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A Characterization of the Small Suzuki Groups by the Number of the Same Element Order

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Abstract

Suppose that G is a finite group. Then the set of all prime divisors of $|G|$ is denoted by $\pi(G)$ and the set of element orders of G is denoted by $\pi_e(G)$. Suppose that $k \in \pi_e(G)$. Then the number of elements of order k in G is denoted by m_k and the sizes of the set of elements with the same order is denoted by $nse(G)$; that is, $nse(G) = \{m_k : k \in \pi_e(G)\}$. In this paper, we prove that if G is a group such that $nse(G) = nse(Sz(n))$, where $n \in \{32,128\}$, then $G \cong Sz(n)$. Here $Sz(n)$ denotes the family of Suzuki simple groups, $n = 2^{2k+1}$, $k \in$. This proves that the second and third member of the family of Suzuki simple groups are characterizable by the set of the number of the same element order.

Keywords: Element order; Sylow subgroup; Simple K_n-group; Suzuki group.

Introduction

Suppose that G is a finite simple group and $|\pi(G)| = n$, where $|\pi(G)|$ denotes the number of prime numbers dividing the order of G. Then G is called a simple K_n-group. Suppose that G is a finite group. Then a Sylow q-subgroup of G is denoted by P_q and the number of Sylow q-subgroups of G is denoted by n_q and the greatest order of elements in P_q is denoted by $\exp(P_q)$. The Euler totient function is denoted by $\varphi(n)$. The set of sizes of conjugacy classes has an essential role in determining of the structure of a finite group. So one might ask whether the set of sizes of elements with the same order has an essential role in determining the structure of a finite group. In [9], it is proved that all simple K_4-groups can be uniquely determined by $nse(G)$ and $|G|$. But in [1,6,10], it is proved that the groups A_4, A_5, A_6, $Sz(8)$ and the groups $L_2(q)$, for $q \in \{7,8,11,13\}$ are uniquely determined only by $nse(G)$. In this paper, we prove

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that if G is a group such that $nse(G) = nse(Sz(n))$, where $n \in \{32,128\}$, then $G \cong Sz(n)$.

Preliminary and Notations

In this section, we bring some lemmas that is need in the proof of main theorem.

Lemma 1.1 [5] If G is a simple K_3-group, then G is isomorphic to one of the following groups:

A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

Lemma 1.2 [8] If G is a simple K_4-group, then G is isomorphic to one of the following groups:

(1) A_7, A_8, A_9, A_{10}.

(2) M_{11}, M_{12}, J_2.

(3) (a) $L_2(r)$, where r is a prime and satisfies

$r^2 - 1 = 2^a \times 3^b \times v^c$

with $a \geq 1$, $b \geq 1$, $c \geq 1$, $v > 3$, v is a prime.

(b) $L_2(2^m)$, where m satisfies

$$\begin{cases} 2^m - 1 = u \\ 2^m + 1 = 3t^b \end{cases}$$

with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, where m satisfies

$$\begin{cases} 3^m + 1 = 4t \\ 3^m - 1 = 2u^c \end{cases} \text{ or } \begin{cases} 3^m + 1 = 4t^b \\ 3^m - 1 = 2u \end{cases}$$

with $m \geq 2$, u, t are odd primes, $b \geq 1$, $c \geq 1$.

(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Lemma 1.3 [3] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \ldots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, \ldots, s\}$:

(1) $q_i^{\beta_i} \equiv 1 \left(\operatorname{mod} p_j\right)$, for some p_j.

(2) The order of some chief factor of G is divisible by $q_i^{\beta_i}$.

Lemma 1.4 [2] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G : g^m = 1\}$, then $m \mid |L_m(G)|$.

Lemma 1.5 [10] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k : k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Lemma 1.6 [7] Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p,m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.

Lemma 1.7 [4] Let G be a solvable group and π be any set of primes. Then

(1) G has a Hall π-subgroup.

(2) If H is a Hall π-subgroup of G and V is any π-subgroup of G, then $V \leq H^g$ for some $g \in G$. In particular, the Hall π-subgroups of G form a single conjugacy class of subgroups of G.

Lemma 1.8 Let G be a finite group which is not solvable. Then there is a normal series $1 (N (M (G$ such that N is a maximal solvable normal subgroup of G and M/N is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups.

Proof. Since G is a finite group, there is chief series $1 = M_0 (M_1 (\ldots (M_{n-1} (M_n = G$. Since G is not solvable, there is a maximal i such that M_{i-1} is solvable and M_i/M_{i-1} is not solvable. On the other hand, we know that every chief factors is a simple group or the direct product of isomorphic simple groups. Therefore M_{i-1} is a maximal solvable normal subgroup of G and M_i/M_{i-1} is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups.

Lemma 1.9 Let G be a group such that $nse(G)=nse(Sz(n))$, where $n\in\{32,128\}$. Then G is finite and for every $i\in\pi_e(G)$,

$$\begin{cases}\varphi(i)\mid m_i\\ i\mid \sum_{d\mid i} m_d\end{cases}$$

and if $i>2$, then m_i is even.

Proof. By Lemma 1.5, G is a finite group. By Lemma 1.4, $i\mid \sum_{d|i} m_d$. We know that the number of elements of order i in a cyclic group of order i is equal with $\varphi(i)$. Hence $m_i=\varphi(i)k$, where k is the number of cyclic subgroups of order i in G. Thus $\varphi(i)\mid m_i$. We know that if $i>2$, then $\varphi(i)$ is even and since $\varphi(i)\mid m_i$, we conclude that m_i is even. □

Results

In this section, we prove two theorems as the main results of our paper. The first theorem is the following theorem:

Theorem 2.1 Suppose that G is a group such that $nse(G)=nse(Sz(32))$. Then $G\cong Sz(32)$.

Proof. By a program written in the *GAP*, we have in

nse(G)=nse(Sz(32))=
{1,31775,1016800,1301504,6507520, 7936000,15744000 }.

We prove this theorem in five steps.

Step 1. $\pi(G)=\{2,5,31,41\}$.

Since 31775 is odd, Lemma 1.9 implies that $2\in\pi(G)$ and $m_2=31775$. Assume that $q\in\pi(G)$ and $q\neq 2$, by Lemma 1.9, $q\mid(1+m_q)$ and $(q-1)=\varphi(q)\mid m_q$, which imply that $q\in\{3,5,7,13,31,41,6507521\}$. If $6507521\in\pi(G)$, then by Lemma 1.9, $m_{6507521}=6507520$. On the other hand, if $13015042=2\times 6507521\in\pi_e(G)$, then by Lemma 1.9, $\varphi(13015042)\mid m_{13015042}$ and $13015042\mid(1+m_2+m_{6507521}+m_{13015042})$, which is a contradiction. Hence $2\times 6507521\notin\pi_e(G)$. Thus $P_{6507521}$ acts fixed point freely on the set of elements of order 2 by conjugation. Therefore $|P_{6507521}|\mid m_2$, which is a contradiction. So $6507521\notin\pi(G)$. If $13\in\pi(G)$, then by Lemma 1.9, $m_{13}=15744000$. On the other hand, if $26=2\times 13\in\pi_e(G)$, then by Lemma 1.9, $\varphi(26)\mid m_{26}$ and $26\mid(1+m_2+m_{13}+m_{26})$, which is a contradiction. Hence $2\times 13\notin\pi_e(G)$. Thus P_{13} acts fixed point freely on the set of elements of order 2 by conjugation. Therefore $|P_{13}|\mid m_2$, which is a contradiction. So $13\notin\pi(G)$. If $7\in\pi(G)$, then by Lemma 1.9, $m_7=15744000$. On the other hand, if $14=2\times 7\in\pi_e(G)$, then by Lemma 1.9, $\varphi(14)\mid m_{14}$ and $14\mid(1+m_2+m_7+m_{14})$, which is a contradiction. Hence $14=2\times 7\notin\pi_e(G)$. Thus P_7 acts fixed point freely on the set of elements of order 2 by conjugation. Therefore $|P_7|\mid m_2$, which is a contradiction. So $7\notin\pi(G)$. Therefore we conclude that $\pi(G)\subseteq\{2,3,5,31,41\}$.

If $\{2,3,5,31,41\}\subseteq\pi(G)$, then by Lemma 1.9, $m_2=31775$, $m_3=1301504$, $m_5=1301504$, $m_{31}=15744000$, $m_{41}=7936000$ and $2^{13},3^3,5^3,31^2,41^2,2\times 31,3\times 41,31\times 41\notin\pi_e(G)$.

Since $2^{13}\notin\pi_e(G)$, we conclude that $\exp(P_2)\in\{2,\ldots,2^{12}\}$. If $\exp(P_2)=2^2$, then by Lemma 1.4 and considering $m=|P_2|$, we conclude that $|P_2|\mid 2^{20}$ otherwise $|P_2|\mid 2^{19}$.

Since $3^3 \notin \pi_e(G)$, we conclude that $\exp(P_3)=3$ or 3^2. There are two cases:

Case 1. If $\exp(P_3)=3$, then by Lemma 1.4 and considering $m=|P_3|$, we conclude that $|P_3|=3$. Hence P_3 is cyclic and $n_3=\frac{m_3}{\varphi(3)}=2^9\times 31\times 41$.

Case 2. If $\exp(P_3)=3^2$, then by Lemma 1.4 and considering $m=|P_3|$, we conclude that $|P_3| \mid 3^3$. If $|P_3|=3^3$, then P_3 is not cyclic. Hence by Lemma 1.6, $9 \mid m_9=15744000$, which is a contradiction. Therefore $|P_3|=3^2$ and $n_3=\frac{m_{3^2}}{\varphi(3^2)}=2^9\times 5^3\times 41$.

Since $5^3 \notin \pi_e(G)$, we conclude that $\exp(P_5)=5$ or 5^2. If $\exp(P_5)=5$, then by Lemma 1.4 and by considering $m=|P_5|$, we conclude that $|P_5|=5$ and $n_5=\frac{m_5}{\varphi(5)}=2^8\times 31\times 41$. If $\exp(P_5)=5^2$, then by Lemma 1.4 and considering $m=|P_5|$, we conclude that $|P_5|=5^2$ and $n_5=\frac{m_{5^2}}{\varphi(5^2)}=2^8\times 31\times 41$.

Since $31^2 \notin \pi_e(G)$, by Lemma 1.4 and considering $m=|P_{31}|$, we conclude that $|P_{31}|=31$ and $n_{31}=\frac{m_{31}}{\varphi(31)}=2^9\times 5^2\times 41$.

Since $41^2 \notin \pi_e(G)$, by Lemma 1.4 and considering $m=|P_{41}|$, we conclude that $|P_{41}| \mid 41^2$.

Now we show that $3 \notin \pi(G)$.

If $3 \in \pi(G)$, then by the above discussion, $n_3=2^9\times 31\times 41$ or $2^9\times 5^3\times 41$. Hence $41 \mid |G|$. Since $3\times 41 \notin \pi_e(G)$, we conclude that P_3 acts fixed point freely on the set of elements of order 41 by conjugation. Hence $|P_3| \mid m_{41}$, which is a contradiction. So $3 \notin \pi(G)$. Therefore $\pi(G) \subseteq \{2,5,31,41\}$.

If $\pi(G)=\{2\}$, then we know that $|nse(G)|=7$. Thus $\exp(P_2)>4$. Hence $|G|=|P_2| \mid 2^{19}$. So $1 \le m_4 \le 2^{19}$, but $m_4 \in \{1016800, 1301504, 6507520, 7936000, 15744000\}$, which is a contradiction.

If $\pi(G)=\{2,41\}$, then we know that 2^{13}, $41^2 \notin \pi_e(G)$ and $|P_2| \mid 2^{20}$, $|P_{41}| \mid 41^2$. Hence $\pi_e(G) \subseteq \{1, 2, \ldots, 2^{12}\} \cup \{41, 41\times 2, \ldots, 41\times 2^{12}\}$. Therefore,

$$|G| = 2^1 \times 41^k = 32537600 + 1016800k_1 + 1301504k_2 + 6507520k_3 + 7936000k_4 + 15744000k_5$$

where $0 \le k_1+k_2+k_3+k_4+k_5 \le 19$, $l \le 20$, $k \le 2$. It is easy to check that this equation has no solution.

If $5 \in \pi(G)$, then $n_5=2^8\times 31\times 41$. We know that $n_5 \mid |G|$. Hence $31 \mid |G|$.

Therefore in any cases we can assume that $31 \in \pi(G)$.

Now we prove that $\pi(G)=\{2,5,31,41\}$. Since $31 \in \pi(G)$, we conclude that $|P_{31}|=31$ and $n_{31}=\frac{m_{31}}{\varphi(31)}=2^9\times 5^2\times 41$. We know that $n_{31} \mid |G|$, hence $2^9\times 5^2\times 41 \mid |G|$. It follows that $\pi(G)=\{2,5,31,41\}$.

Step 2. $|G|=2^k\times 5^l\times 31\times 41$, where $k\le 10$, $l \le 2$.

By the above discussion $|P_{31}|=31$, $|P_5| \mid 5^2$.

Since $62 \notin \pi_e(G)$, we conclude that P_2 acts fixed point freely on the set of elements of order 31 by conjugation. Therefore $|P_2| \mid m_{31}$. Hence $|P_2| \mid 2^{10}$.

Since $1271 \notin \pi_e(G)$, we conclude that P_{41} acts fixed point freely on the set of elements of order 31 by conjugation. Therefore $|P_{41}| \mid m_{31}$. Hence $|P_{41}|=41$.

Step 3. G is not solvable.

If G is solvable, then by Lemma 1.7, G has a Hall

π-subgroup H, where $\pi=\{5,31,41\}$ and all the Hall π-subgroups of G are conjugate and the number of Hall π-subgroups of G is $|G:N_G(H)| \mid 2^{10}$. Since G is solvable, we conclude that H is solvable. Hence by Lemma 1.3, there are non negative integers $\alpha_1,\ldots,\alpha_r$, $\beta_1,\ldots,\beta_s$ such that $n_{31}(H)=5^{\alpha_1+\ldots+\alpha_r}\times 41^{\beta_1+\ldots+\beta_s}$, $5^{\alpha_i}\equiv 1(\bmod 31)$, $41^{\beta_i}\equiv 1(\bmod 31)$. Since $|G|=2^k\times 5^l\times 31\times 41$, where $k\le 10$, $l\le 2$, we conclude that $\alpha_1+\ldots+\alpha_r\le 2$, $\beta_1+\ldots+\beta_s\le 1$. Therefore $n_{31}(H)=1$. So $30\le m_{31}(G)\le(2^{10}\times 30)=30720$, but we have $m_{31}(G)=15744000$, which is a contradiction.

Step 4. $|G|=2^{10}\times 5^2\times 31\times 41$.

Since G is a finite group which is not solvable, there is a normal series $1 \lhd N \lhd M \lhd G$ such that N is a maximal solvable normal subgroup of G and M/N is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups, by Lemma 1.8. Let $M/N\cong S_1\times\ldots\times S_r$, where S_1 is a non-abelian simple group and $S_1\cong\ldots\cong S_r$. Since $1 \lhd N \lhd M \lhd G$ and $|G|=2^k\times 5^l\times 31\times 41$, where $k\le 10$, $l\le 2$, we conclude that $r=1$ and M/N is a simple K_3-group or a simple K_4-group.

If M/N is a simple K_3-group, then by Lemma 1.1 and $|G|=2^k\times 5^l\times 31\times 41$, where $k\le 10$, $l\le 2$, we conclude a contradiction.

If M/N is a simple K_4-group, then by Lemma 1.2 and $|G|=2^k\times 5^l\times 31\times 41$, where $k\le 10$, $l\le 2$, we conclude that $M/N\cong Sz(32)$. Hence $2^{10}\times 5^2\times 31\times 41=|M/N| \mid |G| \mid 2^{10}\times 5^2\times 31\times 41$. So $|G|=|Sz(32)|$.

Step 5. $G\cong Sz(32)$.

Since $1 \lhd N \lhd M \lhd G$, $M/N\cong Sz(32)$ and $|G|=|Sz(32)|$, we can conclude $N=1$, $G=M\cong Sz(32)$ and the proof is completed. □

The second theorem as the main result is the following theorem:

Theorem 2.2 Suppose that G is a group such that $nse(G)=nse(Sz(128))$. Then $G\cong Sz(128)$.

Proof. By a program written in the *GAP*, we have nse(G)=nse(Sz(128))=
{1,2080895,266354560,235126784,1645887488,6583549952,8447918080,16912465920}.

We prove this theorem in four steps.

Step 1. $\pi(G)=\{2,5,29,113,127\}$.

Since 2080895 is odd, Lemma 1.9 implies that $2\in\pi(G)$ and $m_2=2080895$. Assume that $q\in\pi(G)$ and $q\ne 2$ by Lemma 1.9, $q \mid (1+m_q)$ and $(q-1)=\varphi(q) \mid m_q$ which imply that $q\in\{3,5,11,13,29,113,127\}$. If $13\in\pi(G)$, then by Lemma 1.9, $m_{13}=16912465920$. On the other hand, by Lemma 1.9, $13^2\notin\pi_e(G)$. Thus $|P_{13}| \mid (1+m_{13})$. Therefore $|P_{13}|=13$ and $n_{13}=\dfrac{m_{13}}{\varphi(13)}=1409372160$. Since $113 \mid n_{13}$, we deduce that $113\in\pi(G)$. Now by Lemma 1.9, $13\times 113\notin\pi_e(G)$. Thus P_{13} acts fixed point freely on the set of elements of order 113 by conjugation. Therefore $|P_{13}| \mid m_{113}=8447918080$, which is a contradiction. So $13\notin\pi(G)$. Similarly, we can prove that $11\notin\pi(G)$.

If $3\in\pi(G)$, then by Lemma 1.9, $m_3\in\{235126784,1645887488,6583549952\}$. On the other hand, by Lemma 1.9, $3^2\notin\pi_e(G)$. Thus $|P_3| \mid (1+m_3)$. Therefore $|P_3|=3$ and $n_3=\dfrac{m_3}{\varphi(3)}\in\{117563392,822943744,3291774976\}$. Since $127 \mid n_3$, we deduce that $127\in\pi(G)$. Now by Lemma 1.9, $127^2\notin\pi_e(G)$. Thus

$|P_{127}| \mid (1+m_{127}) = (1+16912465920)$. Therefore $|P_{127}| = 127$ and $n_{127} = \frac{m_{127}}{\varphi(127)} = 134225920$. Since $29 \mid n_{127}$, we deduce that $29 \in \pi(G)$. Now by Lemma 1.9, $3\times 29 \notin \pi_e(G)$. Thus P_3 acts fixed point freely on the set of elements of order 29 by conjugation. Therefore $|P_3| \mid m_{29} = 1645887488$, which is a contradiction. So $3 \notin \pi(G)$. If $\{2,5,29,113,127\} \subseteq \pi(G)$, then by Lemma 1.9, $m_2 = 2080895$,

$m_5 = 235126784$,

$m_{29} = 1645887488$,

$m_{113} = 8447918080$,

$m_{127} = 16912465920$

and $2^{18}, 5^2, 29^2, 113^2, 127^2 \notin \pi_e(G)$. Thus by Lemma 1.4 and considering $m = |P_5|$, we conclude that $|P_5| = 5$. Similarly, $|P_{29}| = 29$, $|P_{113}| = 113$ and $|P_{127}| = 127$.

If $\pi(G) = \{2\}$, then since $2^{18} \notin \pi_e(G)$, we conclude that $\pi_e(G) \subseteq \{1,2,2^2,\ldots,2^{17}\}$. Therefore

$$\begin{aligned}|G| = 2^k = 34093383680 &+ 266354560k_1\\ &+ 235126784k_2 + 1645887488\\ &+ 6583549952k_4 + 8447918080k_5\\ &+ 16912465920k_6\end{aligned}$$

where $k, k_1, k_2, k_3, k_4, k_5$ and k_6 are non-negative integers and $0 \le k_1+k_2+k_3+k_4+k_5+k_6 \le 10$. It is easy to check that this equation has no solution.

If $5 \in \pi(G)$, then $|P_5| = 5$ and $n_5 = \frac{m_5}{\varphi(5)} = \frac{235126784}{4} = 58781696$. We know that $n_5 \mid |G|$. Hence $127 \in \pi(G)$. Similarly, we can prove that if $29 \in \pi(G)$ or $113 \in \pi(G)$, then $127 \in \pi(G)$. So in any cases, we can assume that $127 \in \pi(G)$.

Now we prove that $\pi(G) = \{2,5,29,113,127\}$. Since $127 \in \pi(G)$, we conclude that $|P_{127}| = 127$ and $n_{127} = \frac{m_{127}}{\varphi(127)} = 134225920$. We know that $n_{127} \mid |G|$. Hence $134225920 \mid |G|$. It follows that $\pi(G) = \{2,5,29,113,127\}$.

Step 2. $|G| = 2^k \times 5 \times 29 \times 113 \times 127$, where $13 \le k \le 14$.

By the above discussion $|P_5| = 5$, $|P_{29}| = 29$, $|P_{113}| = 113$ and $|P_{127}| = 127$.

By Lemma 1.9, $2\times 127 \notin \pi_e(G)$. Thus P_2 acts fixed point freely on the set of elements of order 127 by conjugation. Therefore $|P_2| \mid m_{127}$. Hence $|P_2| \mid 2^{14}$. On the other hand, since $n_{127} \mid |G|$, we deduce that $2^{13} \mid |G|$. Hence $2^{13} \mid |P_2|$.

Step 3. G is not solvable.

If G is solvable, then by Lemma 1.7, G has a Hall π-subgroup H, where $\pi = \{5,29,113,127\}$ and all the Hall π-subgroups of G are conjugate and the number of Hall π-subgroups of G is $|G : N_G(H)| \mid 2^{14}$. Since G is solvable, we conclude that H is solvable. Hence by Lemma 1.3, there are nonnegative integers $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s,\gamma_1,\ldots,\gamma_t$ such that $n_5(H) = 29^{\alpha_1+\ldots+\alpha_r} \times 113^{\beta_1+\ldots+\beta_s} \times 127^{\gamma_1+\ldots+\gamma_t}$, $29^{\alpha_i} \equiv 1 \pmod 5$, $113^{\beta_i} \equiv 1 \pmod 5$, $127^{\gamma_i} \equiv 1 \pmod 5$.

Since $|G| = 2^k \times 5 \times 29 \times 113 \times 127$, where $13 \le k \le 14$, we conclude that $\alpha_1+\ldots+\alpha_r \le 1$, $\beta_1+\ldots+\beta_s \le 1$, $\gamma_1+\ldots+\gamma_t \le 1$. Therefore $n_5(H) = 1$. So $4 \le m_5(G) \le (2^{14}\times 4) = 65536$, but we have $m_5(G) = 235126784$, which is a contradiction.

Step 4. $G \cong Sz(128)$.

Since G is a finite group which is not solvable, there is a normal series $1 (N (M (G$ such that N is a maximal solvable normal subgroup of G and M/N is a non-abelian simple group or the direct product of isomorphic non-abelian simple groups, by

Lemma 1.8. Let $M/N \cong S_1 \times \ldots \times S_r$, where S_1 is a non-abelian simple group and $S_1 \cong \ldots \cong S_r$. Since $1(N(M(G$ and $|G| = 2^k \times 5 \times 29 \times 113 \times 127$, where $13 \le k \le 14$, we conclude that $r = 1$ and M/N is a non-abelian simple group. Since $3 \nmid |G|$, we deduce that $3 \nmid |M/N|$. We know that the group $Sz(q)$ is only non-abelian simple group such that $3 \nmid |Sz(q)|$. Hence $M/N \cong Sz(128)$ and since $|G| = 2^k \times 5 \times 29 \times 113 \times 127$, where $13 \le k \le 14$, we deduce that $|N| = 1$ and $G = M \cong Sz(128)$.

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