Earth Observation and Geomatics Engineering 1(2) (2017) 71–81

EARTH OBSERVATION AND GEOMATICS ENGINEERING
website: https://eoge.ut.ac.ir

The generalized F and G series for the satellite orbit propagation

Mohammad Reza Seif *

Department of Civil Engineering, Imam Hossein University (IHU), Tehran, Iran

Article history:
Received: 16 March 2017, Received in revised form: 4 September 2017, Accepted: 20 September 2017

ABSTRACT

In this paper, an advanced version of the Lagrange method, F and G series, is proposed for the many applications in the celestial mechanics and space science such as initial orbit determination and satellite orbit propagation. In this development, the Lagrange coefficients were developed from a gravitational field of an inhomogeneous attractive body to all the perturbing accelerations acting on an orbiter. The efficiency of the method is tested for the satellite orbit propagation. This assessment is based on the comparison between the Lagrange solution and the analytical one for Keplerian motion and numerically integrated orbit for non-Keplerian motion. The discrepancy at centimeter and sub-centimeter accuracy shows the performance of the developed algorithm for MEO and LEO satellites orbit propagation. The results of computational time showed that the Lagrange method is as time-consuming as the multi-step methods where it is faster than the single-step methods. Besides the CPU-time, the stability test of the Lagrange method shows that it is as stable as the single-step and is more stable than the multi-step methods at the equivalent orders. Therefore, the Lagrange method offers the advantages of the single- and multi-step methods.

KEYWORDS

Lagrange coefficients
Satellite
Orbit Propagation
F and G functions
LEO satellite
MEO satellites

1. Introduction

The dynamic orbit propagation is still one of the most significant discussions in the satellite geodesy and celestial mechanics. Dynamic orbit is the solution of the equations of motion of satellites or celestial bodies without using any observations (Seeber, 2003). In general, different numerical methods have been categorized into the single- and multi-step integrators to solve the Initial Value Problem (IVP) such as the equations of motion. As an alternative, the Taylor series could be used to solve the IVP. The advantages of the Taylor series over other numerical methods were demonstrated by Montenbruck (1992). Like the multi-step methods, only one function evolution per step is required for the series expansion. Then, increasing the order (terms of the Taylor series) does not impressively change the CPU-time. Like the single-step methods, increasing the order leads to the stability increase and accuracy improvements in the Taylor series approach. Therefore, the Taylor series method combines all the advantages of the single- and multi-step methods with the additional freedom to choose the order in accordance with the runtime and accuracy requirements (Montenbruck, 1992). In this paper, we try to solve the orbit propagation problem using the Taylor series method.

The Taylor series representation of the Lagrange F and G functions are called F and G series. The Lagrange method is based on the expansion of the solution of the equation of motion into F and G series (Beutler, 2004). The Lagrange method has been traditionally used for the celestial body orbit determination using ground-based observations in the classic celestial mechanics based on the hypothesis of the Keplerian motion of satellite (Curtis, 2005; Escobal, 1965).

* Corresponding author
E-mail addresses: m.r.seif@ut.ac.ir (M.R. Seif)
DOI: 10.22059/eoge.2017.235787.1008

The initial orbit determination methods like Gauss and Lambert are formulated based on the F and G Taylor series (Escobal, 1965, Bem & Szczodrowska-Kozar, 1995). In (Sconzo et al., 1965), the expansion series of the F and G were symbolically determined in a Keplerian gravitational field using FORMAC computer program. A recursive formulation of the F and G series was developed for Keplerian motion by (Bond, 1985). (Bem & Szczodrowska-Kozar, 1995) derived expansion coefficients of the F and G series up to the degree 20 in the central field. They determined the coefficients for the two- and three-body problems. The idea was implemented for reduced-dynamic orbit determination of the LEO satellites using the GPS-based observations in (Feng, 2001).

Very few attempts have been executed for generalizing this method from the Keplerian motion to the perturbed motion. As a first try, the Lagrange coefficients have been developed by (Lin & Xin, 2003) by taking into account the Earth's oblateness. In that research, the Laplace's method of the initial orbit determination using the angular observations was developed based on the F and G series. (M. A. Sharifi & Seif, 2011) have developed the Lagrange coefficients from J2 field (the field of the attractive body by considering oblateness) to the gravitational field of an inhomogeneous attracting body (full gravitational field). These coefficients are only restricted to the gravitational field of an attractive body and other perturbations have to be taken into account. In addition to the satellite orbit propagation and initial orbit determination, this method could be utilized to solve various problems such as the 3-body problem (Steffensen, 1956) and N-body problem (Broucke, 1971). As another application, the Lagrange method could represent the continuous solution for the equations of motion over a time-span. It makes the method well suited for problems requiring dense output e.g., ephemeris calculation (Montenbruck, 1992). The method of the Lagrange might be classified as a semi-analytical approach based on the terms of the F and G series that we could approach into analytical solution by using the highest-order series. This new semi-analytical formulation could be used for the satellite motion analysis (Sharifi et al., 2013). Recently, the Lagrange coefficients are extended to solve the Stark problem (Pellegrini et al., 2014). In this paper, an attempt is made to represent a formulation to include the non-static forces (non-Earth gravitational and non-gravitational perturbations) into the Lagrange coefficients. Besides, the stability test of the Lagrange method has been carried out and compared with the numerical integrators. To analyze the accuracy of the method, the orbit propagated using the Lagrange method was compared with the numerically integrated one.

2. Methodology

This section begins by giving a brief overview of the basic formulation of the Lagrange coefficients as a base for generalizing the Lagrange method. It continues by representing the algorithm of the Lagrange coefficients computation. This algorithm is divided into two parts. At the first step, the Lagrange coefficients are computed in the Earth's gravitational field and the other perturbations are added in the next step.

2.1 Basic formulation

Different approaches have been introduced for solving the equations of motion of a satellite. The method of the Lagrange coefficients is classically used for orbit determination of the planets and celestial bodies. The solution of the orbit propagation problem is equivalent to the determination of the Lagrange coefficients (Goodyear, 1965). In general, due to the presence of the non-Keplerian terms, the classical representation of the scalar Lagrange coefficients is rewritten in a matrix form as follows (M. A. Sharifi & Seif, 2011):

$$\begin{cases} \underline{r}(t) = \underline{F}(t)\underline{r}(t_0) + \underline{G}(t)\underline{\dot{r}}(t_0) \\ \underline{\dot{r}}(t) = \underline{\dot{F}}(t)\underline{r}(t_0) + \underline{\dot{G}}(t)\underline{\dot{r}}(t_0) \end{cases} \tag{1}$$

where $\underline{F}(t)$ and $\underline{G}(t)$ are the matrices of the Lagrange coefficients reformulated for the orbital motion in a non-Keplerian gravitational field.

$$\begin{aligned} \underline{F}(t) &= \sum_{q=0}^{\infty} \frac{1}{q!} \underline{F}_{(q)} \Big|_{t=t_0} (t-t_0)^q \\ \underline{G}(t) &= \sum_{q=0}^{\infty} \frac{1}{q!} \underline{G}_{(q)} \Big|_{t=t_0} (t-t_0)^q \end{aligned} \tag{2}$$

And their time-derivatives are:

$$\begin{aligned} \underline{\dot{F}}(t) &= \sum_{q=1}^{\infty} \frac{1}{(q-1)!} \underline{F}_{(q)} \Big|_{t=t_0} (t-t_0)^{q-1} \\ \underline{\dot{G}}(t) &= \sum_{q=1}^{\infty} \frac{1}{(q-1)!} \underline{G}_{(q)} \Big|_{t=t_0} (t-t_0)^{q-1} \end{aligned} \tag{3}$$

The single and double dotted variables indicate the first and second-time derivatives respectively. For higher order time derivatives of order q, superscript variable in parenthesis is used. In general, q-order time derivatives of position vector are derived as functions of position and velocity vectors at epoch t_0 with linear combination coefficients of $\underline{F}_{(q)}$ and $\underline{G}_{(q)}$ as follows:

$$\frac{\partial^q \underline{r}(t)}{\partial t^q} = \underline{F}_{(q)}(t)\underline{r}(t) + \underline{G}_{(q)}(t)\underline{\dot{r}}(t) \tag{4}$$

By differentiating Eq. (4) with respect to t:

$$\frac{\partial^{(q+1)}\underline{r}(t)}{\partial t^{(q+1)}} = \underline{\dot{F}}_{(q)}(t)\underline{r} + \underline{F}_{(q)}(t)\underline{\dot{r}} + \underline{\dot{G}}_{(q)}(t)\underline{\dot{r}} + \underline{G}_{(q)}(t)\underline{\ddot{r}} \quad (5)$$

As an alternative, the $(q+1)$-order derivative could be easily obtained by advancing the sequence in Eq. (4) as:

$$\frac{\partial^{(q+1)}\underline{r}(t)}{\partial t^{(q+1)}} = \underline{F}_{(q+1)}(t)\underline{r} + \underline{G}_{(q+1)}(t)\underline{\dot{r}} \quad (6)$$

The comparison between Eq. (4) and Eq. (6) gives the recursive coefficients definitions:

$$\begin{aligned} \underline{F}_{(q+1)}(t) &= \underline{\dot{F}}_{(q)}(t) + \underline{G}_{(q)}(t)\underline{F}_{(2)} \\ \underline{G}_{(q+1)}(t) &= \underline{F}_{(q)}(t) + \underline{\dot{G}}_{(q)}(t) + \underline{\dot{G}}_{(q)}\underline{G}_{(2)} \end{aligned} \quad (7)$$

Eq. (7) is the fundamental formula for computing $\underline{F}_{(q)}$ and $\underline{G}_{(q)}$. The initial terms are $\underline{F}_{(0)} = \underline{I}$ and $\underline{G}_{(0)} = \underline{O}$, where $\underline{I}$ is identity and $\underline{O}$ is zeros matrices. The next terms of this sequence could be easily generated from Eq. (7), $\underline{F}_{(1)} = \underline{O}$ and $\underline{G}_{(1)} = \underline{I}$. To find the next terms of the sequence, we need some information about the acceleration acting on the orbiter, static and non-static forces. By decomposing the perturbing acceleration into position and velocity vectors, $\underline{F}_{(2)}$ and $\underline{G}_{(2)}$ could be resulted, e.g. in the central gravity field $\underline{F}_{(2)} = -GM/r^3$ and $\underline{G}_{(2)} = \underline{O}$

2.2 Formulation in the static gravitational field

The Lagrange method was previously extended to the Earth's gravitational field. However, it needs a little development to consider all of the Earth's Orientation Parameters in the orbit propagation process. In this section, it has been tried to represent the new formulation for the full gravitational field as summary as possible.

Based on the equation represented in (M. A. Sharifi & Seif 2011), the Earth's gravitational acceleration in the Earth-fixed frame (ECF) could be formulated as:

$$\underline{\ddot{r}}_g^{ECF} = [(a-b)\underline{I} + \underline{A}]\underline{r}^{ECF} \quad (8)$$

where $\underline{I}$ is identity matrix and $\underline{A}$ is a diagonal matrix with three different diagonal entries.

$$\underline{A} = \begin{bmatrix} -l & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & e \end{bmatrix} \quad (9)$$

The first and second terms in Eq. (8) show the radial and out-of-plane components of the gravitational acceleration. The unknown coefficients of Eqs. (8) and (9) are expressed in terms of the position and the partial derivatives of gravitational potential towards the curvilinear coordinates; see Appendix A for more details. It should be noted that the Newton's law used for solving the equations of motion is only valid in an inertial reference frame. Therefore, the gravitational acceleration of the Earth has to be transferred from the Earth-fixed (ECF) to the Earth-centered inertial frame (ECI) to solve the equations of motion. According to the well-known relationship of these two reference frames, one can write (Montenbruck & Gill, 2000, p. 247):

$$\underline{\ddot{r}}_g^{ECI} = \underline{T}(t)^{-1}\underline{\ddot{r}}_g^{ECF} = \underline{T}(t)^{-1}[(a-b)\underline{I} + \underline{A}]\underline{r}^{ECF} \quad (10)$$

where $\underline{T}(t)$ is the transformation matrix between the Earth-fixed and Inertial frames. It is constructed based on the Earth's Orientation Parameters (EOPs), GAST, Nutation, Precision and Polar motion (McCarthy & Petit, 2003). Similarly, the position vector in the Inertial frame could be transformed to the Earth-fixed frame as:

$$\underline{r}^{ECF} = \underline{T}(t)\underline{r}^{ECI} \quad (11)$$

For convenience, $\underline{r}$ is used as position vector in the Inertial coordinate system, $\underline{r}^{ECI}$. This decision is considered for its derivatives $\underline{\ddot{r}}_g$ too. Finally, by substituting Eq. (11) into Eq. (10), the Earth's gravitational acceleration in the Inertial frame is:

$$\underline{\ddot{r}}_g^{ECI} = \underline{T}(t)^{-1}[(a-b)\underline{I} + \underline{A}]\underline{T}(t)\ \underline{r}^{ECI} \quad (12)$$

it could be resulted from decomposing Eq. (12) that $\underline{G}_{(2)}^g = \underline{O}$ and $\underline{F}_{(2)}^g = \underline{T}(t)^{-1}[(a-b)\underline{I} + \underline{A}]\underline{T}(t)$.

The terms of Taylor series $\underline{F}_{(q)}^g$ and $\underline{G}_{(q)}^g$ are calculated based on the recursive Eq. (7) in the static gravitational field. Table (1) lists the first five terms of the sequence, beginning with $q = 0$**.**

Table 1. The Lagrange coefficients in the full gravitational field

q	$\underline{F}_{(q)}^g$	$\underline{G}_{(q)}^g$
0	$\underline{I}$	$\underline{O}$
1	$\underline{O}$	$\underline{I}$
2	$\underline{F}_{(2)}^g$	$\underline{O}$
3	$\underline{\dot{F}}_{(2)}^g$	$\underline{F}_{(2)}^g$
4	$\underline{\ddot{F}}_{(2)}^g + \underline{F}_{(2)}^g\underline{F}_{(2)}^g$	$2\underline{\dot{F}}_{(2)}^g$

where $\underline{F}_{(2)}^g$ and their first and second-time derivatives are given through:

$$\begin{aligned}
\underline{F}^{g}_{(2)} &= \underline{T}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\underline{T}(t) \\
\dot{\underline{F}}^{g}_{(2)} &= \dot{\underline{T}}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\underline{T}(t) \\
&\quad +\underline{T}(t)^{-1}[(\dot{a}-\dot{b})\underline{I}+\dot{\underline{A}}]\underline{T}(t)+\underline{T}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\dot{\underline{T}}(t) \\
\ddot{\underline{F}}^{g}_{(2)} &= \ddot{\underline{T}}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\ \underline{T}(t) \\
&\quad +\underline{T}(t)^{-1}[(\ddot{a}-\ddot{b})\underline{I}+\ddot{\underline{A}}]\underline{T}(t)+\underline{T}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\ddot{\underline{T}}(t) \\
&\quad +2\ \dot{\underline{T}}(t)^{-1}[(\dot{a}-\dot{b})\underline{I}+\dot{\underline{A}}]\underline{T}(t) \\
&\quad +2\ \dot{\underline{T}}(t)^{-1}[(a-b)\underline{I}+\underline{A}]\dot{\underline{T}}(t)+2\ \underline{T}(t)^{-1}[(\dot{a}-\dot{b})\underline{I}+\dot{\underline{A}}]\dot{\underline{T}}(t)
\end{aligned} \quad (13)$$

Satellites' motion in the real world is governed by not only the static gravitational field but also the non-static accelerations e. g., air drag, third body, solid and ocean tide, solar radiation (Seeber, 2003). In the next section, the effect of the other perturbations has been considered in the F and G series.

2.3 Non-static forces

We divided the acceleration acting on a satellite into two parts, the static force (Earth's gravitational acceleration) and non-static forces e.g. solid and ocean tide, third body, solar radiation, and air-drag. In this paper, EGM96 was applied as the geopotential model (Lemoine et al., 1998). The ephemerides of the Moon and Sun are respectively calculated using the theory ELP-2000/82 represented by (Chapront-Touzé & Chapront, 1983) and analytical formulas (Montenbruck, 1989). A dynamic global of the Earth's atmosphere, NRLMSISE-00, was used for the air-drag force computations (Picone et al., 2002). The solar radiation pressure is obtained from the calculation of the sunlight percentage (Montenbruck & Gill, 2000). IERS formulations are used for solid Earth and ocean tide modeling (McCarthy & Petit, 2003). The acceleration acting on a satellite is formulated as:

$$\ddot{\underline{r}} = \ddot{\underline{r}}_g + \ddot{\underline{r}}_{op} \quad (14)$$

where $\ddot{\underline{r}}_g$ is the Earth's gravitational acceleration and $\ddot{\underline{r}}_{op}$ is the non-static accelerations acting on a satellite. The non-static accelerations acting on a satellite could be formulated as:

$$\ddot{\underline{r}}_g^{ECF} = [(a-b)\underline{I}+\underline{A}]\underline{r}^{ECF} \quad (15)$$

The radial, tangential and normal terms of the non-static perturbing accelerations have been respectively described by the first, second and third terms of the Eq. (15). In order to compute the Lagrange coefficients, the q-th derivatives of the position vector should be expanded to the position and velocity vector. Then, the out-of plane term should be reformulated based on the position and velocity vector.

The vector cross product also can be expressed with matrix multiplication as the product of a skew-symmetric matrix and a vector (Liu, 2008): (Liu, 2008 #341).

$$\underline{r}\times\dot{\underline{r}} = [\underline{r}]_\times \dot{\underline{r}} \quad (16)$$

where $[\underline{r}]_\times$ is the cross product matrix of $\underline{r}$, the operator $[\underline{r}]_\times$ is equivalent to (Taylor & Kriegman, 1994).

$$[\underline{r}]_\times = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \quad (17)$$

Finally, by substituting Eq. (16) into Eq. (15), the vector of other perturbing accelerations could be rewritten as:

$$\ddot{\underline{r}}_{op} = (\alpha_R\,\underline{I} + \alpha_N\,\underline{R})\underline{r} + \alpha_T\,\dot{\underline{r}} \quad (18)$$

Due to the presence of the non-gravitational accelerations, the along-track component, tangential term, of $\ddot{\underline{r}}_{op}(t)$ is not zero. By decomposing $\ddot{\underline{r}}_{op}(t)$ based on Eq. (4) gives:

$$\underline{F}^{op}_{(2)}(t) = \alpha_R\,\underline{I} + \alpha_N\,\underline{R}$$

$$\underline{G}^{op}_{(2)}(t) = \alpha_T\,\underline{I} \quad (19)$$

Like the Earth's gravitational acceleration, $\underline{F}^{op}_{(2)}(t)$ consists of the radial and out-of-plane terms. The first and second parts of $\underline{F}^{op}_{(2)}(t)$ are, the radial and out-of-plane terms of the non-static accelerations, respectively. The time derivative of non-static acceleration is

$$\underline{r}^{op}_{(3)} = \frac{\partial\ddot{\underline{r}}_{op}}{\partial\underline{r}}\frac{\partial\underline{r}}{\partial t} + \frac{\partial\ddot{\underline{r}}_{op}}{\partial\dot{\underline{r}}}\frac{\partial\dot{\underline{r}}}{\partial t} + \frac{\partial\ddot{\underline{r}}_{op}}{\partial t} \quad (20)$$

Like the non-static acceleration vector, its time derivative could be decomposed into the position and velocity vectors too as:

$$\underline{r}^{op}_{(3)} = (\tilde{\alpha}_R\,\underline{I} + \tilde{\alpha}_N\,\underline{R})\underline{r} + \tilde{\alpha}_T\,\dot{\underline{r}} \quad (21)$$

Finally, by decomposing $\underline{r}^{op}_{(3)}(t)$ based on the initial position and velocity vectors, we have:

$$\underline{F}^{op}_{(3)}(t) = \tilde{\alpha}_R\,\underline{I} + \tilde{\alpha}_N\,\underline{R},$$

$$\underline{G}^{op}_{(3)}(t) = \tilde{\alpha}_T\,\underline{I}\,. \quad (22)$$

The higher terms of the $\underline{F}^{op}$ and $\underline{G}^{op}$ are neglected due to their complexity and negligible contributions of the non-static perturbations in the orbiter motion with respect to the attractive body gravitational acceleration. The $\underline{F}_{(q)}$ and

$\underline{G}_{(q)}$ needed for computing the Lagrange coefficients in Eqs. (2) and (3) could be evaluated by:

$$\underline{F}_{(q)} = \underline{F}^{g}_{(q)} + \underline{F}^{op}_{(q)}, \qquad \underline{G}_{(q)} = \underline{G}^{g}_{(q)} + \underline{G}^{op}_{(q)} \tag{23}$$

Using $\underline{F}_{(q)}$ and $\underline{G}_{(q)}$, the matrices of the Lagrange coefficients could be computed based on Eqs. (2) and (3). Finally, the satellite orbit is propagated using Eq. (1). The accuracy of the satellite orbit obtained from the Lagrange method will be assessed in section 4. Similar to another numerical solver, the stability of the Lagrange method will be administrated in the next sections.

3. Stability Analysis

Like other differential equation, the equations of motion have been solved using different numerical approaches. These methods are different in the stability. An eligibility criterion for analysis of integrators methods is their ability to preserve the stability of a stable equilibrium. It is a very important property that should be investigated for a satellite propagator such as the Lagrange method. *Stability region* is a standard tool for this analysis. A small stability region reveals that very small step size is needed (Hairer & Wanner, 1991). Stability analysis of an integrator usually is performed by considering the most frequently used function for this purpose $\dot{y} = \lambda y, y_0 = 1$, where λ is a constant complex number with negative real part (Butcher, 1987). The region of stability of an integrator method is that set of (complex) values of $z = \lambda h$ for which all numerically obtained solutions of the test problem will remain bounded (Dahlquist & Björk, 1974). Small stability regions indicate that very small step sizes might be necessary for numerical integration (Hairer & Wanner, 1991). The stability region of the Lagrange method truncated up to 4 terms is equivalent to the Runge-Kutta of order 4. As it is obvious in Figure 1, the Lagrange method is more stable than the multi-step methods .

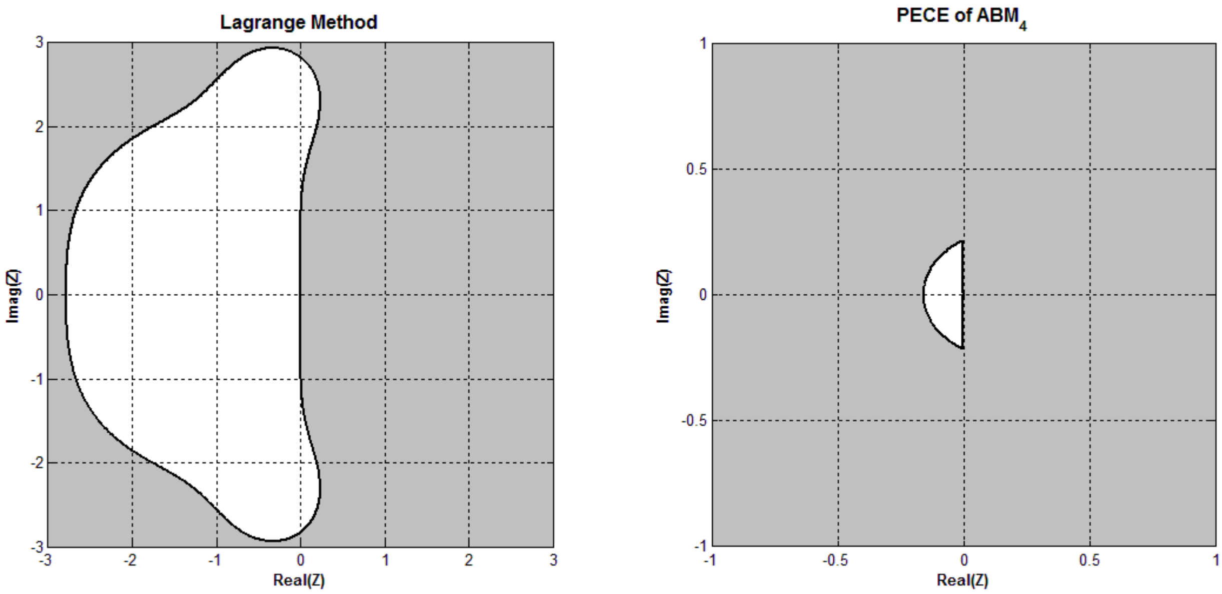

Figure 1. Stability regions (un-shaded region) of the PECE algorithm of Adams-Bashforth-Moulton integrator and the Lagrange method of order 4

4. Numerical Analysis

This study set out with the aim of assessing the efficiency and precision of the proposed Lagrange method by some experimental results for LEO and MEO satellite orbit propagation. The best way for testing the accuracy of a propagator is comparing the solution of the exact solution of the differential system when it is available. The analytical solution of the equations of motion of satellites and planets is easily computed in the central force field. The simplest form of an orbiter motion is called the Keplerian motion . At first, we are interested to compare three classes of the integrators, single-step, multi-step and Taylor series (Lagrange method), in solving the equations of motion. As an example, the satellite orbit propagation was carried out in a 30-day time span using the Runge-Kutta (RK), Adams-Bashforth-Moulton (ABM) and Lagrange methods. All methods are considered of order 4. The numerical study is based on a comparison of a 30-day span of a CHAMP-like satellite orbit (Reiberg et al., 2002) in the central field of the Earth. The obtained results are compared with the analytical solutions. The Euclidian norm of analytical and numerically obtained position vector differences in three dimensions (3D-difference) is considered as a criterion for error comparison. The computational time demand and the propagation error versus time span were compared for the methods. Figure 2 shows the logarithm of the propagation error versus the CPU time, the logarithm of error vs. time span in days, and the CPU time vs. the integration time span.

As it was expected, the Lagrange method represents more accurate result with respect to the ones of the common integrators, especially to the multi-step method. From the viewpoint of computational time, the F and G series approach are almost as time-consuming as ABM4 integrator, because both methods need just one function evaluation in each step. Compared to the RK4, it is less time-consuming due to the relatively lower number of elementary operations which is used in the propagation process. Unlike the Keplerian motion, in the non-Keplerian motion, an exact solution does not exist, since the system is non-integrable. Then, the dynamic orbit computed using the well-known error controlled numerical integration methods has been considered as reference orbit for the efficiency test of the Lagrange method. In particular, it was interesting that the numerical solution of Lagrange coefficients is compared against the classical numerical integration of the equations of motion for LEO and MEO satellite orbit propagation. The error controlled numerical integration method is used as a reference for comparison since it is the most popular numerical orbit propagator in the aerospace community.

The numerical studies are based on a comparison between orbits obtained from the Lagrange method and numerical integrators for GRACE A, TOPEX-Poseidon, Spot 6 and GPS satellites over one day. Numerically integrated orbits have been computed using MATLAB routine ODE45, based on an explicit Runge-Kutta (4,5) with error control of about 10^{-16} and automated step size (Dormand & Prince, 1978). When error controlled integrators are used, the accuracy of integrated orbit could be modified by decreasing tolerance of error in integrator option, e.g. 10^{-16}. The initial conditions for dynamic orbit propagation are listed in table (2) for GRACE A, Spot 6, TOPEX-Poseidon and GPS satellites. For all comparisons, these values are used as the initial conditions.

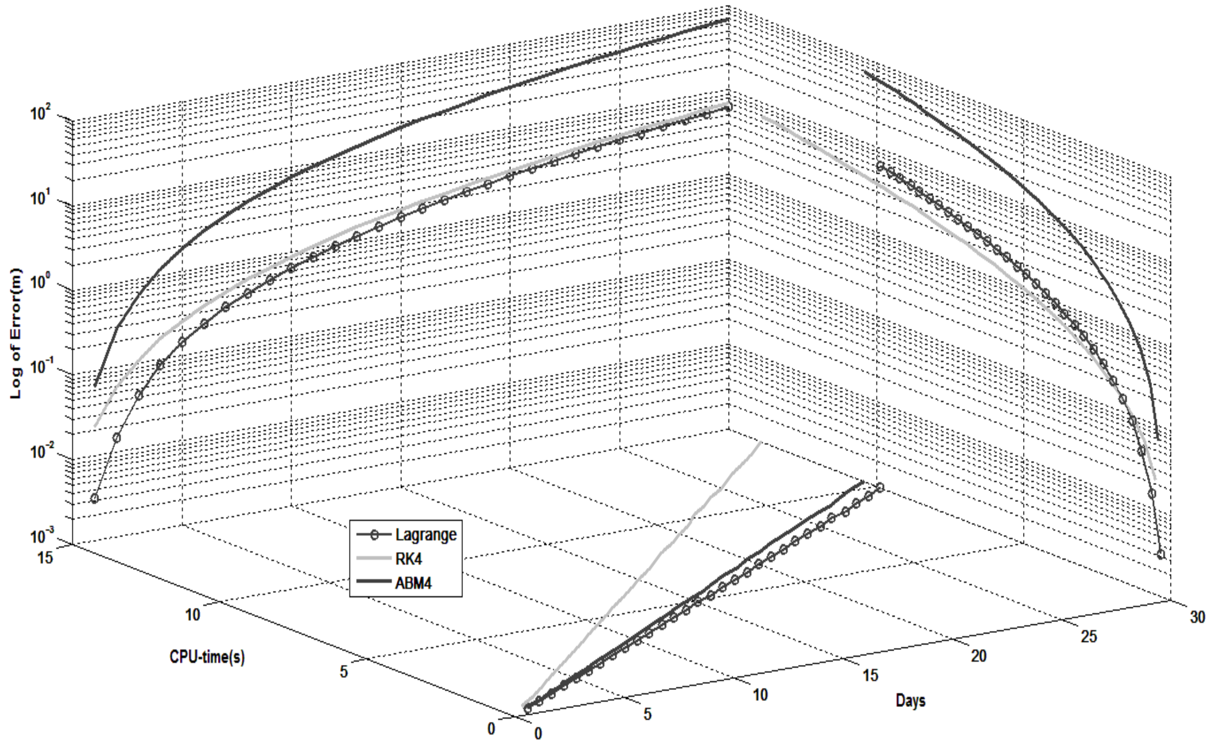

Figure 2. The comparison of the numerical integrations and Lagrange error versus CPU-time in a one-month span.

Table 2. The initial conditions setup in the ECI frame

Parameters	GRACE A	TOPEX-Poseidon	Spot 6	GPS 01Satellite
Date **GPS-Time (Hour, Min, Sec)**	13 Nov 2008 00:00:00	19 Jan 2013 20:23:23	20 Jan 2013 10:43:20	7 Jul 2000 02:00:00
$\underline{r}$ **(m)**	-3237459.157	1810698.864	-36966.704	-15230002.91
	-2113675.500	-2601714.005	7076046.367	7143903.358
	-5642313.514	7040607.196	16295.146	20669207.44
$\dot{\underline{r}}$ **(m/s)**	5389.603	5286.109	1068.601	-2589.7828
	3250.112	4843.011	-12.405	-2689.171
	-4315.634	431.015	7428.766	-984.0158
Semi-major axis (m)	6835241.546	7715222.067	7075985.418	26560603.802
Eccentricity	0.000916	0.0007888	0.0001200	0.003512
Inclination (degree)	88.8085°	66.0379°	98.1850°	54.7001°
Right-Ascension of the ascending node (degree)	0.5540°	224.0263°	90.3183°	35.3233°
Argument of perigee (degree)	1.6981°	274.1652°	102.1214°	270.8742°
Mean anomaly (degree)	2.4136°	172.0756°	258.0119°	197.3823°

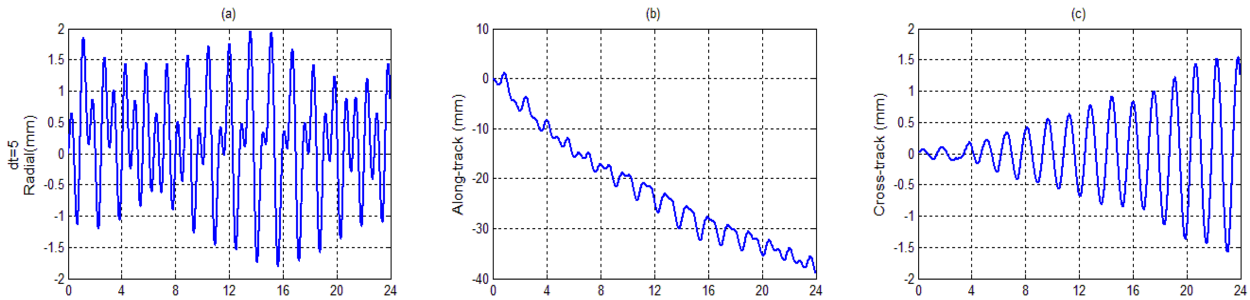

Figure 3. The difference between the reference and Lagrange orbits for GRACE A satellite over one day

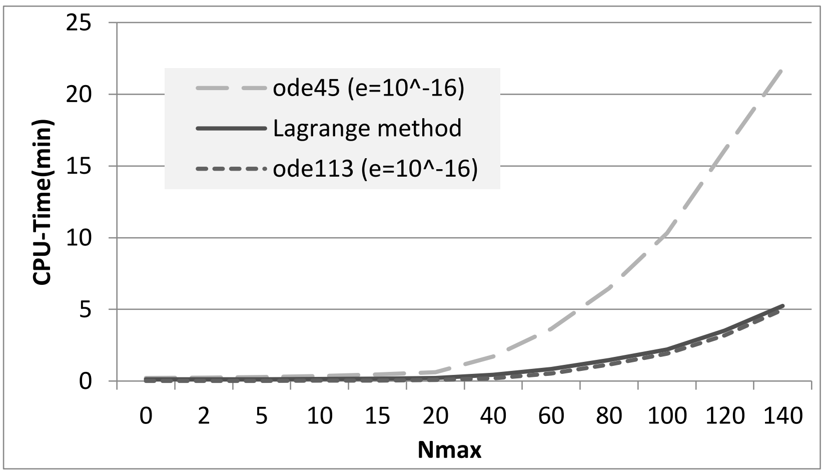

Figure 4. The computation time needed for propagating over one day of the GRACE A satellite using ode45, ode113, and the Lagrange method

Figures represented in this section describe the influence of the rounding error and truncation one produced by truncating infinite Taylor series at the certain terms. This certain terms differ for different perturbing accelerations because of their different impact effects. As described in the previous section, the five terms (q=4) are considered for the Earth's gravity field (static part), q=3 for other perturbations in Taylor series of the Lagrange coefficients. Based on the analysis carried out in (M. A. Sharifi and Seif, 2011), the accuracy of the orbit propagation can be improved by setting $f_i^{(5)}$ and $g_i^{(5)}$ to the J2 coefficients instead of zero. The higher terms of Taylor series could be used for the central field acceleration up to q=7 for improving the accuracy of the orbit propagation of a LEO satellite. Acquired precision at q=7 will be sufficient for many applications in satellite geodesy. At first, the accuracy of dynamic orbit obtained by the Lagrange method called the Lagrange orbit was tested for the satellite GRACE A. Figure 3 describes the influence of truncation errors produced by truncating infinite Taylor series of the Lagrange coefficients for the orbit propagation of the GRACE A satellite with *dt*=5s. Figure 3a, 3b, 3c at the radial, along-track, and cross-track directions over 24 hours. As shown in Figure 3, the maximum difference between the reference and Lagrange orbits for a GRACE-like satellite is about 4 cm over one day.

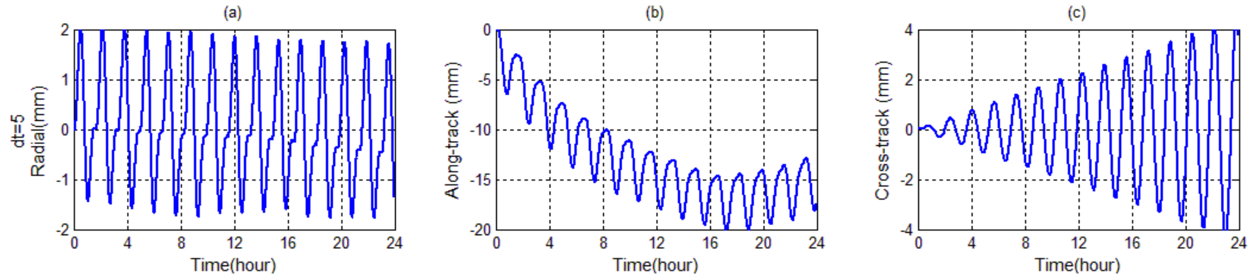

Figure 5. The difference between the reference and Lagrange orbits for Spot 6 satellite with *dt*=5s over one day

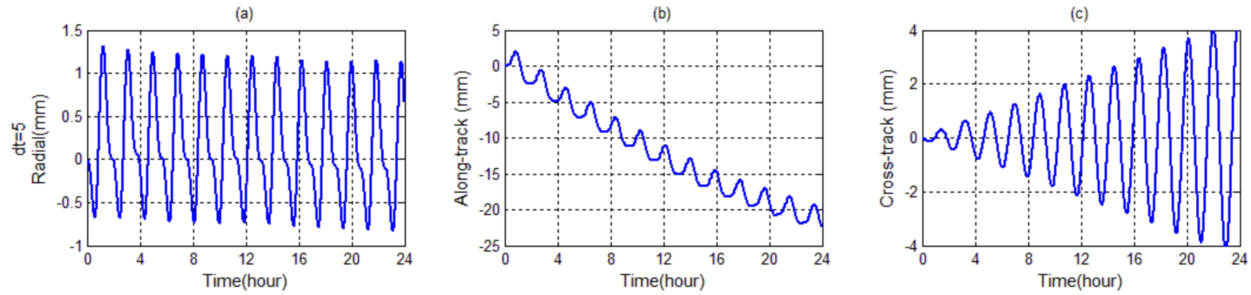

Figure 6. The difference between the reference and Lagrange orbits for TOPEX-Poseidon satellite with *dt*=5s over one day

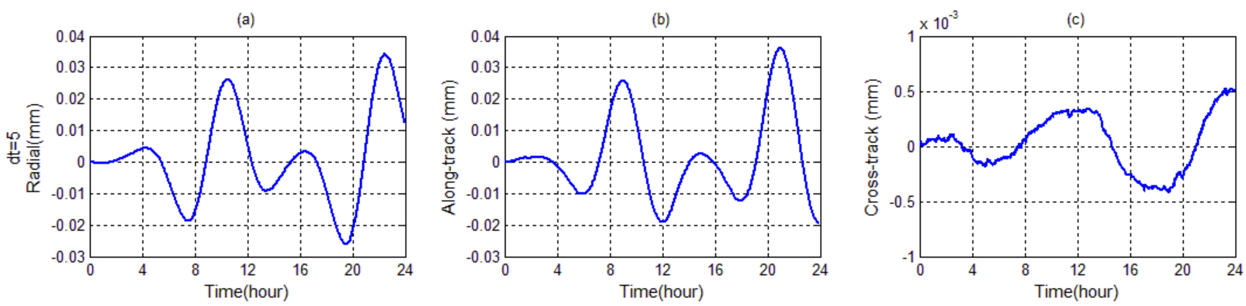

Figure 7. The difference between the reference and Lagrange orbits for GPS 01 satellite with *dt*=5s over one day

It is sufficient and suitable accuracy for many applications in the satellite geodesy and celestial mechanics. Please note that the obtained accuracy is over one day with *dt*=5s and for more accurate results, the time interval could be chosen shorter for the orbit propagation. As an eligibility criterion, the CPU-time analysis was carried out for the Lagrange method with *dt*=5s, and two well-known MATLAB routines ode45 (as an error controlled single step method), ode113 (as an error controlled multi-step method (Shampine, 2005; Shampine & Reichelt, 1997) with error control of about 10^{-16}. Figure 4 describes the CPU-time of these methods for different Nmax (i.e., the maximum degree of the gravity field) for a GRACE-like satellite over one day. As it was expected, the Lagrange method is almost as time-consuming as ode113, because both methods need just one function evaluation in each step. Compared to the ode45, it is less time-consuming due to the relatively lower number of elementary operations which is used in the propagation process. The number of function evaluations for ode45 integrator is about four times larger than the Lagrange method at each step of the propagation process. For small Nmax, it is nearly doubled and it will be increased by a factor of 4 for large Nmax over one day .In addition to GRACE A, the Lagrange method was tested for two further LEO satellites launched at higher altitudes inner and outer atmosphere of the Earth. This proposed method was used for propagating the Spot 6 and TOPEX-Poseidon orbits .

Figure 5 illustrates the accuracy of the Lagrange coefficients for Spot 6 orbit propagation with *dt*=5s Figure 5a, 5b, 5c) at radial, along-track, and cross-track directions over 24 hours. This analysis was carried out for TOPEX-Poseidon satellite too. Figure 6 presents the accuracy of the Lagrange orbit TOPEX-Poseidon with *dt*=5s Figure 5a, 5b, 5c at radial, along-track, and cross-track directions over 24 hours. The last two Figures show that the difference between the Lagrange and integrated orbits remains at centimeter level over one day for two different LEO satellites, Spot-like and TOPEX-Poseidon-like. In addition to LEO satellite, it was highly interesting to continue this section with testing the proposed method for any satellite in higher altitude e.g. a MEO satellite. Figure 7 shows the differences between the numerically integrated and the Lagrange orbits in over one day for a GPS satellite with *dt*=5s, Figure 7a, 7b, 7c at radial, along-track, and cross-track directions. Besides the advantages of the Lagrange method described in the introduction, the high accuracy obtained from the Lagrange method in this section proves that it is an efficient alternative for the satellite orbit integration not only for the MEO but also for the LEO satellites.

5. Conclusion

In conclusion, this paper is another attempt to apply Taylor series method to solve equations of the perturbed motion. This paper sets out to develop the Lagrange method, F and G series, from a gravitational field of an inhomogeneous attractive body to all perturbing accelerations. At first, the accuracy of the Lagrange method is tested besides other propagators based on the comparison with the analytical solution. The results show that the Taylor series based method, F, and G series, could obtain a more accurate result with respect to the traditional numerical integrators at the same situation. The result of the CPU-time analysis shows that the Lagrange method is as time-consuming as multi-step methods and faster than single-step methods. In addition to CPU-time analysis, the stability analysis demonstrates that the Lagrange method is more stable than the multi-step method and is equal to the single-step method in stability property. As it was expected, like other Taylor series based methods, the Lagrange method combines the advantages of both methods of the single- and the multi-step methods.

For testing the accuracy of the Lagrange method in the non-Keplerian motion, the Lagrange orbit is compared with the numerically propagated one computed using the well-known error-controlled integration methods. This comparison is made by considering all perturbing accelerations for a few LEO satellites (GRACE A, Spot 6, TOPEX-Poseidon) and a MEO-type satellite (GPS). The results show that the Lagrange method leads to a nearly identical solution to that of the numerical integration with a maximum difference of about 0.04 millimeter for GPS satellite in one day. It is about 5 centimeters for the GRACE A satellite. The Lagrange coefficients yield centimeter accuracy in terms of position for the LEO satellite. This analysis shows that the accuracy of the Lagrange method is under 3 centimeters for Spot 6, and about 3 centimeters for TOPEX-Poseidon satellites. It is highly recommended that further studies be undertaken to compute more terms of the Lagrange coefficient series for more accuracy. Using variable step-size in an error-controlled algorithm could be another strategy for increasing the accuracy of the Lagrange method.

Appendix A

The method of computing the first and second order derivatives of a,b,e,h and l is described in the following paragraphs. These scalars are:

$$
\begin{aligned}
a &= \frac{1}{r} U_r \\
b &= \left(\frac{z}{r^2\sqrt{x^2+y^2}}\right) U_\varphi \\
e &= \frac{1}{z\sqrt{x^2+y^2}} U_\varphi \\
h &= \frac{x}{y(x^2+y^2)} U_\lambda \\
l &= \frac{y}{x(x^2+y^2)} U_\lambda
\end{aligned}
\tag{24}
$$

All of the scalars (a,b,e,h and l) have this form:

$$H(x,y,z)=K(x,y,z)\frac{\partial U(r,\phi,\lambda)}{\partial\alpha} \quad (25)$$

where H and K are known functions and $\alpha\in[r,\phi,\lambda]$. The first and second order derivatives of H are:

$$\dot{H}=\frac{dK}{dt}\frac{\partial U}{\partial\alpha}+K\frac{d}{dt}(\frac{\partial U}{\partial\alpha})$$
$$\ddot{H}=\frac{d^2K}{dt^2}\frac{\partial U}{\partial\alpha}+2\frac{dK}{dt}\frac{d}{dt}(\frac{\partial U}{\partial\alpha})+K\frac{d^2}{dt^2}(\frac{\partial U}{\partial\alpha}) \quad (26)$$

By using Einstein's summation convention:

$$\frac{dK}{dt}=\frac{\partial K}{\partial x_i}\dot{x}_i$$
$$\frac{d^2K}{dt^2}=\frac{\partial^2K}{\partial x_i\partial x_j}\dot{x}_i\dot{x}_j+\frac{\partial K}{\partial x_i}\ddot{x}_i \quad (27)$$

where x_i represents the Cartesian coordinate system in the ECF frame. The chain role should be used for computing first- and second order time derivatives of $\frac{\partial U}{\partial\alpha}$.

$$\frac{d}{dt}(\frac{\partial U}{\partial\alpha})=\frac{\partial^2U}{\partial\alpha\,\partial\theta_j}\dot{\theta}_j$$
$$\frac{d^2}{dt^2}(\frac{\partial U}{\partial\alpha})=\frac{\partial^3U}{\partial\alpha\,\partial\theta_j\partial\theta_k}\dot{\theta}_j\dot{\theta}_k+\frac{\partial^2U}{\partial\alpha\,\partial\theta_j}\ddot{\theta}_j \quad (28)$$

where θ_i represent the curvilinear coordinate in ECF frame.

The value of mixed derivatives is independent of the order in which the derivatives are taken for continuous functions, based on Schwartz's theorem about mixed derivatives. For example:

$$\frac{dU_\phi}{dt}=U_{r\phi}\dot{r}+U_{\phi\phi}\dot{\phi}+U_{\phi\lambda}\dot{\lambda}$$
$$\frac{d^2U_\phi}{dt^2}=U_{r\phi}\ddot{r}+u_{\phi\phi}\ddot{\phi}+U_{\phi\lambda}\ddot{\lambda}$$
$$+(U_{rr\phi}\dot{r}+U_{r\phi\phi}\dot{\phi}+U_{r\phi\lambda}\dot{\lambda})\dot{r}$$
$$+(U_{r\phi\phi}\dot{r}+U_{\phi\phi\phi}\dot{\phi}+U_{\phi\phi\lambda}\dot{\lambda})\dot{\phi} \quad (29)$$
$$+(U_{r\phi\lambda}\dot{r}+U_{\phi\phi\lambda}\dot{\phi}+U_{\phi\lambda\lambda}\dot{\lambda})\dot{\lambda}$$

Like the derivative of K function mentioned above, the derivatives of the spherical coordinates are:

$$\dot{r}=(\frac{\partial r}{\partial x_i})\dot{x}_i$$
$$\dot{\phi}=(\frac{\partial\phi}{\partial x_i})\dot{x}_i \quad (30)$$
$$\dot{\lambda}=(\frac{\partial\lambda}{\partial x_i})\dot{x}_i$$

At last the second derivatives of the spherical coordinates are:

$$\ddot{r}=(\frac{\partial^2r}{\partial x_i\partial x_j})\dot{x}_i\dot{x}_j+(\frac{\partial r}{\partial x_i})\ddot{x}_i$$
$$\ddot{\phi}=(\frac{\partial^2\phi}{\partial x_i\partial x_j})\dot{x}_i\dot{x}_j+(\frac{\partial\phi}{\partial x_i})\ddot{x}_i \quad (31)$$
$$\ddot{\lambda}=(\frac{\partial^2\lambda}{\partial x_i\partial x_j})\dot{x}_i\dot{x}_j+(\frac{\partial\lambda}{\partial x_i})\ddot{x}_i$$

where x_i represents the Cartesian coordinate in the ECF frame.

References

Bem, J., & Szczodrowska-Kozar, B. (1995). High order F and G power series for orbit determination. Astronomy and Astrophysics Supplement Series, 110, 411.

Beutler, G. (2004). Methods of celestial mechanics volume ii: application to planetary system, geodynamics and satellite geodesy [M]: Berlin, Heidelberg: Springer-Verlag.

Butcher, J. (1987). The Numerical Analysis of Ordinary Differential Equations: Runge–Kutta and General Linear Methods: John Wiley & Sons, Chichester, New York.

Chapront-Touzé, M., & Chapront, J. (1983). The lunar ephemeris ELP 2000. Astronomy and Astrophysics, 124, 50-62.

Curtis, H. D. (2005). Orbital mechanics for engineering students (Third edition. ed.).

Dahlquist, G., & Björk, Å. (1974). Translated by N. Anderson. 1974. Numerical Methods: Prentice Hall, New Jersey.

Dormand, J., & Prince, P. (1978). New Runge-Kutta algorithms for numerical simulation in dynamical astronomy. Celestial Mechanics, 18(3), 223-232.

Escobal, P. R. (1965). Methods of orbit determination. New York: Wiley, 1965, 1.

Feng, Y. (2001). An alternative orbit integration algorithm for gps-based precise leo autonomous navigation. GPS Solutions, 5(2), 1-11.

Goodyear, W. H. (1965). Completely general closed-form solution for coordinates and partial derivative of the two-body problem. The Astronomical Journal, 70, 189.

Hairer, E., & Wanner, G. (1991). Solving ordinary differential equations, vol. II: Springer Verlag, Berlin.

Lemoine, F. G., Kenyon, S. C., Factor, J. K., Trimmer, R. G., Pavlis, N. K., Chinn, D. S., Torrence, M. H. (1998). The Development of the Joint NASA GSFC and the

National Imagery and Mapping Agency(NIMA) Geopotential Model EGM 96. NASA(19980218814).

Lin, L., & Xin, W. (2003). A method of orbit computation taking into account the earth's oblateness. Chinese Astronomy and Astrophysics, 27(3), 335-339. doi: http://dx.doi.org/10.1016/S0275-1062(03)90056-7

Taylor, C. J., & Kriegman, D. J. (1994). Minimization on the Lie group SO (3) and related manifolds. Yale University.

McCarthy, D. D., & Petit, G. (2003). IERS conventions. Paper presented at the IAU Joint Discussion.

Montenbruck, O. (1989). Practical ephemeris calculations (A. H. Armstrong, Trans.): Springer-verlag Heidelberg.

Montenbruck, O. (1992). Numerical integration of orbital motion using Taylor series. Spaceflight mechanics 1992, 1217-1231.

Montenbruck, O., & Gill, E. (2000). Satellite orbits: Springer.

Pellegrini, E., Russell, R., & Vittaldev, V. (2014). F and G Taylor series solutions to the Stark and Kepler problems with Sundman transformations. Celestial Mechanics and Dynamical Astronomy, 118(4), 355-378. doi: 10.1007/s10569-014-9538-7

Picone, J., Hedin, A., Drob, D. P., & Aikin, A. (2002). NRLMSISE-00 empirical model of the atmosphere: Statistical comparisons and scientific issues. Journal of Geophysical Research: Space Physics (1978–2012), 107(A12), SIA 15-11-SIA 15-16.

Schaub, H., & Junkins, J. L. (2003). Analytical mechanics of space systems: Aiaa.

Sconzo, P., LeSchack, A., & Tobey, R. (1965). Symbolic computation of F and G series by computer. The Astronomical Journal, 70, 269.

Seeber, G. (2003). Satellite geodesy: foundations, methods, and applications: Walter de Gruyter.

Shampine, L. F. (2005). Error estimation and control for ODEs. Journal of Scientific Computing, 25(1), 3-16.

Shampine, L. F., & Reichelt, M. W. (1997). The matlab ode suite. SIAM journal on scientific computing, 18(1), 1-22.

Sharifi, M. A., & Seif, M. R. (2011). Dynamic orbit propagation in a gravitational field of an inhomogeneous attractive body using the Lagrange coefficients. Advances in Space Research, 48(5), 904-913. doi: http://dx.doi.org/10.1016/j.asr.2011.04.021

Sharifi, M. A., Sneeuw, N., Seif, M. R., & Farzaneh, S. (2013). A semi-analytical formulation of the Earth's flattening on the satellite formation flying observables using the Lagrange coefficients. Paper presented at the Hotine-Marussi Symposium, Rome, Italy.

Steffensen, J. F. (1956). On the restricted problem of three bodies. Mat.-fys. medd.; Bd 30.

Tapley, B., Schutz, B., & Born, G. H. (2004). Statistical orbit determination: Academic Press.

Taylor, C. J., & Kriegman, D. J. (1994). Minimization on the Lie group SO and related manifolds. Yale University.