**Journal of Sciences, Islamic Republic of Iran 26(4): 349 - 354 (2015)** http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

# New explicit and Soliton Wave Solutions of Some Nonlinear Partial Differential Equations with Infinite Series Method

A. Aasaraai*

*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan P. O. Box 41335-19141, Rasht, Islamic Republic of Iran.*

Received: 26 May 2015 / Revised: 21 September 2015 / Accepted: 3 October 2015

## Abstract

To start with, having employed transformation wave, some nonlinear partial differential equations have been converted into an ODE. Then, using the infinite series method for equations with similar linear part, the researchers have earned the exact soliton solutions of the selected equations. It is required to state that the infinite series method is a well-organized method for obtaining exact solutions of some nonlinear partial differential equations. In addition, it is worth mentioning that this method can be applied to non-integrable equations as well as integrable ones. This direct algebraic method is also used to construct the new exact solutions of the three given examples. It can also be claimed that any equation matching the special form which has been made in this article, will be solved immediately by means of infinite series method.

**Keywords**: Infinite Series method; Davey-Stewartson (2+1)-dimensional equation; Generalized Hirota-Satsuma coupled KdV equation; Phi-four equation.

## Introduction

Investigation of solutions to nonlinear evolution has become an interesting subject in nonlinear science field. Moreover, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery [1].

It was not until the mid-1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a "thing" displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought. For a more detailed and technical description of the solitary wave, see [2]. In recent years, other methods have been developed, such as the Backlund transformation method [3], Darboux transformation [4], Tanh method [5, 6], Extended tanh function method [7], Exp-function method [8], Generalized hyperbolic function

* Corresponding author: Tel: +981316664408; Fax: +981316666427; Email: Aasaraai@gmail.com

[9] and Modified F-expansion method [10]. In this paper, by using Infinite Series method, the soliton solution is solely obtained. Certainly, the purpose of this paper is to find exact soliton solutions.

Here, the solutions are developed as series in real exponential functions which physically corresponds to mixing of elementary solutions of the linear part due to nonlinearity. Hereman method falls into the category of direct methods for nonlinear partial differential equations. In addition, depending on the number of nonlinear terms in the partial differential equation with arbitrary coefficients, it is sometimes necessary to specialize the particular values of the velocity in order to find closed form solutions. On the other hand, Hereman series method does give a systematic means of developing recursion relations [11].

### *1. Description of Method*

Consider a general nonlinear partial differential equation in the form:

(1) $$F\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}, \dots\right) = 0$$

Where $u = u\left(x, t\right)$ is the solution of nonlinear PDE Eq (1). Furthermore, the transformations which are used are as follows:

(2) $$u\left(x, t\right) = U\left(\xi\right), \qquad \xi = x - \lambda t.$$

Where $\lambda$ is a constant. Using the transformation, it can be found that

(3) $$\frac{\partial}{\partial t}(.) = -\lambda \frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.), \ \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.), \dots$$

At present, the relations (3) is employed to change the nonlinear PDE equation (1), into a nonlinear ordinary differential equation, say

(4) $$G\left(U\left(\xi\right), U_{\xi}\left(\xi\right), U_{\xi\xi}\left(\xi\right), \dots\right) = 0$$

Now, Hereman's approach can be applied. At first, we solve the linear terms then solution is to assume the following form:

(5) $$U\left(\xi\right) = \sum_{n=1}^{\infty} a_n f^{\,n}(\xi).$$

Where $f\,(\xi)$ is a solution of linear terms and the coefficients of the expansion $a_n$ $(n = 1, 2, 3, \dots)$, should be determined. To deal with the nonlinear terms, we need to apply the extension of Cauchy's product rule to multiple series.

**Theorem 1.** (Extension of Cauchy's product rule)
There exists:

(6)
$$\prod_{j=1}^{I} F^{(j)} = \sum_{n=1}^{\infty} \sum_{r=I-1}^{n-1} \sum_{p=I-2}^{q-1} \dots \sum_{m=2}^{s-1} \sum_{l=1}^{m-1} a_l^{(1)} a_{m-l}^{(2)} \dots a_{r-p}^{(I-1)} a_{n-r}^{(I)},$$

That: $$F^{(j)} = \sum_{\vartheta=1}^{I} a_{\vartheta}^{(j)} .$$

$I$ represents infinite convergent series [11].

Substituting (5) into (4), a recursion relation is obtained which gives the values of the coefficients.

### *2. Applications of infinite series method*

Let's assume that Equation (1) based on transformations (2), can be written as:

(7) $$\phi_{\xi\xi}\left(\xi\right) - A\phi\left(\xi\right) - \nu B\phi^3\left(\xi\right) = 0 .$$

Where $A$ and $B$ are constants and $\nu = \pm 1$. In this section, the exact solutions of nonlinear partial differential equations in the form (7) will be obtained. Due to this condition, the solution to linear part of (7) is as the following

$$f\,\left(\xi\right) = \exp(\sqrt{A}\,\xi).$$

Thus, the solution of (1), will be found that as follows

(8) $$\phi\left(\xi\right) = \sum_{n=1}^{\infty} a_n \exp\left(n\sqrt{A}\,\xi\right).$$

Substituting (2) into (1) and by Lemma 1, it is obtained that

(9)
$$\sum_{n=1}^{\infty}\left[\left(n^2 - 1\right) A\, a_n - \upsilon B \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} a_l a_{m-l} a_{n-m}\right] \exp\left(\sqrt{A}\,\xi\right) = 0 \cdot$$

Regarding (9), the recursion relation is as the following

$$a_n = \nu \frac{B}{A} \frac{1}{n^2 - 1}\left(\sum_{m=2}^{n-1} \sum_{l=1}^{m-1} a_l a_{m-l} a_{n-m}\right), \qquad n \ge 3 \tag{10}$$

By assuming that $a_1$ is arbitrary and $a_2 = 0$ in (10), it can be obtained that

(11) $$a_{2k} = 0,$$

$$a_{2k+1} = \left(\nu\right)^k \left(\frac{B}{A}\right)^k \frac{a_1^{2k+1}}{2^{3k}}. \qquad k \in \mathrm{N}$$

Substituting (11) into (8) gives:

(12)
$$\phi(\xi)=\sum_{k=0}^{\infty}(\nu)^k\left(\frac{B}{A}\right)^k\frac{a_1^{2k+1}}{2^{3k}}\exp\left((2k+1)\sqrt{A}\,\xi\right)=\frac{a_1 exp\left(\sqrt{A}\,\xi\right)}{1-(\nu)\frac{Ba_1^2}{8A}exp\left(2\sqrt{A}\,\xi\right)}$$

If $a_1=2\sqrt{\frac{2A}{B}}$, and replacing this condition with (12), it can be written as:

(13) $$\phi(\xi)=\frac{2\sqrt{\frac{2A}{B}}exp\left(\sqrt{A}\,\xi\right)}{1-\nu exp\left(2\sqrt{A}\,\xi\right)}.$$

**Case I:** If $A>0$,

In (13), if $\nu=-1$ and according to the definition, $\operatorname{sech}t=\frac{2e^t}{1+e^{2t}}$, it is obtained:

(14) $$\phi(\xi)=\sqrt{\frac{2A}{B}}sech\left(\sqrt{A}\,\xi\right).$$

Likewise, if $\nu=1$ and according to the definition, $\operatorname{csch}t=\frac{2e^t}{e^{2t}-1}$, it is obtained:

(15) $$\phi(\xi)=-\sqrt{\frac{2A}{B}}csch\left(\sqrt{A}\,\xi\right).$$

**Case II:** If $A<0$,

In (13), if $\nu=-1$ and according to the definition, $\sec t=\frac{2e^{it}}{1+e^{2it}}$, it can be obtained that

(16) $$\phi(\xi)=\sqrt{\frac{2A}{B}}sec\left(\sqrt{A}\,\xi\right).$$

Likewise, if $\nu=1$ and according to the definition $\csc t=\frac{2e^{it}}{e^{2it}-1}$, it is obtained:

(17) $$\phi(\xi)=-\sqrt{\frac{2A}{B}}csc\left(\sqrt{A}\,\xi\right).$$

As it can be seen, the exact solutions (14), (15), (16), and (17) are soliton solutions.

## Results

### *3-1. Davey-Stewartson (2+1)-dimensional Equation*

Consider the following Davey-Stewartston Equation (DSE) in two spatial functions involves a complex field $u$ and a real field $v$ :

(18) $$iu_t+c_0u_{xx}+u_{yy}-c_1|u|^2u-c_2uv_x=0,$$
$$v_{xx}+c_3v_{yy}-\left(|u|^2\right)_x=0.$$

In fluid dynamics, the Davey-stewartson Equation (DSE) was introduced in a paper by Davey and Stewartson (1974) to describe the evolution of a 3-dimensional wave-packet on finite depth water.

By using the transformation:

(19) $$u(x,y,t)=e^{i\theta}U(\xi),\ v(x,y,t)=V(\xi),$$
$$\theta=\alpha x+\beta y+\gamma t \text{ and } \xi=k(x+ly-\lambda t).$$

Where $\alpha$, $\beta$, $\gamma$, $k$, $l$ and $\lambda$ are real constants, it is obtained that $\lambda=2(c_0\alpha+\beta l)$, substituting (19) into (18), it changes has been obtained

(20)
$$\left(c_0k^2+k^2l^2\right)U_{\xi\xi}(\xi)-\left(\gamma+\beta^2+c_0\alpha^2\right)U(\xi)-c_1U^3(\xi)-c_{2k}U(\xi)V_\xi(\xi)=0$$
,

(21) $$\left(k^2+c_3k^2l^2\right)V_{\xi\xi}(\xi)-k\left(U^2(\xi)\right)_\xi=0.$$

Where by integrating of (21) once respect to $\xi$, it can be found that

(22) $$V_\xi(\xi)=\frac{U^2(\xi)}{k(1+c_3l^2)}+\frac{c}{k(1+c_3l^2)}.$$

Where $c$ is integration constant, substituting (22) into (20), the following equation is obtained

(23)
$$U_{\xi\xi}(\xi)-\frac{\gamma+\beta^2+c_0\alpha^2}{c_0k^2+k^2l^2}U(\xi)-\frac{c_1(1+c_3l^2)+c_2+c}{(1+c_3l^2)(c_0k^2+k^2l^2)}U^3(\xi)=0$$

Equation (23) coincides with (7), where A, B and $\nu$ are defined by the relations

(24) $$A=\frac{\gamma+\beta^2+c_0\alpha^2}{c_0k^2+k^2l^2},$$
$$B=\frac{c_1(1+c_3l^2)+c_2+c}{(1+c_3l^2)(c_0k^2+k^2l^2)},\qquad \nu=1.$$

Substituting in (15) from (24), then the solution of (23) can be obtained that

(25)
$$U(\xi)=-\sqrt{\frac{2\left(\left(\gamma+\beta^2+c_0\alpha^2\right)(1+c_3l^2)\right)}{c_1(1+c_3l^2)+c_2+c}}csch\left(\sqrt{\frac{\gamma+\beta^2+c_0\alpha^2}{c_0k^2+k^2l^2}}\xi\right)$$

By integrating of (22) once respect to $\xi$, it can be found that

(26) $$V(\xi)=\frac{U^3(\xi)}{k(1+c_3l^2)}+\frac{c}{k(1+c_3l^2)}\xi.$$

Therefore, the exact solutions of the Davey-Stewartson (2+1)-dimensional will be found as follows:

$$u(x,y,t)=-\sqrt{\frac{2\left(\left(\gamma+\beta^2+c_0\alpha^2\right)(1+c_3l^2)\right)}{c_1(1+c_3l^2)+c_2+c}}csch\left(\sqrt{\frac{\gamma+\beta^2+c_0\alpha^2}{c_0k^2+k^2l^2}}\,(k(x+ly-\lambda t))\right)e^{i(\alpha x+\beta y+2(c_0\alpha+\beta l)t)}$$
,
$$v(x,y,t)=\frac{u^3(x,y,t)}{k(1+c_3l^2)}+\frac{c}{k(1+c_3l^2)}\left(k(x+ly-\lambda t)\right).$$

### *3-2. Generalized Hirota-Satsuma coupled KdV equation*

Consider the following Hirota-Satsuma coupled KdV system [11]

(27) $$u_t=\frac{1}{4}u_{xxx}+2uu_x+3\left(w-v^2\right)_x,$$
$$v_t=\frac{1}{2}v_{xxx}-3uv_x,$$
$$w_t=-\frac{1}{2}w_{xxx}-3uw_x.$$

Where $w=0$, Equations (27) reduces to be the well-known Hirota-Satsuma coupled KdV system [12]. The transformations which are used are as follows

(28)
$$u(x,t)=U(\xi);\quad v(x,t)=V(\xi);\quad w(x,t)=W(\xi);\quad \xi=\alpha(x-\beta t)$$

Substituting (28) into (27), it is obtained that

(29)
$$-\alpha\beta U_\xi(\xi)=\frac{1}{4}\alpha^3U_{\xi\xi\xi}(\xi)+3\alpha U(\xi)U_\xi(\xi)+3\alpha\left(W(\xi)-V^2(\xi)\right)_\xi$$

(30)
$$-\alpha\beta V_\xi(\xi)=-\frac{1}{2}\alpha^3V_{\xi\xi\xi}(\xi)-3\alpha U(\xi)V_\xi(\xi),$$
.

(31) $$-\alpha\beta W_\xi(\xi)=-\frac{1}{2}\alpha^3W_{\xi\xi\xi}(\xi)-3\alpha U(\xi)W_\xi(\xi)$$

Let's suppose

(32) $$U(\xi)=A_0V^2(\xi)+B_0,$$
$$W(\xi)=CV(\xi)+D.$$

where $A_0,B_0,C$ and $D$ are constants. Substituting (32) into (30) and (31), integrating once, it is clear that (30) and (31) give rise to the same equation (33)

$$V_{\xi\xi}(\xi)-\frac{2(\beta-3B_0)}{\alpha^2}V(\xi)+\frac{2A_0}{\alpha^2}V^3(\xi)=0.$$

Equation (33) coincides with (7), where $A, B$ and $\nu$ are defined by

(34) $$A=\frac{2(\beta-3B_0)}{\alpha^2},\ B=-\frac{2A_0}{\alpha^2},\ \nu=-1.$$

Substituting (34) into Eq. (14), then the solution of (33), can be obtained that

(35) $$V(\xi)=\sqrt{\frac{2(3B_0-\beta)}{A_0}}sech\left(\sqrt{\frac{2(\beta-3B_0)}{\alpha^2}}\xi\right),$$

That $\beta>3B_0$. By means of (32) and (28), the solutions of (27) are given by

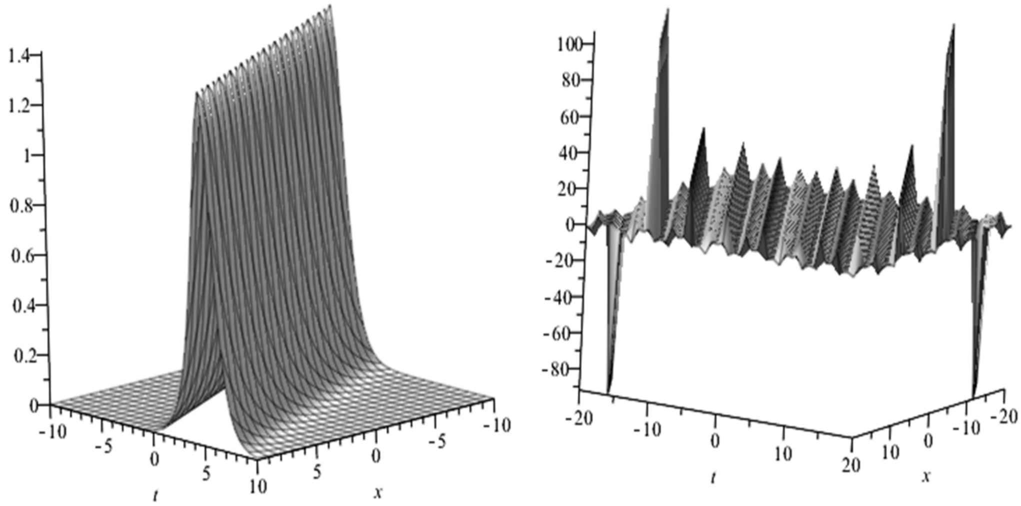
(36)
$$u(x,t)=2(3B_0-\beta)\mathrm{sech}^2\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right)+B_0,$$
$$v(x,t)=\sqrt{\frac{2(3B_0-\beta)}{A_0}}sech\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right),$$
$$w(x,t)=C\sqrt{\frac{2(3B_0-\beta)}{A_0}}sech\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right)+D\cdot$$

If $\beta<3B_0$, it can be written as the following

(37) $$V(\xi)=\sqrt{\frac{2(3B_0-\beta)}{A_0}}sec\left(\sqrt{\frac{2(\beta-3B_0)}{\alpha^2}}\xi\right).$$

Substituting (34) into (16), also by means of (32) and (28), the solutions of (27) are given by

(38) $$u(x,t)=2(3B_0-\beta)\sec^2\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right)+B_0,$$
$$v(x,t)=\sqrt{\frac{2(3B_0-\beta)}{A_0}}sec\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right),$$




**Figuer 1.** (a) Graphic of soliton solution (43) (b) Graphic of soliton solution (44)

$$w(x,t)=C\sqrt{\frac{2(3B_0-\beta)}{A_0}}\,sec\left(\sqrt{2(\beta-3B_0)}(x-\beta t)\right)+D\cdot$$

If $B_0=0$ and $D=0$ then, the exact solutions (36) and (38) of the Hirota-Satsuma coupled KdV system (27) are soliton solutions.

### *3-3. Phi-four equation*

We consider the Phi-four equation [13]

(39) $u_{tt}-au_{xx}-u+u^3=0, \qquad a>0$.

Where $a$ is real constant.

With the transformation defined by

(40) $u(x,t)=U(\xi),\ \xi=\alpha(x-\beta t)$.

It is obtained that

(41) $U_{\xi\xi}(\xi)-\frac{1}{\alpha^2(\beta^2-a)}U(\xi)+\frac{1}{\alpha^2(\beta^2-a)}U^3(\xi)=0\cdot$

Equation (41) coincides with (7), where $A, B$ and $\nu$ are defined by

(42) $A=-B=\frac{1}{\alpha^2(\beta^2-a)},\nu=-1$.

Substituting (42) into (14), and $\beta^2>a$, then the solution of (39) can be obtained that

(43) $u(x,t)=\sqrt{2}\,sech\left(\sqrt{\frac{1}{\beta^2-a}}(x-\beta t)\right)\cdot$

If $\beta^2<a$, then substituting in (16) from (42) the exact solution of (39) can be obtained that

(44) $u(x,t)=\sqrt{2}\,sec\left(\sqrt{\frac{1}{\beta^2-a}}(x-\beta t)\right)\cdot$

At hand solutions of the Phi-four equation (39) are considered as soliton solutions. If $\alpha=1$ and $\beta=2$ are selected, the figures of these solutions of (39), are provided as follows

## Discussion

In this study, Infinite series method was employed to solve the special form (Elliptic-like). Accordingly, the exact solutions were obtained to three selected equations, with the aid of a simple transformation technique. Besides, it was shown that the Davey-Stewartson (2+1)-dimensional Equations, generalized Hirota-Satsuma coupled KdV system and the Phi-four equation, can be reduced to the special form (Elliptic-like) with a specific solution.

In spite of the fact that these new solutions may be important for physical problems, this method can be utilized to solve many systems of nonlinear partial differential equation arising in the theory of soliton and other related areas of research. Finally, it is worthwhile to mention that the proposed method is straightforward and concise.

## Acknowledgement

The authors are very grateful to the referees for their detailed comments and generous help.

## References

1. Scott Russell J., Report on waves, Rept. Fourteenth Meeting of the British Association for the Advancement of Science, John Murray, London. **57**: 311-390 (1844).
2. Scott Alwyn C., The nonlinear universe, Springer-Verlag, Berlin, 150-281 (2007).
3. Miura M. R., Backlund Transformation, Springer-Verlag, Berlin, (1978).
4. Matveev V. B., Salle M. A., Darboux transformations and solitons. Berlin, Heideelberg. Springer. (1991).
5. Evans D. J., Raslan K. R., The tanh function method for solving some important non-linear partial differential equation. Int. J. comput. Math. **82**: 897-905 (2005).
6. Parkes E. J., Duffy B. R., An automated tanh-function method for finding solitary wave solution to nonlinear evolution equations. Comput. Phys. Commun. **98**: 288-300 (1998).
7. Fan E., Extended tanh-function method and its applications to nonlinear equations. Phys. Lett. A. **277**: 212-218 (2000).
8. Biazar J., Ayati Z., Extension of the Exp-function method for systems of two-dimensional Burgers equations , Computers and Mathematics with Applications . **58**: 2103-2106 (2009).
9. Gao Y. T., Tian B., Generalized hyperbolic-function method with computerized symbolic computation to construct the solitonic solutions to nonlinear equations of mathematical physics. Comput. Phys. Commun. **133**: 158-164 (2001).
10. Aasaraai A., Mehrlatifan M. B., Khaleghizadeh S., Modified F-expansion method applied to coupled system of equation. J. of Sci, Islamic Republic of Iran, **24**: 347-354 (2013).
11. Hereman W., Banerjee P. P., Korpel A., Assanto G., Van Immerzeele A., and Meerpole A., Exact solitary wave solutions of nonlinear evolution and wave equations using a direct algebraic method**,** J. Phys. A. Math. Gen, **19**: 607-628 (1986).
12. Hirota R., Satsuma J., Soliton solutions of a coupled Korteweg-de Vriesequation, Phys. Lett. A **85**: 407-408 (1981).
13. Wazwaz, A. M., A sine–cosine method for handling nonlinear wave equations. Math. and Comput. Modeling. **40**: 499-508 (2004).