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Estimation of Scale Parameter Under a Bounded Loss Function

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Abstract

The quadratic loss function has been used by decision-theoretic statisticians and economists for many years. In this paper the estimation of scale parameter under a bounded loss function, which is adequate for assessing quality and quality improvement, is considered with restriction to the principles of invariance and risk unbiasedness. An implicit form of minimum risk scale equivariant estimator and Bayes estimators are obtained. Fisher's problem of the Nile as an example is included.

Keywords: Best invariant estimator; Bayes estimator; Scale parameter; Bounded loss function; Fisher's problem of the Nile.

Introduction

A loss function has been $L(\delta,\theta)$ represents the amount by which a statistician is penalized when θ is the true state of nature and δ is the statistican's action. In the literature, $L(\delta,\theta)$ is usually taken to be convex in δ and even in $(\delta-\theta)$. For example, let $X_1,\ldots,X_n$ be a random sample of size n from a density $\frac{1}{\tau}f(\frac{x}{\tau})$, where f is known and τ is an unknown scale parameter. In this case, the commonly used quadratic loss is given by

$$L(\delta,\tau)=(\frac{\delta}{\tau}-1)^2 \qquad (1.1)$$

This loss function has been criticized by some researchers e. g. Rukhin and Ananda [7], Dey, Ghosh and Srinivasan [3], Akaike [1] and [2]. They motivated the entropy loss as an asymmetric loss function for estimating an unknown scale parameter, but this loss with its infinite maximum value, is not appropriate in describing , for example, the loss associated with a product. The arguments suggest that bounded loss functions are more appropriate than the unbounded one [10] and [11]. Also unbounded asymmetric losses, such as square error loss, are widely employed in decision theory but their application is often justified by their nice mathematical properties, not their appropriateness in representing a true loss structure. The nature of many decision problems, such as reliability analysis, requires the use of asymmetric losses. For a scale parameter estimation,we use a loss function of the form

$$L(\delta,\tau)=b\{1-e^{a(2-\frac{\delta}{\tau}-\frac{\tau}{\delta})}\} \qquad (1.2)$$

where a>0 is a shape parameter and b>0 is the maximum loss parameter. The general form of the loss function is illustrated in Figure [1]. This is obviously a

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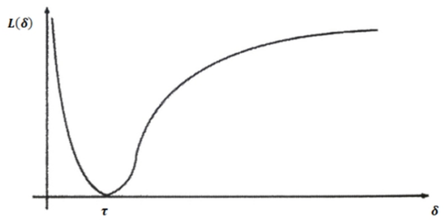

Figure 1. The Loss Function (1.2)

bounded loss function with respect to varying δ and τ and adequate for assessing quality and quality improvement [10] . For the sake of simplicity we take $a = b = 1$ in the rest of paper.

In this paper, we study the problem of estimation of a scale parameter, using the loss (1.2). In section 2 we introduce the best invariant estimator of the scale parameter τ using the loss (1.2). In section 3 we consider a subclass of the expossessing quality andnential family and obtain the Bayes estimates of τ using the loss (1.2). In section 4, the Fisher's problem of the Nile as an example is included.

Best Scale Invariant Estimator

Consider a random sample $X_1, X_2, \ldots, X_n$ from $\frac{1}{\tau} f\left(\frac{x}{\tau}\right)$, where f is a known function, and τ is an unknown scale parameter. It is desired to estimate τ under the loss function (1.2). The class of scale-invariant estimators of τ is of the form [12], [13]

$$\delta(X) = \delta_0(X)/W(Z)$$

Where δ_0 is any scale-invariant estimator, $X = (X_1, \ldots, X_n)$, and $Z = (Z_1, \ldots, Z_n)$ with $Z_i = \frac{X_i}{X_n}$; $i = 1, \ldots, n-1$, $Z_n = \frac{X_n}{|X_n|}$. Moreover the best scale-invariant (minimum risk equivariant (MRE)) estimator δ^* of τ is given by $\delta^*(X) = \delta_0(X)/\omega^*(Z)$

Where w*(z) is a function of Z which maximizes

$$E_{\tau=1}\left[e^{2 - \frac{\delta_0(X)}{\omega(Z)} - \frac{\omega(Z)}{\delta_0(X)}} \middle| Z = z \right]$$

Or minimizing

$$E_{\tau=1}\left[e^{\frac{\delta_0(X)}{\omega(Z)} + \frac{\omega(Z)}{\delta_0(X)}} \middle| Z = z \right] . \tag{2.1}$$

In the presence of location parameter as a nuisance parameter, the MRE estimator of τ is of the form

$$\delta^*(X) = \delta_0(Y)/\omega^*(Z),$$

where $\delta_0(Y)$ is any finite risk scale-invariant estimator of τ, based on $Y = (Y_1, \ldots, Y_{n-1})$ with $Y_i = X_i - X_n ; i = 1, \ldots, n-1, Z = (Z_1, \ldots, Z_{n-1})$, $Z_i = \frac{Y_i}{Y_{n-1}}$; $i = 1, \ldots n-2$, $Z_{n-1} = \frac{Y_{n-1}}{|Y_{n-1}|}$ and $\omega^*(Z)$ is any function of Z maximizing

$$E_{\tau=1}\left[e^{2 - \frac{\delta_0(Y)}{\omega(Z)} - \frac{\omega(Z)}{\delta_0(Y)}} \middle| Z = z \right] \tag{2.2}$$

In many cases ,when $\tau = 1$,we can find an equivariant estimator $\delta_0(X)$ or $\delta_0(Y)$ which has the gamma distribution with known parameters ν, η and is independent of Z. see for instance Rahman, M.S., and Gupta, R.P[7].

It follows that $\delta^* = \frac{\delta_0}{\omega^*}$ is the MRE estimator of τ where ω^* is a number which maximizes

$$\begin{aligned} g(\omega) &= \int_0^\infty e^{2 - \frac{x}{\omega} - \frac{\omega}{x}} \left\{ \frac{\eta^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\eta x} \right\} dx \\ &= \frac{2e^2 \eta^\nu \omega^\nu}{\Gamma(\nu)(1+\eta\omega)^{\frac{\nu}{2}}} k_\nu\left(2\sqrt{1+\omega\eta}\right) \\ &= c(\eta,\nu) \frac{\omega^\nu}{(1+\eta\omega)^{\frac{\nu}{2}}} k_\nu\left(2\sqrt{1+\omega\eta}\right) \end{aligned} \tag{2.3}$$

where c is a function of η and ν , and $k_f(.)$ is the modified Bessel function of the third kind (Gradshteyn and Ryzhik[5]) and Kariya[8]. Now we can show by differentiating $g(\omega)$ with respect to ω and using the recurrence relation $k_{\nu-1}(z) - k_{\nu+1}(z) = -\frac{2\nu}{z} k_\nu(z)$ that ω^* must satisfy the following equation

$$\frac{\omega^{*2}}{1+\omega^*\eta} k_{\nu+1}\left(2\sqrt{1+\omega^*\eta}\right) = k_{\nu-1}\left(2\sqrt{1+\omega^*\eta}\right). \tag{2.4}$$

Hence we have the following result.

Theorem 2.1: If $\delta_0(X)$ is a finite risk scale-invariant estimator of τ, which has the gamma distribution with known parameters v, η, when $\tau = 1$. Then the MRE (minimum risk equivariant) estimator of τ under the loss function (1.2) is $\delta^*(\mathrm{X}) = \frac{\delta_0(X)}{\omega^*}$, where ω^* must satisfy the equation (2.4).

Example 2.1: (Exponential) Let $X_1, ..., X_n$ be a random sample from $E(0, \lambda)$ with density $\frac{1}{\lambda} e^{-\frac{x}{\lambda}}; x > 0$, and consider the estimation of λ under the loss (1.2).

$\delta_0(X) = \sum_{i=1}^{n} X_i$ is an equivariant estimator which has Ga(n,1)-distribution when $\lambda = 1$.It follows from the Basu's theorem that δ_0 is independent of Z, hence the MRE estimator

of λ under the loss(1.2) is $\delta^*(X) = \frac{\sum_{i=1}^{n} X_i}{\omega^*}$,

where ω^* must satisfy the following

$$\frac{\omega^{*2}}{1+\omega^*} k_{n+1}\left(2\sqrt{1+\omega^*}\right) = k_{n-1}\left(2\sqrt{1+\omega^*}\right). \tag{2.5}$$

Example 2.1: (Continued) Suppose that $X_1, ..., X_n$ is a random sample of $E(\theta, \lambda)$ with density $\frac{1}{\lambda} e^{-(x-\theta)/\lambda}$, $x > \theta$ and consider the estimation of λ when θ is unknown.

We know that $\left(X_{(1)}, \sum_{i=1}^{n} (\mathrm{X}_i - X_{(1)})\right)$ is a complete sufficient statistics for (θ, λ).It follows that $\delta_0(\mathrm{Y}) = 2\sum_{i=1}^{n} (X_i - X_{(1)})$ has $Ga(\mathrm{n}-1, \frac{1}{2})-$ distribution, when $\lambda = 1$, and from the Basu's theorem $\delta_0(Y)$ is independent of Z and hence $\delta^*(\mathrm{Y}) = \frac{\sum_{i=1}^{n} (X_i - X_{(1)})}{\omega^*}$ is the MRE estimator of λ under the loss (1.2), where ω^* must satisfies the following equation

$$\frac{\omega^{*2}}{1+\omega^*} k_n\left(2\sqrt{1+\omega^*}\right) = k_{n-2}\left(2\sqrt{1+\omega^*}\right). \tag{2.6}$$

Example 2.2: (Normal variance) Let $X_1, ..., X_n$ be a random sample of $N(0, \sigma^2)$ and consider the estimation of σ^2. $\delta_0(X) = \sum_{i=1}^{n} X_i^2$ is a finite risk scale-invariant estimator of σ^2 and is independent of Z, and when $\sigma^2 = 1$, $\delta_0(X)$ has $Ga(\frac{n}{2}, \frac{1}{2})$-distribution and hence $\delta^*(X) = \frac{\sum_{i=1}^{n} X_i^2}{\omega^*}$ is the MRE estimator of σ^2, where ω^* must satisfies the equations (2.5).

Example 2.2: (Continued) Let $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$, with a nuisance parameter μ. In estimating σ^2 under the loss (1.2), it follows that $\delta_0(X) = \sum_{i=1}^{n} (X_i - \bar{X})^2$ is independent of Z, and when $\sigma^2 = 1$, the distribution of $\delta_0(Y)$ is $Ga(\frac{n-1}{2}, \frac{1}{2})$. Therefore, $\delta^*(X) = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}{\omega^*}$ is the MRE estimator of σ^2, where ω^* must satisfies the equation (2.6).

Example 2.3: (Inverse Gaussian with zero drift) Let $X_1, ..., X_n$ be a random sample of $IG(\infty, \lambda)$ with density

$$f(x \mid \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2x}} \qquad if \quad x > 0$$

and consider the estimation of λ.

$\delta_0(X) = \sum_{i=1}^{n} X_i^{-1}$ has $Ga(\frac{n}{2}, \frac{1}{2})$-distribution and is independent of Z and hence $\delta^*(X) = \frac{\sum_{i=1}^{n} X_i^{-1}}{\omega^*}$ is the MRE estimator of λ, where ω^* must satisfies the equation(2.5).

The Bayes Estimator

In this section, we consider the Bayesian estimation of the scale parameter τ in a subclass of one- parameter exponential families in which the complete sufficient statistic $\delta_0(X)$ has $Ga(\mathrm{v}, \frac{\eta}{\tau})$- distribution, where $v > 0, \eta > 0$ are known.

Assume that the conjugate family of prior distribution for $\beta = \frac{1}{\tau}$ is the family of Gamma distributions $Ga(\alpha, \xi)$. Now, that posterior distribution of β is $Ga(v+\alpha, \xi + \eta\delta_0(x))$ and the Bayes estimate of τ is a function $\delta(x)$ which maximizes the function

$$E\left[e^{2-\delta\beta-\frac{1}{\delta\beta}} \mid x\right] =$$

$$\frac{e^2(\eta\delta_0(x)+\xi)^{v+\alpha}}{T(v+\alpha)} \int_0^\infty \beta^{v+\alpha-1} e^{-\frac{1}{\delta\beta}-\beta(\delta+\xi+\eta\delta_0(x))} d\beta,$$

Or the function,

$$g(\delta) = (\delta^2 + \xi\delta + \eta\delta\delta_0(x))^{-\frac{v+\alpha}{2}} k_{v+\alpha}(2\sqrt{\frac{\delta+\xi+\eta}{\delta}}).$$

δ is obtained from the relation $\frac{dg(\delta)}{d\delta} = 0$ .Hence, the maximized δ must satisfies the following equation,

$$k_{v+\alpha-1}\left(2\sqrt{\frac{1}{\delta}(\delta+\xi+\eta\delta_0(x))}\right) = \frac{\delta}{\delta+\xi+\eta\delta_0(x)} k_{v+\alpha+1}\left(2\sqrt{\frac{1}{\delta}(\delta+\xi+\eta\delta_0(x))}\right) \tag{3.1}$$

So all estimators satisfying (3.1) are also Bayes estimators.

Example 3.1: In examples 2.1, 2.2 and 2.3, all estimators, satisfying (3.1), where $\delta_0(X)$ is the complete sufficient statistic, are Bayes estimator of the scale parameter τ .

Application to the Fisher Nile's Problem

The classical example of an ancillary statistic is known as the problem of Nile, originally formulated by Fisher [4]. Assume that X and Y are two positive valued random variables with the joint density function

$$f(x,y;\tau) = e^{-(\tau x + \frac{1}{\tau}y)} \quad ; x>0, y>0 \ \tau>0 \tag{4.1}$$

and that is (X_i, Y_i) , $i = 1,\dots,n$ a random sample of n paired observation on (X, Y) .Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $T = \sqrt{\frac{\bar{Y}}{\bar{X}}}$, $U = \sqrt{\bar{X}\bar{Y}}.T$ is the MLE of τ and the pair (T, U) is a jointly sufficient, but not complete statistic for τ and U is ancillary.

Consider a nonrandomized rule $\delta(T,U)$ based on the sufficient statistic $(\bar{X}, \bar{Y})$ which is equivariant under the transformation

$$\begin{pmatrix} z \\ \omega \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \quad ; c>0$$

For $\delta(T,U)$ to be scale equivariant, we must have

$$c\delta(T,U) = \delta(cT,U) \quad ; \quad \forall c>0 \tag{4.2}$$

Following Lehman and Casella [9] a necessary and sufficient condition for an estimator δ to be scale equivariant is that it is of the form $\delta = \delta_0 Z$, where δ_0 satisfies (4.2), and Z is invariant under scale changes. Note that T satisfies (4.2), hence $\delta_0 = T$, $Z = \phi(U)$. We see that all the scale equivariant estimators $\delta(T,U)$ must have the form

$$\delta(T,U) = T\phi(U) \tag{4.3}$$

using the loss function (1.2), and the fact that the joint distribution of $\left(\frac{T}{\tau}, U\right)$ is independent of τ , we can evaluate the risk at $\tau = 1$ .Hence

$$R(\tau, T\phi(U)) = E_U[E(1 - e^{2 - T\phi(U) - \frac{1}{T\phi(U)}}) \mid U].$$

It follows that $R(\tau, T(\phi(U))$ is minimized by minimizing the inner expectation. Hence, the minimum risk scale equivariant estimator is $\hat{\tau}_{MRE} = T\phi_*(U)$, where $\phi_*(U)$ must satisfy the following equation

$$\phi_*^2(u)\left(\frac{u+\varphi_*^{-1}(u)}{u+\varphi_*(u)}\right)^{\frac{3}{2}} k_1(2\sqrt{(u+\phi_*^{-1}(u))(u+\phi_*(u))} = K_{-1}\left(2\sqrt{(u+\phi_*^{-1}(u))(u+\phi_*(u))}\right) , \tag{4.4}$$

where we use the fact that the joint density function of (T, U) is g(t,u), when $\tau = 1$, but Joshi and Nabar [6]

$$g(t,\frac{u}{\tau})=\begin{cases}\dfrac{2e^{-nu(\frac{t}{\tau}+\frac{\tau}{t})}u^{2n-1}}{n^{-2n}[(n-1)!]^2t} & if \quad t>0, u>0\\ 0 & otherwise\end{cases}$$

Note that function $k_r(z)$ are tabulated so their values can be computed. For deriving the Bayes estimator of τ, let us consider the Inverted Gamma distribution as a prior distribution

$$\pi_{\alpha,\lambda}(\tau)=\frac{\lambda^\alpha e^{-\lambda/\tau}}{\tau^{\alpha+1}\Gamma(\alpha)} \quad ; \quad \tau>0\ ,\ \lambda>0$$

Therefore the unique Bayes estimator $\delta_{Bayes}(\alpha,\lambda)=\delta_B$ which is admissible under the loss (1.2) must satisfies the following equation

$$K_{-\alpha-1}\left(2\sqrt{(\delta_B-ut^{-1})(\delta_B^{-1}-\lambda-ut)}\right)=$$
$$\delta_B^2\left(\frac{\delta_B^{-1}-\lambda-ut}{\delta_B-ut^{-1}}\right)k_{-\alpha+1}\left(2\sqrt{(\delta_B-ut^{-1})(\delta_B^{-1}-\lambda-ut)}\right)$$

Note that $\hat{\tau}_{MRE}=\hat{\tau}_{Bayes}(\circ,\circ)$. This means that when the loss function is scale invariant loss (1.2), than $\hat{\tau}_{MRE}$ is a generalized Byes rule against the scale invariant improper prior $\pi(\tau)=\dfrac{1}{\tau}$; $\tau>0$ and is therefore minimax Kariya[8].

Results

The estimation of scale parameter under a bounded loss function is considered with restriction to the principles of invariance and risk unbiasedness. An implicit form of minimum risk scale equivariant estimators and Bayes estimators are obtained. Fisher's problem of the Nile as an example is included.

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