appear in series (57), (59-61) satisfies $-1 < \kappa < 1$ when α_1 and α_2 are positive real numbers (as is the case here since they represent the mechanical impedances of the layer and the half-space, respectively). This means that $|\kappa| < 1$ and the sequence $\{|\kappa|^n\}$ is a strictly decreasing sequence of real numbers. Physically, this indicates the fact that wavefronts created in the elastic medium are subjected to a kind of decaying phenomenon, which is completely correct. The reason is that as wavefronts, which are travelling forward (in the positive direction of x-axis) in the upper layer, collide with the interface of the half-space, they are partly reflected back in the layer and partly transmitted to the lower half-space. The latter portion of the wavefront starts to travel to infinity in the half-space and never returns, meaning that a portion of the initial energy of the incident wave is, then, radiated away and lost. Since this form of energy loss occurs not continuously over time but only on specific moments when a transmission of waves to the half-space takes place, it is fundamentally different from the continuous energy dissipation which occurs, namely, in mechanical systems that have viscous dampers. The process of radiation repeats every time, when a wavefront in the layer collides the interface of the half-space. This kind of decaying phenomenon, which is due to the radiation of mechanical waves, is not observed in the case 1 as one may recognise in Eq. (52), for there is no half-space in that problem to let waves radiate away. The energy is, thus, fully *trapped* in the finite layer in the case 1.

Now, let us investigate in detail the case 3 of the problem. It is seen from Eqs. (34) and (35) that

$$\begin{aligned}\Psi_1(s) &= \frac{\cosh\left(\frac{x}{c_1}s\right)\sinh(\beta_2 s)}{\alpha_1\sinh(\beta_1 s)\sinh(\beta_2 s) + \alpha_2\cosh(\beta_1 s)\cosh(\beta_2 s)},\\ \Psi_2(s) &= \frac{\cosh(\beta_1 s)\sinh\left(\beta_2 s + \frac{H_1 - x}{c_2}s\right)}{\alpha_1\sinh(\beta_1 s)\sinh(\beta_2 s) + \alpha_2\cosh(\beta_1 s)\cosh(\beta_2 s)},\end{aligned} \tag{62}$$

where the definition of real constants β_1 and β_2 have been previously given in (27). By some algebraic manipulations, one may rewrite $\Psi_1(s)$ and $\Psi_2(s)$ as:

$$\begin{aligned}\Psi_1(s) &= \frac{\hat{h}(\beta_1,\beta_2)}{\alpha_1+\alpha_2}\left[e^{-\left(\beta_1 - \frac{x}{c_1}\right)s} + e^{-\left(\beta_1 + \frac{x}{c_1}\right)s} - e^{-\left(\beta_1 + 2\beta_2 - \frac{x}{c_1}\right)s} - e^{-\left(\beta_1 + 2\beta_2 + \frac{x}{c_1}\right)s}\right],\\ \Psi_2(s) &= \frac{\hat{h}(\beta_1,\beta_2)}{\alpha_1+\alpha_2}\left[e^{-\left(\frac{x-H_1}{c_2}\right)s} + e^{-\left(2\beta_1 + \frac{x-H_1}{c_2}\right)s} - e^{-\left(2\beta_2 - \frac{x-H_1}{c_2}\right)s} - e^{-\left(2\beta_1 + 2\beta_2 - \frac{x-H_1}{c_2}\right)s}\right],\end{aligned} \tag{63}$$

where

$$\hat{h}(\beta_1,\beta_2) = \left[1 - \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}e^{-2\beta_1 s} - \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}e^{-2\beta_2 s} + e^{-2\beta_1 s}e^{-2\beta_2 s}\right]^{-1}, \tag{64}$$

can be developed in the form of a general Dirichlet series. Obviously, a simple geometric series, as was used in the first two cases is not applicable here. However, multivariable Taylor expansions can be utilized here. To this end, one may make the substitutions $\zeta_1 = e^{-2\beta_1 s}$, $\zeta_2 = e^{-2\beta_2 s}$ and $\kappa = (\alpha_1 - \alpha_2)/(\alpha_1 + \alpha_2)$ in Eq. (64) to obtain a new function $h(\zeta_1,\zeta_2) = \hat{h}(\beta_1,\beta_2)$ given by

$$h(\zeta_1,\zeta_2) = \frac{1}{1-\kappa\zeta_1-\kappa\zeta_2+\zeta_1\zeta_2}. \tag{65}$$

A two-variable Taylor expansion of the function h at the vicinity of the point $(\zeta_1,\zeta_2)=(0,0)$ can now be written in the form of (Apostol, 1974)

$$h(\zeta_1,\zeta_2) = 1+\kappa\zeta_1+\kappa\zeta_2+\kappa^2\zeta_1^2+\kappa^2\zeta_2^2 + (2\kappa^2-1)\zeta_1\zeta_2+\cdots. \tag{66}$$

Therefore, $\hat{h}(\beta_1,\beta_2)$ can be written in the form of an infinite series given by

$$\hat{h}(\beta_1,\beta_2) = 1+\kappa e^{-2\beta_1 s}+\kappa e^{-2\beta_2 s} + \kappa^2 e^{-4\beta_1 s}+\kappa^2 e^{-4\beta_2 s} + (2\kappa^2-1)e^{-(2\beta_1+2\beta_2)s} + \cdots, \tag{67}$$

In which, with an appropriate rearrangement of terms, is a general Dirichlet series. Hence, looking back at the Eq. (63), it can be deduced that $\Psi_1(s)$ and $\Psi_2(s)$ are indeed expressible as general Dirichlet series

$$\Psi_1(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}, \qquad \Psi_2(s) = \sum_{n=0}^{\infty} b_n e^{-\mu_n s}. \tag{68}$$

The rest is similar to the previous cases and the solutions $u_1(x,t)$ and $u_2(x,t)$ are expressible in the form of d'Alembert series. Using the results in (63) and (67), and then by using the Eq. (47), one finds that

$$\begin{aligned} u_1(x,t) = \frac{1}{\alpha_1+\alpha_2}\Big[& \tilde{f}\left\langle t+\frac{x}{c_1}-\frac{H_1}{c_1}\right\rangle + \tilde{f}\left\langle t-\frac{x}{c_1}-\frac{H_1}{c_1}\right\rangle + (\kappa-1)\tilde{f}\left\langle t+\frac{x}{c_1}-\frac{H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle \\ & + (\kappa-1)\tilde{f}\left\langle t-\frac{x}{c_1}-\frac{H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle + \kappa\tilde{f}\left\langle t+\frac{x}{c_1}-\frac{3H_1}{c_1}\right\rangle + \kappa\tilde{f}\left\langle t-\frac{x}{c_1}-\frac{3H_1}{c_1}\right\rangle \\ & - \kappa\tilde{f}\left\langle t+\frac{x}{c_1}-\frac{3H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle - \kappa\tilde{f}\left\langle t-\frac{x}{c_1}-\frac{3H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle \\ & + (\kappa^2-\kappa)\tilde{f}\left\langle t+\frac{x}{c_1}-\frac{H_1}{c_1}-\frac{4H_2}{c_2}\right\rangle + (\kappa^2-\kappa)\tilde{f}\left\langle t-\frac{x}{c_1}-\frac{H_1}{c_1}-\frac{4H_2}{c_2}\right\rangle + \cdots \Big], \end{aligned} \tag{69}$$

for $0<x<H_1$ and $t>0$. In a similar manner one obtains

$$\begin{aligned} u_2(x,t) = \frac{1}{\alpha_1+\alpha_2}\Big[& \tilde{f}\left\langle t-\frac{x}{c_2}+\frac{H_1}{c_2}\right\rangle + (1+\kappa)\tilde{f}\left\langle t-\frac{x}{c_2}+\frac{H_1}{c_2}-\frac{2H_1}{c_1}\right\rangle - \tilde{f}\left\langle t+\frac{x}{c_2}-\frac{H_1}{c_2}-\frac{2H_2}{c_2}\right\rangle \\ & - \tilde{f}\left\langle t+\frac{x}{c_2}-\frac{H_1}{c_2}+\frac{2H_1}{c_1}+\frac{2H_2}{c_2}\right\rangle + \kappa\tilde{f}\left\langle t-\frac{x}{c_2}+\frac{H_1}{c_2}-\frac{4H_1}{c_1}\right\rangle \\ & - \kappa\tilde{f}\left\langle t+\frac{x}{c_1}-\frac{H_1}{c_2}-\frac{2H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle - \kappa\tilde{f}\left\langle t+\frac{x}{c_2}-\frac{H_1}{c_2}-\frac{4H_1}{c_1}-\frac{4H_2}{c_2}\right\rangle \\ & + \kappa\tilde{f}\left\langle t-\frac{x}{c_2}+\frac{H_1}{c_2}-\frac{2H_2}{c_2}\right\rangle + \kappa\tilde{f}\left\langle t-\frac{x}{c_1}+\frac{H_1}{c_2}-\frac{2H_1}{c_1}-\frac{2H_2}{c_2}\right\rangle \\ & - \kappa\tilde{f}\left\langle t+\frac{x}{c_2}-\frac{H_1}{c_2}-\frac{H_1}{c_2}-\frac{4H_2}{c_2}\right\rangle + \cdots \Big], \end{aligned} \tag{70}$$

for $H_1<x<H_1+H_2$ and $t>0$. The inverse Laplace transform for $U_1(x,s)$ and $U_2(x,s)$ in case 4 is somewhat similar to the case 3. Indeed, by investigating the Eqs. (37) and (38), one can find that

$$\begin{aligned} \Psi_1(s) &= \frac{1}{\Delta(s)}\cosh\left(\frac{x}{c_1}s\right)[\alpha_2\cosh(\beta_2 s)+\alpha_3\sinh(\beta_2 s)], \\ \Psi_2(s) &= \frac{1}{\Delta(s)}\cosh(\beta_1 s)\left[\alpha_2\cosh\left(\beta_2 s+\frac{H_1-x}{c_2}s\right)+\alpha_3\sinh\left(\beta_2 s+\frac{H_1-x}{c_2}s\right)\right], \end{aligned} \tag{71}$$

where

$$\Delta(s) = \alpha_1\alpha_2 \sinh(\beta_1 s)\cosh(\beta_2 s) + \alpha_1\alpha_3 \sinh(\beta_1 s)\sinh(\beta_2 s)$$
$$+\alpha_2\alpha_3 \cosh(\beta_1 s)\cosh(\beta_2 s) + \alpha_2^2 \cosh(\beta_1 s)\sinh(\beta_2 s). \quad (72)$$

After some algebraic manipulations, one may rewrite $\Psi_1(s)$ and $\Psi_2(s)$ in Eq. (71) as:

$$\Psi_1(s) = \frac{\hat{h}(\beta_1,\beta_2)}{\alpha_1+\alpha_2}\left[e^{-\left(\beta_1-\frac{x}{c_1}\right)s} + e^{-\left(\beta_1+\frac{x}{c_1}\right)s} + \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-\left(\beta_1+2\beta_2-\frac{x}{c_1}\right)s} + \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-\left(\beta_1+2\beta_2+\frac{x}{c_1}\right)s}\right],$$
$$\Psi_2(s) = \frac{\hat{h}(\beta_1,\beta_2)}{\alpha_1+\alpha_2}\left[e^{-\left(\frac{x-H_1}{c_2}\right)s} + e^{-\left(2\beta_1-\frac{x-H_1}{c_2}\right)s} + \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-\left(2\beta_2-\frac{x-H_1}{c_2}\right)s} + \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-\left(2\beta_1+2\beta_2-\frac{x-H_1}{c_2}\right)s}\right], \quad (73)$$

in which

$$\hat{h}(\beta_1,\beta_2) = \left[1 - \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}e^{-2\beta_1 s} + \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-2\beta_2 s} - \frac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3}e^{-2\beta_1 s}e^{-2\beta_2 s}\right]^{-1}. \quad (74)$$

As before, upon making the substitutions $\zeta_1 = e^{-2\beta_1 s}$, $\zeta_2 = e^{-2\beta_2 s}$, $\kappa_1 = (\alpha_1-\alpha_2)/(\alpha_1+\alpha_2)$ and $\kappa_2 = (\alpha_2-\alpha_3)/(\alpha_2+\alpha_3)$ one can define a new function $h(\zeta_1,\zeta_2) = \hat{h}(\beta_1,\beta_2)$ given by

$$h(\zeta_1,\zeta_2) = \frac{1}{1-\kappa_1\zeta_1+\kappa_2\zeta_2-\kappa_2\zeta_1\zeta_2}, \quad (75)$$

which can be written in the form of a two-variable Taylor series as it has been done for the case 3. This leads to development of $\Psi_1(s)$ and $\Psi_2(s)$ as general Dirichlet series. The representation of solutions $u_1(x,t)$ and $u_2(x,t)$ as d'Alembert series will then follow directly from the Eq. (47); the details are omitted here. Finally, by noticing the Eq. (39), the solution $u_3(x,t)$ can be immediately derived by simply using the shifting Eq. (44). It is noteworthy that if in either cases 3 or 4, one had $\beta_1 = \beta_2$ then a simpler single-variable Taylor series could be utilised instead of a two-variable Taylor series. This is easily seen by setting $\zeta_1 = \zeta_2 = \zeta$ in Eq. (65) or (75).

Looking back at the Eqs. (25), (26) and (28) it can be seen that three different parameters β_1, β_2 and β_3 appear in the denominators of functions $\Psi_1(s)$, $\Psi_2(s)$ and $\Psi_3(s)$ in the original sample problem that involves three different layers of finite thickness. It can be intuitively inferred that in this case, the problem can be analysed with the aid of a three-variable Taylor expansion. More generally, if N different parameters β_m $(m = 1, 2, \ldots, N)$ are present in a problem, one has to utilise an N-variable Taylor expansion. Accordingly, the analytical solution of any one-dimensional transient wave propagation problem in a multi-layered elastic medium is possible, in principle, with the methods developed in this paper.

A VERIFICATION EXAMPLE

We have examined in detail the solution of the related multi-layered IBVPs. It is also interesting at this stage to examine a simple example whose solution is given in standard references on the elastodynamic theory, for instance (Achenbach, 1975). This verification example, although very simple, would be beneficial for validation purposes of the methods hitherto developed in this paper. In this section the solutions for the wave motions (either longitudinal or transverse in-plane or anti-plane motions) of a homogeneous elastic half-space due to arbitrary surface excitation $f(t)$ are obtained.

Since the matrix method presented in Section 0 for obtaining the solution in Laplace domain was originally developed for media composed of elastic layers with finite depths, we first consider an elastic layer ($N =$

1) of finite depth H, upon which a time-variable uniform traction $f(t)$ is exerted (and thus $F_0(s) = F(s)$). The bottom of the elastic layer can be assumed either fixed on a rigid base ($Q_1(s)$=0) or, alternatively, traction-free ($F_1(s) = 0$). The choice makes no difference here since the limit of the solution in the case $H \to \infty$ needs to be considered ultimately. We, therefore, move on with the rigid base assumption $Q_1(s) = 0$. From the general matrix equation (19) one obtains the matrix equation

$$\begin{bmatrix} F(s) \\ F_1(s) \end{bmatrix} = \begin{bmatrix} X(s) & Y(s) \\ Y(s) & X(s) \end{bmatrix} \begin{bmatrix} Q_0(s) \\ 0 \end{bmatrix}, \tag{76}$$

from which $F(s) = X(s)Q_0(s)$ and thus from Eq. (16) we have

$$Q_0(s) = \frac{F(s)}{X(s)} = \frac{F(s)}{\alpha s \coth(\frac{H}{c}s)} = \frac{\sinh\left(\frac{H}{c}s\right)}{\alpha \cosh\left(\frac{H}{c}s\right)} \frac{F(s)}{s}, \tag{77}$$

where the subscript $m = 1$ has been omitted since the problem consists of a single layer only. Now, by using Eq. (21) with $Q_0(s)$ as above, $Q_1(s) = 0$ (as assumed earlier), $x_0 = 0$ and $x_1 = H$, one readily obtains

$$U(x, s) = \frac{\sinh\left(\frac{H-x}{c}s\right)}{\alpha \cosh\left(\frac{H}{c}s\right)} \frac{F(s)}{s}, \quad 0 < x < H, \tag{78}$$

which is the Laplace domain solution for an elastic layer of depth H, subjected to dynamic traction $f(t)$ on top and fixed on a rigid base at bottom. Finally, by taking the limit of the above solution as $H \to \infty$, the solution for an elastic half-space is acquired as:

$$U(x, s) = \frac{1}{\alpha} e^{-\frac{x}{c}s} \frac{F(s)}{s}, \qquad 0 < x < \infty. \tag{79}$$

Obviously, the term $e^{-\frac{x}{c}s}/\alpha$ above can be interpreted as a general Dirichlet series, which is degenerated to a single term. By a very simple Laplace transform inversion of the above solution via Eq. (47), the time domain solution for the half-space is obtained as:

$$u(x, t) = \frac{1}{\alpha} \tilde{f} \left\langle t - \frac{x}{c} \right\rangle, \qquad x, t > 0. \tag{80}$$

Therefore, the time domain solution is obtained as a d'Alembert series which is again degenerated to a single term (this term being a single wavefront travelling in the positive x-direction. This is indeed plausible since in the half-space, the wavefront generated by the disturbance at the surface does not ever get reflected and hence no new wavefronts can be generated in the medium.

Should one perceive $f(t)$ as a normal pressure exerted upon the surface of an elastic half-space, then longitudinal waves shall be generated in the half-space that travel with the speed $c_L = \sqrt{(\lambda + 2\mu)/\rho}$, λ and μ being the Lamé elastic constants of the medium. Therefore $c = c_L$ and $\alpha = \rho c_L$ in (80) and with the use of the definition of the function $\tilde{f}$ and the Macaulay brackets given in Eqs. (43) and (48) one may obtain an alternative representation of solution (80) as:

$$u(x, t) = \begin{cases} 0, & \text{for } t < \dfrac{x}{c_L}, \\ \dfrac{c_L}{\lambda + 2\mu} \displaystyle\int_0^{t - x/c_L} f(\tau) d\tau, & \text{for } t > \dfrac{x}{c_L}. \end{cases} \tag{81}$$

This form of solution exactly matches the closed-form solution given in (Achenbach, 1975) for a homogeneous elastic half-space, initially at rest and disturbed by a uniform pressure $f(t)$ on its boundary. The above solution was obtained by other methods in this reference. Accordingly, we have successfully validated the methods and notions developed in this paper by a simple verification example.

CONVERGENCE OF THE SERIES REPRESENTATIONS OF THE SOLUTIONS

Thus far, emphasis has been placed upon the effective methods for acquiring the Laplace domain and time domain solutions of the IBVPs formulated in Section 0 as infinite series. Specifically, we have used a formal term-by-term Laplace transform inversion of the Laplace domain series solutions to obtain the time domain series solutions. Although justifications of the final solutions have been made for some simpler cases (as given in paragraphs following Eqs. (53) and (61)), however a general discussion on the convergence of the Laplace and time domain solutions is obviously beneficial in terms of being mathematically sound and rigorous. It is therefore appropriate to devote this section to a thorough and general discussion on the convergence of our previously obtained series solutions.

It has been realised in the preceding section that for the IBVPs at hand, the general Dirichlet series expansion of the typical function $\Psi_m(s)$ in the form given in Eq. (42) can always been found. For simpler cases in the problems consisting of a single layer of finite depth (or a finite-depth layer resting on a half-space), this is simply done by a geometric series expansion. For more complicated cases consisting of several finite-depth layers, the general Dirichlet series expansion of $\Psi_m(s)$ can be found via a multi-variable Taylor series expansion. Therefore, power series expansions are effective in all cases and since all (non-trivial) power series absolutely converge in some region, the absolute convergence of $\Psi_m(s)$ and thus the Laplace domain solution $U_m(x,s)$ can be established in some region of the complex s-plane. For instance, it was shown that for the solutions of cases 1 and 2 of the sample problem that the Laplace domain series solutions absolutely converge in the half-planes $\Re(s) > 0$ and $\Re(s) > (1/2b)\log|\kappa|$, respectively, in the complex s-plane.

Based upon the above discussions and by inspecting the general form of Laplace domain solutions of several cases discussed in the previous section, it can be concluded that in all cases, the Laplace domain solution is expressible in the general form of:

$$U_m(x,s) = \Phi(s)\sum_{n=0}^{\infty} a_n e^{-(\omega+\xi n)s}. \tag{82}$$

which is but a special case of a representation (45) with λ_n replaced by $\omega + \xi n$ and the lower limit of the summation shifted to 0. Here, ω and ξ are real numbers in such a way that $\omega + \xi n \geq 0$ for all n. This is typical for the IBVPs studied in this paper. The above equation can be readily manipulated to obtain

$$\begin{aligned} U_m(x,s) &= \Phi(s)e^{-\omega s}\sum_{n=0}^{\infty} a_n\left(e^{-\xi s}\right)^n \\ &= \Phi(s)e^{-\omega s}\sum_{n=0}^{\infty} a_n \zeta^n \end{aligned} \tag{83}$$

where $\zeta = e^{-\xi s}$. Certainly, the infinite series $\sum_{n=0}^{\infty} a_n\zeta^n$ above is a power series in term of ζ and if it is non-trivial, it must have a radius of convergence $r > 0$ in the complex ζ-plane. Thus within the open disc $|\zeta| < r$, the Laplace domain solution (83) converge absolutely, with x being treated as a positive parameter as usual. Therefore, in the complex s-plane, the region of absolute convergence is the right half-plane $\Re(s) > (1/\xi)\log(1/r)$, since one has

$$\begin{aligned} |\zeta| = \left|e^{-\xi s}\right| &= \left|e^{-\xi(s_{\mathrm{R}}+is_{\mathrm{I}})}\right| \\ &= \left|e^{-\xi s_{\mathrm{R}}}\right|\left|e^{-i\xi s_{\mathrm{I}}}\right| \\ &= e^{-\xi s_{\mathrm{R}}} < r \end{aligned} \tag{84}$$

where $s_{\mathrm{R}} = \Re(s)$ and $s_{\mathrm{I}} = \Im(s)$ denote the real and imaginary parts of s, respectively. Accordingly, the absolute convergence of Laplace domain solutions is established for general case of N-layered IBVP.

On the other hand, the time domain solutions of the IBVPs in this research are expressed as d'Alembert series with terms that are of much more complicated and general forms. It can be seen from the time-domain solutions (52-53), (57), (59), (60-61), (69-70) that based upon the excitation function $f(t)$, the time domain d'Alembert series solutions can take a wide variety of complicated forms. Hence, a direct investigation of the general convergence properties of d'Alembert series may not be easily possible. Nevertheless, it is proposed that the convergence of the time domain series solutions can be investigated indirectly by establishing the uniform convergence of the original Laplace domain series solutions.

Looking back at (83), we again focus on the power series $\sum_{n=0}^{\infty} a_n \zeta^n$. It can be shown by means of Cauchy's convergence condition (Apostol, 1974) for infinite series that a power series in this form with a radius of (absolute) convergence equal to $r > 0$ also uniformly converge in every closed disc $|\zeta| \leq R$ in the complex ζ-plane, where $R < r$ (Kreyszig, 2011). Therefore, it follows that the series representing $U_m(x,s)$ in (82) also uniformly converge within every half-plane $\Re(s) \geq (1/\xi)\log(1/R)$, where $R < r$.

General theorems in complex analysis state that a uniformly convergent series of analytic functions, converges in some region to a sum (another function) which can be differentiated (or integrated) term-by-term (Ahlfors, 1966). The series $\sum_{n=0}^{\infty} a_n \left(e^{-\xi s}\right)^n$ in Eq. (83) thus converges uniformly to an analytic function, because it is composed of analytic terms. In addition, since the excitation functions $f(t)$ (or $q(t)$) have been assumed to be of exponential order as $t \to \infty$ (so that their Laplace transforms exist), the function $\Phi(s) = F(s)/s$ (or $Q(s)$) is analytic in some right half-plane of the complex s-plane (Churchill, 1958). Finally, since $e^{-\omega s}$ in Eq. (83) is also analytic, the resulting sum $U_m(x,s)$ in Eq. (83) is indeed analytic (in some right half-plane). Now the complex-valued function

$$\frac{1}{2\pi i} U_m(x,s) e^{st} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} a_n e^{-(\omega+\xi n)s} \Phi(s) e^{st}, \tag{85}$$

is again analytic in some right-half plane in the complex s-plane; the convergence being uniform. The inversion Eq. (7) is nothing more than a complex integration along the Bromwich contour of the above function. Since each term of the corresponding series in Eq. (85) is analytic in some right half-plane, a sufficiently large positive number γ for the Bromwich contour can always be found and thus inverse Laplace transform (Bromwich integration) can be taken term-by-term. When this is done, the left hand side of the above equation becomes $u_m(x,t)$ and the right-hand side becomes the corresponding d'Alembert series solution in the time domain. In practice, it is easier to use the series representation (82) for $U_m(x,s)$ and the shifting property of the Laplace transform to obtain $u_m(x,t)$, as extensively examined in the previous section. The foregoing discussions show that the d'Alembert series solutions do indeed converge. Accordingly, we have indirectly established the convergence of series solutions in the time domain.

ILLUSTRATIONS OF THE NUMERICAL COMPUTATIONS

In order to acquire a better physical conception of the exact solutions derived in the previous sections, it is helpful to plot a few of them for some special forms of traction functions $f(t)$ and specific values of layer thickness, densities and elastic moduli.

Generally, the time-history of stress and displacement at any point in a multi-layered elastic medium is greatly influenced by the type of excitation, mechanical properties and the geometry of layers. Not only the nature of reflection and transmission of wavefronts at an interface of two layers depends upon the ratio of the mechanical impedance of the two layers (Achenbach, 1975), but also the thickness of layers and the geometry of the medium significantly influence the overall wave propagation pattern in the medium. For instance, a relatively thin layer in a medium gives rise to more frequent wavefront collisions with the interface planes and thus a wave motion that is oscillatory with higher frequencies even if the excitation is not of oscillatory nature. In addition, as it has been discussed in the previous sections, the presence of a layer having infinite thickness (a half-space) beneath the other layers, gives rise to an eventual decay in time of all wave motions in the medium due of the phenomenon of one-dimensional mechanical radiation.

We first consider the results of case 1 of the sample problem. The function $f(t)$ can be chosen arbitrarily. Here, a function $f(t)$ with an irregular time-history, typical of earthquake excitations, is used (as plotted in Figure 7). The function $f(t)$ has been fabricated here as an analytic expression using simpler trigonometric, rational and exponential functions in the form of (in MPa)

$$\begin{aligned} f(t) = \operatorname{sinc}\big(10(t-1)\big) & - 0.5\operatorname{sinc}\big(20(t-1.5)\big) \\ & + 0.1\,e^{-t}\sin(50t) \\ & + \operatorname{sinc}\big(50(t-3)\big) \\ & - \operatorname{sinc}\big(30(t-2)\big) \\ & - 0.8\operatorname{sinc}\big(60(t-1)\big) \\ & + \operatorname{sinc}\big(100(t-1.3)\big) \\ & + 0.1\frac{\sin(200t)}{t^2+1} \\ & + 0.2e^{-0.6t}\cos^2(120t), \end{aligned} \tag{86}$$

where $\operatorname{sinc}(x) = \sin(x)/x$ denotes the cardinal sine function. A numerical table of data instead, could be used for $f(t)$. The mechanical and geometric properties of the elastic layer can also be chosen arbitrarily. For instance, here the values $\rho = 1800\ \text{kg/m}^3$, $c = 800\ \text{m/s}$ and $H = 100\ \text{m}$ are chosen. It is of primary engineering interest to have the time-history of stress at an arbitrary point in the medium. Here, for instance, the time-history of stress at the point $x = H/2$ (midpoint of the layer) is plotted in Figure 8. As illustrated in this figure, no energy dissipation occurs in case 1 and the medium is set to a state of perpetual oscillatory motion by the excitation function.

Case 2 of the sample problem considers the case where a finite layer is situated upon an elastic half-space. For the purpose of comparison, we use the same excitation function and the same mechanical and geometric properties for the top layer. Thus $\rho_1 = 1800\ \text{kg/m}^3$, $c_1 = 800\ \text{m/s}$ and $H = 100\ \text{m}$. A somewhat denser and stiffer half-space is chosen here with the parameters of $\rho_2 = 2000\ \text{kg/m}^3$ and $c_2 = 1000\ \text{m/s}$. The time-history of stress at the points $x = H/2$ (midpoint of the layer) and $x = 10H$ (deep into the half-space) are plotted in Figures 9 and 10, respectively.

As can be seen in the both figures, the wave motions are significantly different from those of case 1, since in the case 2, the intensities of waves become vanishingly small, eventually, due to the mechanical radiation of wavefronts via the half-space. Figure 10 vividly manifests the latency in arrival of wavefronts to the points that are relatively far away from the source of excitation.

For cases 3 and 4 of the sample problem, another excitation function defined as:

$$f(t) = \begin{cases} 10\ \text{MPa}, & 0 < t < 0.01, \\ 0, & \text{otherwise}, \end{cases} \tag{87}$$

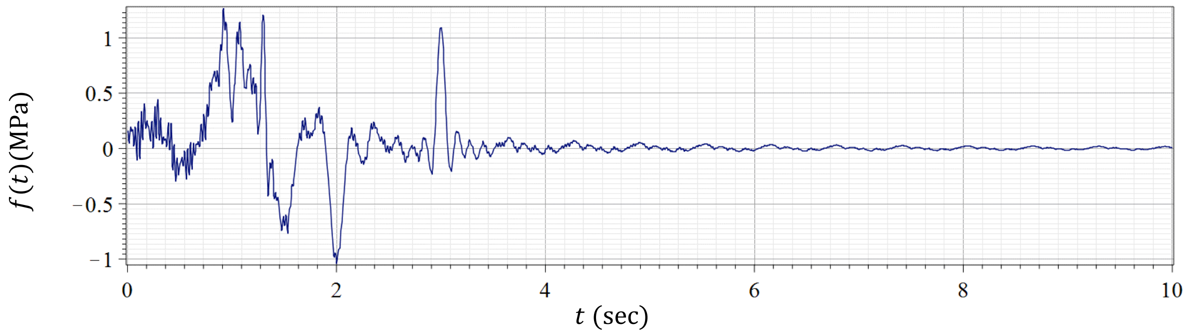

Fig. 7. Time-history of the excitation function $f(t)$

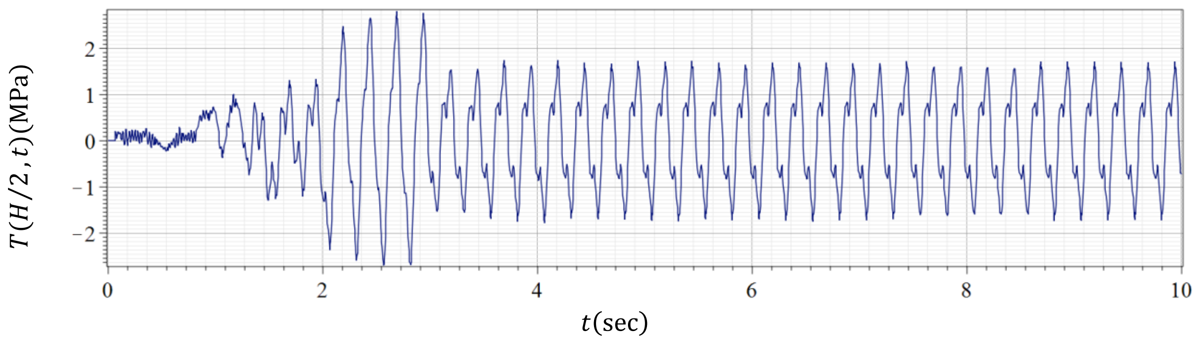

Fig. 8. Time-history of $T(H/2, t)$ in case 1

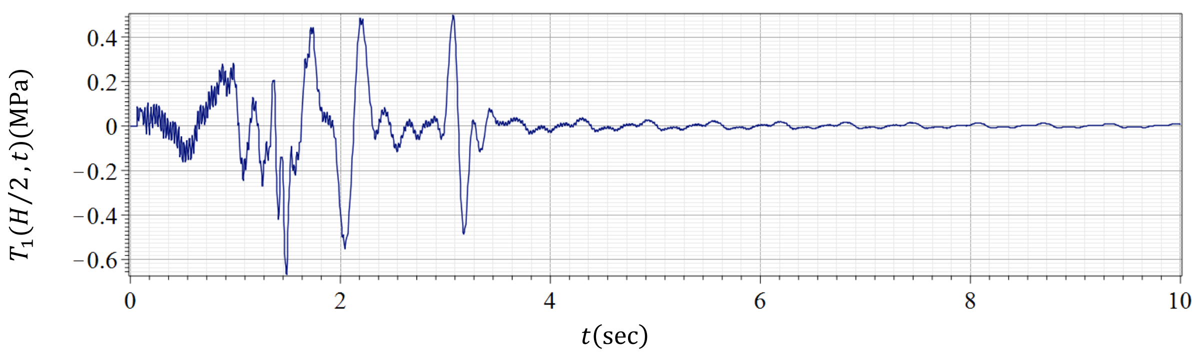

Fig. 9. Time-history of $T_1(H/2, t)$ in case 2

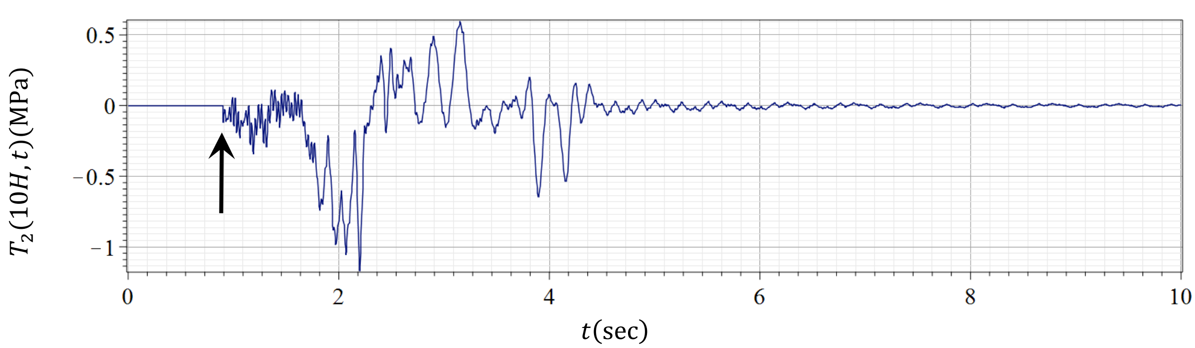

Fig. 10. Time-history of $T_2(10H, t)$ in case 2. The arrow indicates the arrival of the first wavefront at $t = 0.9$ sec

is used. The function $f(t)$ here is, thus, a simple impulsive traction of rectangular shape and of a very short duration. The mechanical properties and the geometry of the two upper layers in cases 3 and 4 are as follows: $\rho_1 = 1800\ \mathrm{kg/m^3}$, $c_1 = 800\ \mathrm{m/s}$, $\rho_2 = 2000\ \mathrm{kg/m^3}$, $c_2 = 1000\ \mathrm{m/s}$, $H_1 = 100\ \mathrm{m}$ and $H_2 = 50\ \mathrm{m}$. The mechanical properties of the half-space in case 4 are given by $\rho_3 = 2300\ \mathrm{kg/m^3}$ and $c_3 = 1300\ \mathrm{m/s}$.

Figure 11 illustrates the time-history of displacement at point $x = 0$ (free surface of the medium) in case 3. As shown in this figure, an impulsive excitation of a very simple shape can induce a complicated pattern of wave motion in a multi-layered medium. This behaviour results from interferences of individual wavefronts. Increasing the number of the layers results in more elaborate patterns of transient motions.

As a final illustration, Figure 12 depicts the displacement time-history of the point $x = 0$ (free surface of the medium) in case 4 of the sample problem due to the same impulsive excitation function $f(t)$ used in case 3. It is again obvious in this figure that the wave motions become vanishingly small after a short period of time because of energy loss through the lower half-space. The residual constant displacement seen in Figure 12 stems from the rigid motion of the whole medium. This was expected since the IBVP defined in case 4 of the sample problem has Neumann conditions (Kreyszig, 2011; Brown and Churchill, 2012) throughout its boundary.

The illustrations for results of cases 3 and 4 of the sample problem demonstrate that even a simple impulsive excitation can induce complicated patterns of wave motion in a multi-layered medium.

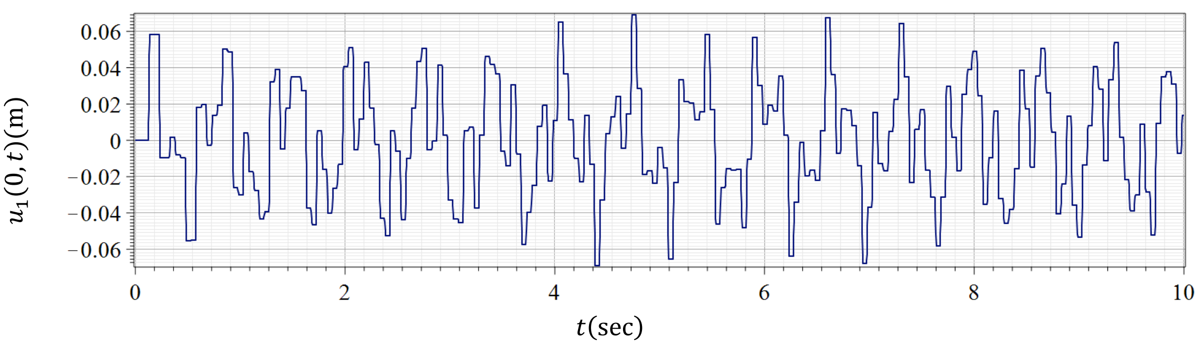

Fig. 11. Time-history of $u_1(0,t)$ in case 3

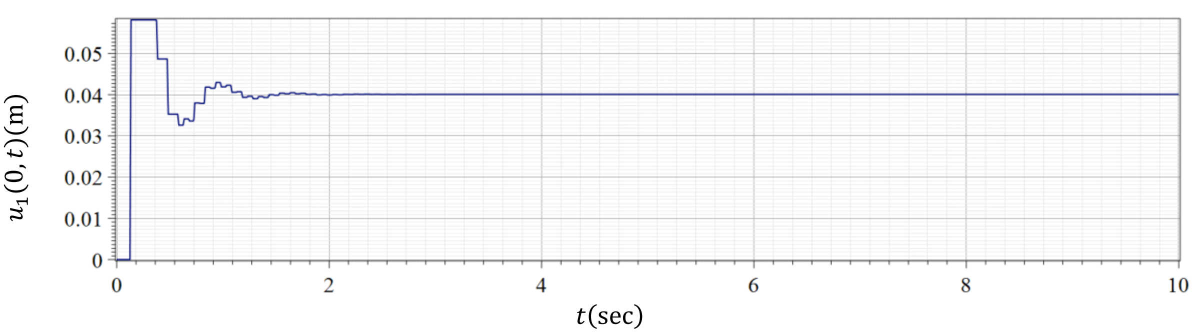

Fig. 12. Time-history of $u_1(0,t)$ in case 4

CONCLUSIONS

Transient one-dimensional wave propagation in a multi-layered elastic medium has been considered in this paper, where the source of excitation is either arbitrarily prescribed tractions or displacements at the interfaces and/or boundaries. The IBVP has been solved with the use of Laplace integral transform. Via the systematic method presented in this paper, one is able to find the solution of any multi-layered problem of one-dimensional wave propagation with arbitrary configurations in the Laplace domain. The Laplace transform inversion has been taken analytically with the use of expansions of complex-valued functions in the Laplace domain (complex s-plane) in the form of general Dirichlet series. This, in turn, leads to an elegant representation of exact solutions in time domain in the form of d'Alembert series, i.e. an infinite succession of d'Alembert type wave functions of varying amplitudes which travel back and forth in the medium. The methods presented here can be extended for three-dimensional wave propagation problems. It is emphasised that the infinite series of d'Alembert type wave functions is reduced to a finite sum for finite time in which, one is usually interested. The solutions have been verified by justification of the resulting formulae with regard to the boundary and continuity conditions. In addition, the convergence of the series solutions in both Laplace domain and time domain were examined. An increase in the number of layers gives rise to solutions of more complicated forms. It has been shown via illustrations that the time-histories of stress or displacement of the medium are highly dependent upon the geometrical and mechanical properties of layers as well as the general nature of the excitation functions.

ACKNOWLEDGEMENT

The partial support from the University of Tehran through Grant No. 27840/1/08 to the second author during this work is gratefully acknowledged. The authors would also like to express their sincere gratitude to the reviewers, whose valuable comments and suggestions led to certain improvements in the manuscript.

REFERENCES

Achenbach, J.D. (1975). *Wave propagation in elastic solids*, Applied Mathematics and Mechanics, Vol. 16, Amsterdam: North-Holland, Elsevier Science Publishers.

Achenbach, J.D., Hemann, J.H. and Ziegler, F. (1968). "Tensile failure of interface bonds in a composite body subjected to compressive loads", *AIAA Journal* (*The American Institute of Aeronautics and Astronautics*), 6(10), 2040-2043.

Ahlfors, L.V. (1966). *Complex analysis*, International Series in Pure and Applied Mathematics, Second Edition, McGraw-Hill, Inc.

Apostol, T.M. (1974). *Mathematical analysis.* Second Edition. Addison-Wesley Publishing Company, Inc.

Ardeshir-Behrestaghi, A., Eskandari-Ghadi, M. and Vaseghi-Amiri, J. (2013). "Analytical solution for a two-layer transversely isotropic half-space affected by an arbitrary shape dynamic surface load", *Civil Engineering Infrastructures Journal*, 46(1), 1-14.

Beddoe, B. (1966). "Vibration of a sectionally uniform string from an initial state", *Journal of Sound and Vibration*, 4(2), 215-223.

Berlyand, L. and Burridge, R. (1995). "The accuracy of the O'Doherty-Anstey Approximation for wave propagation in highly disordered stratified media", *Wave Motion*, 21(4), 357-373.

Boström, A. (2000). "On wave equations for elastic rods", *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 80(4), 245-51.

Brown, J.W. and Churchill, R.V. (2012). *Fourier series and boundary value problems*, 8th Edition, McGraw-Hill, Inc.

Bruck, H.A. (2000). A one-dimensional model for designing functionally graded materials to manage stress waves", *International Journal of Solids and Structures*, 37(44), 6383-6395.

Burridge, R. (1988). "One-dimensional wave propagation in a highly discontinuous medium", *Wave Motion,* 10(1), 19-44.

Cheshmehkani, S. and Eskandari-Ghadi, M. (2017). "Three-dimensional dynamic ring load and point load Green's functions for continuously inhomogeneous viscoelastic transversely isotropic half-space", *Engineering Analysis with Boundary Elements*, 76, 10-25.

Cheung, Y.K., Tham, L.G. and Lei, Z.X. (1995). "Transient response of single piles under horizontal excitations", *Earthquake Engineering and Structural Dynamics*, 24(7), 1017-1038.

Chiu, T.C. and Erdogan, F. (1999). "One-dimensional wave propagation in a functionally graded elastic medium", *Journal of Sound and Vibration,* 222(3), 453-487.

Churchill, R.V. (1936). "Temperature distribution in a slab of two layers", *Duke Math Journal*, 2(1), 405-414.

Churchill, R.V. (1937). "The inversion of Laplace transformation by a direct expansion in series and its application to boundary-value problems", *Mathematische Zeitschrift,* 42(1), 567-579.

Churchill, R.V. (1958). *Operational mathematics*, Second Edition, McGraw-Hill.

Cohen, A.M. (2007). *Numerical methods for Laplace transform inversion,* Springer.

De Hoop, A.T. (1960). "A modification of Cagniard's method for solving seismic pulse problems", *Applied Scientific Research, Section B,* 8(1), 349-356.

Duffy, D.G. (1993). 'On the numerical inversion of Laplace transforms: Comparison of three new methods on characteristic problems from applications", *ACM Transactions on Mathematical Software (TOMS),* 19(3), 333-359.

Durbin, F. (1974). Numerical inversion of Laplace transforms: An efficient improvement to Dubner and Abate's method", *The Computer Journal*, 17(4), 371-376.

Eskandari-Ghadi, M., Hassanpour Charmhini, A. and Ardeshir-Behrestaghi, A. (2014). "A method of function space for vertical impedance function of a circular rigid foundation on a transversely isotropic ground", *Civil Engineering Infrastructures Journal*, 47(1), 13-27.

Eskandari-Ghadi, M., Rahimian, M., Mahmoodi, A. and Ardeshir-Behrestaghi, A. (2013). "Analytical solution for two-dimensional coupled thermoelastodynamics in a cylinder", *Civil Engineering Infrastructures Journal,* 46(2), 107-123.

Eskandari-Ghadi, M. and Sattar, S. (2009). "Axisymmetric transient waves in transversely isotropic half-space", *Soil Dynamics and Earthquake Engineering,* 29, 347-355.

Hardy, G.H. (1915). *The general theory of Dirichlet's series,* Cambridge Tracts in Mathematics and Mathematical Physics, No. 18, Cambridge: Cambridge University Press.

Hassani, S. (2013). *Mathematical physics: A modern introduction to its foundations,* 2nd Edition, Springer International Publishing Switzerland.

Kobayashi, M.H. and Genest, R. (2014). "On an extension of the d'Alembert solution to initial–boundary value problems in multi-layered, multi-material domains", *Wave Motion*, 51(5), 768-784.

Kreyszig, E. (2011). *Advanced engineering mathematics,* Tenth Edition, John Wiley & Sons, Inc.

Lang, S. (1999). *Complex analysis,* Graduate Texts in Mathematics, 103, Fourth Edition, Springer.

Lin, W.H. and Daraio, C. (2016). Wave propagation in one-dimensional microscopic granular chains", *Physical Review E*, 94(5), 052907(1-6).

Macaulay, W.H. (1919). "Note on the deflection of beams. *The Messenger of Mathematics*, 48, 129-130.

O'Doherty, R.F. and Anstey, N.A. (1971). "Reflections on amplitudes", *Geophysical Prospecting,* 19(3), 430-458.

Ponge, M.F. and Croënne, C. (2016). "Control of elastic wave propagation in one-dimensional piezomagnetic phononic crystals, *The Journal of the Acoustical Society of America*, 139(6), 3288-3295.

Raoofian Naeeni, M. and Eskandari-Ghadi, M. (2016). "A potential method for body and surface wave propagation in transversely isotropic half- and full-spaces", *Civil Engineering Infrastructures Journal,* 49(2), 263-288.

Raoofian Naeeni, M. and Eskandari-Ghadi, M. (2016). "Analytical solution of the asymmetric transient wave in a transversely isotropic half-space due to both buried and surface impulses", *Soil Dynamics and Earthquake Engineering,* 81, 42-57.

Shafiei, M. and Khaji, N. (2015). "An adaptive physics-based method for the solution of one-dimensional wave motion problems", *Civil Engineering Infrastructures Journal*, 48(2), 217-234.

Sneddon, I.N. (1995). *Fourier transforms,* New York: Dover Publications, Inc., (reprinted).

Thambiratnam, D.P. (1986). "Transient waves in a rod subjected to impulsive end loading", *Earthquake Engineering and Structural Dynamics*, 14(3), 475-485.

Van Der Hijden, J.H. (2016). *Propagation of transient elastic waves in stratified anisotropic media,* Applied Mathematics and Mechanics, Vol. 32, Amsterdam: North-Holland, Elsevier Science Publishers.

Weinberger, H.F. (1995). *A first course in partial differential equations with complex variables and transform methods,* New York: Dover Publications, Inc. (reprinted).

Yang, H. and Yin, X. (2015). "Transient responses of girder bridges with vertical poundings under near-fault vertical earthquake", *Earthquake Engineering & Structural Dynamics,* 44(15), 2637-2657.

Yang, K. (2008). "A unified solution for longitudinal wave propagation in an elastic rod", *Journal of Sound and Vibration,* 314(1), 307-329.

NOMENCLATURE

u:	transverse or normal elastic displacement	T:	shearing or normal stress component
x:	position (depth) variable	t:	time variable
c:	mechanical wave speed	ρ:	mass density
H:	thickness of the layer	N:	total number of the layers
α:	mechanical impedance ($\alpha = \rho c$)	β:	the ratio H/c in a layer
q:	displacement at boundary/interface	f:	traction at boundary/interface
$\mathfrak{L}$:	Laplace transform operator (with respect to t)	$\mathfrak{L}^{-1}$:	inverse Laplace transform operator
s:	complex parameter of Laplace transform	U:	Laplace transform of u
Q:	Laplace transform of q	F:	Laplace transform of f
Θ:	Heaviside unit step function	δ:	Dirac delta distribution
$\mathfrak{R}, \mathfrak{I}$:	real and imaginary parts of a complex number	$(\cdot)^{\mathrm{T}}$:	transpose of a matrix