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A Kind of Non-commuting Graph of Finite Groups

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Abstract

Let g be a fixed element of a finite group G. We introduce the g-noncommuting graph of G whose vertex set is whole elements of the group G and two vertices x,y are adjacent whenever [x,y] $\neq$ g and [y,x] $\neq$ g. We denote this graph by Γ_G^g . In this paper, we present some graph theoretical properties of g-noncommuting graph. Specially, we investigate about its planarity and regularity, its clique number and dominating number. We prove that if G, H are isoclinic groups with |Z (G)|=|Z (H)|, then their associated graphs are isomorphic.

Keywords: g-noncommuting graph; commutator; non-commuting graph; isoclinism.

Introduction

The powerful combinatorial methods found in graph theory have been used to prove fundamental results in other areas of pure mathematics. One of the interesting tools in mathematics is to give a connection between some different areas of general mathematics which create some new interdisciplinary branches in mathematics for instance algebraic topology, algebraic geometry, differential geometry and algebraic graphs. By this method we can consider a problem in some different views of mathematics and find some more solutions for a problem. Group theory in mathematics, probability in statistics and graph theory in applied mathematics are the subject which we are going to state a relation between them.

Let G be a group. The probability of commuting two random elements of a group was investigated by Erdos and Turan in [2]. It is called commutativity degree and is denoted by d(G). By this new concept (commutativity degree) we may improve some known results in group theory. For instance we know that if G is abelian, then G is nilpotent. In 1995, Lescot [8] proved that if the commutativity degree of a group is bigger than half, then G is nilpotent. Another relation between group theory and graph theory is to associate a graph to the group G, which is denoted by Γ_G. The non-commuting graph Γ_G was first introduced by Paul Erdos. He associated to the group G a graph whose vertices is G\ Z (G) and two vertices are joined by an edge whenever they do not commute, where Z (G) is the center of G. The non-commuting graph has been studied in a couple of papers (for instance see [1, 9, 11]). In fact, the number of edges of the non-commuting graph displays that how much a group associated to the graph, is far from to be an abelian group. Similarly the number of edges of the non-commuting graph and commutativity degree are in the opposite proportion.

Creating a graph by a group, semigroup or ring is a topic which is increasingly interested by authors (see [3,4]). Pournaki and Sobhani generalized the

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commutativity degree to the probability of the commutator of two randomly chosen elements in a finite group G is equal to a given element g in G (see [10]).

In this paper we extend the notion of non-commuting graph. We assign to the group G and its certain element g∈ G, a graph Γ_G^g namely g-noncommuting graph of G with vertex set G such that two vertices are adjacent if their commutator is not equal to g and g^{-1}. We prove that there is no g-noncommuting tree graph. The bounds for domination number, clique number, chromatic number and independence number of the g-noncommuting graph are also presented here. Moreover, all the groups G for which Γ_G^g is planar will be determined.

In order to classify the groups, P. Hall in [6] introduced the concept of isoclinism which is weaker than isomorphism. It is an equivalence relation on the class of all groups. Let us state its definition.

Definition 1.1. Let G and H be two groups. A pair (φ,ψ) is called an isoclinism of groups G and H if φ is an isomorphism from G/Z(G) to H/Z(H), ψ is also an isomorphism from G' to H' and $\psi([g_1,g_2])=[h_1,h_2]$ whenever $h_i \in \varphi(g_i Z(G))$, for all $g_i \in G$, $h_i \in H$, $i \in \{1,2\}$. If there is an isoclinism from G to H, we say that G and H are isoclinic and denote it by G⊔ H.

Finally, it is proved that if (φ,ψ) is an isoclinism between two groups G and H such that the order of their centers are equal, then $\Gamma_G^g \cong \Gamma_H^{\psi(g)}$, where g∈ G'.

Throughout the paper, graphs are simple and all the notations and terminologies about the graphs are standard (for instance see [5]).

Results

Definition 2.1. Let G be a group and g a fixed element of G. We denote the g-noncommuting graph of G by Γ_G^g as the graph with vertex set G and two distinct vertices x and y join by an edge if $[x,y] \neq g$ and $[y,x] \neq g$.

It is clear that $\Gamma_G^g = \Gamma_G^{g^{-1}}$ and the non-commuting graph Γ_G is an induced subgraph of Γ_G^1. Also the commuting graph of G is an induced subgraph of Γ_G^g when $g \neq 1$. It is easy to see that diam(Γ_G^g)=2 and girth(Γ_G^g)=3, where $g \neq 1$. Indeed, Γ_G^1 is disconnected, because elements of the center of G are isolated. However, if we consider its subgraph Γ_G, then it would be connected and diam(Γ_G)=2 (see [1]).

In the following lemma we give the degree of each vertex in Γ_G^g for all cases.

Lemma 2.2. Let x∈ G.

(i) If $g^2 \neq 1$, then deg(x)=|G|-ε|C_G (x)|-1, where ε=1 if x is conjugate to xg or xg^{-1},but not both, and ε=2 if x is conjugate to xg and xg^{-1}.

(ii) If g^2= 1 and $g \neq 1$, then deg(x) =|G|-|C_G (x)|-1 whenever xg is conjugate to x. For g=1 we have deg(x) =|G|-|C_G (x)|.

(iii) If xg and x$g^{-1}$are not conjugate to x, then deg(x) =|G|-1.

Proof. Suppose there is an element y in the group G such that x^y=xg. This means we have $[x,y]=g$. Now we must answer to this question that how many elements like y exist in the group G? Obviously these elements are not adjacent to x. If we consider the conjugates of x, then how many of them are equal to xg? It is clear that $x^{y_1}=x^{y_2}$, whenever $y_1y_2^{-1} \in C_G$ (x). Hence there are |C_G (x)| elements like y which does not join x.

(i) If x and xg are conjugate, then |{y∈ G: x^y =xg}|=|C_G (x)|. Also if x and xg^{-1} are conjugate, then |{y∈ G: x^y=xg^{-1}}|=|C_G (x)|. The second and third part is clear. ■

A dominating set for a graph Γ is a subset D of V(Γ) such that every vertex which does not belong to D joins to at least one member of D by an edge. The domination number $\gamma(\Gamma)$ is the number of vertices in a smallest dominating set for Γ.

Proposition 2.3. Let g be an element of the group G of even order. Then $\gamma(\Gamma_G^g) = 1$.

Proof. The singleton {g} is a dominating set, because if x is a vertex which is not adjacent to g, then [g,x]=g. Thus |g|=|g^2|, which is a contradiction.■

Proposition 2.4. The g-noncommuting graph of a non-trivial group G is not a tree, unless |G|=2 and $g \neq 1$.

Proof. Since Γ_G^1 is a disconnected graph, it is certainly not a tree. Assume that $g \neq 1$. If G is abelian, then Γ_G^g is a complete graph which is a tree only if

|G|=2. Suppose that G is a non-abelian group. If $g^2=1$, then there is no vertex of degree one. Assume otherwise x is a vertex of degree one, then 1=deg(x) =|G|-|C_G (x)|-1 which implies that |G|=4, a contradiction. In the case that $g^2 \neq 1$, by the same method, we can deduce that |G|=4 or 6. Since G is non-abelian, we have G$\cong S_3$. However, $\Gamma_{S_3}^{(123)}$ is not a tree. ■

The above proposition implies that there is no g-noncommuting star graph, except for a group of order 2 and a non-trivial element g.

Theorem 2.5. If G is a non-abelian group and Γ is the induced subgraph of Γ_G^g with vertex set G\ Z (G), then Γ is not a tree.

Proof. Suppose Γ is a tree. If g=1, then there is a vertex x in Γ such that deg(x) =|G|-|C_G (x)|=1, which is a contradiction. Suppose that g$\neq$ 1. If g^2=1 and x is a vertex such that deg(x) =|G|-|Z (G)|-| C_G (x)|-1=1, then |Z (G)|=1 or 2. If |Z (G)|=1, then (k-1)| C_G (x)|=3, for the positive integer k. Consequently k=2 and G$\cong$ S_3. But for all g$\in S_3$ such that g^2=1, Γ is complete. Now if |Z (G)|=2, then G$\cong$ D_8 or Q_8. Consider D_8 =< a,b: $a^4=b^2=1$, $a^b=a^{-1}$> and Q_8={±1, ±i, ±j, ±k}, where $(-1)^2$=1, $i^2=j^2=k^2$=-1,ij=-ji=k, jk=-kj=i, ki=-ik=j. In both cases Γ is disconnected. If $g^2 \neq 1$, then there is a vertex x such that deg(x) =|G|-|Z (G)|- ε|C_G (x)|-1=1, where ε=1 or 2. For the case ε =1, we have a similar argument as above. If ε=2, then |Z (G)|=1 or 2. Let |Z (G)|=1. Then |G|=9, which is impossible. Suppose that |Z (G)|=2. Then G$\cong$ D_{12} or T, where D_{12}=<a,b: $a^6=b^2=1$, $a^b=a^{-1}$> and T=< a,b: a^6=1, $b^2=a^3$, $a^b=a^{-1}$>. Suppose that G$\cong$ D_{12} or T. We have G'= $\{1, a^2, a^4\}$ and Z (G)={1, a^3 }. Thus in $\Gamma_G^{a^2}=\Gamma_G^{a^4}$, the vertices a, a^2 and a^4 make a cycle. Thus Γ is not a tree. Hence the assertion follows. ■

Let us recall that we use the notations ω(X), χ(X) and α(X) to denote clique, chromatic and independence number of the graph X.

Central elements of the group G join to all other vertices and make a clique for Γ_G^g whenever g$\neq$ 1.

Proposition 2.6. If g is a non-central element of the group G then $\omega(\Gamma_G^g)\geq$|Z(G)|+2.

Proof. We claim that there exists x $\in$ G\ Z(G) adjacent to g and so Z(G) $\cup$ {g,x} would be a clique in Γ_G^g. Suppose on the contrary that [x,g]= g^{-1} for all x$\in$ G\ Z(G), x$\neq$ g. Thus C_G (g) = {g}$\cup$Z(G) and so |Z(G)|=1. Hence |C_G (g)|=2 and g^2=1 which is a contradiction. ■

The above proposition implies that if g is a non-central element of the group G, then $\chi(\Gamma_G^g)\geq$ |Z(G)|+2 and $\alpha(\Gamma_G^g)\leq$|G|-|Z(G)|-2. Furthermore, if g $\neq$ 1, then $\alpha(\Gamma_G^g)\geq$max{|x|-1: x$\in$ G}.

Proposition 2.7. Let g and h be two conjugate elements of G, then $\Gamma_G^g \cong \Gamma_G^h$.

Proof. Suppose h= g^x for some x$\in$ G. The bijection ψ : V(Γ_G^g)$\rightarrow$V(Γ_G^h) which maps t to t^x for every t$\in$ G, preserves edges. ■

One can see that if G is abelian, then Γ_G^g is regular, for every g$\in$G. The following theorem deals with regularity of non-abelian groups.

Proposition 2.8. Let G be a non-abelian group. Then Γ_G^g is a regular graph if and only if g$\notin$K(G), where K(G)={[x,y]: x,y$\in$ G}.

Proof. It is clear that Γ_G^1 is not regular. Suppose g $\neq$ 1 and Γ_G^g is a regular graph. Since degree of the identity element is |G|-1, so deg(t)=|G|-1 for every t$\in$ G. This means [t,t'] $\neq$ g for all t,t'$\in$ G and g$\notin$K(G). The converse is clear. ■

Theorem 2.9. Let Γ be the induced subgraph of Γ_G^g with vertex set G\ Z (G), where G is a non-abelian group. Then

(i) Γ is not 2-regular.

(ii) Γ is 3-regular if and only if G is a group of order 16 and |Z(G)|=4, for g$\neq$ 1.

(iii) Γ is 4-regular if and only if G$\cong$ D_8 or Q_8 and g=1.

(iv) Γ is 5-regular if and only if G is a group of order 27 and |Z(G)|=3, for g$\neq$ 1.

Proof. (i) Suppose Γ is 2-regular. By Lemma 2.2 we have all the possibilities for the degree of vertices. If g=1, then deg(x)=|G|-|C_G (x)| for all x$\in$ V(Γ). Therefore |C_G (x)|=2 and |G|=4 which is a contradiction. If xg and x$g^{-1}$are not conjugate to x, then deg(x) =|G|-|Z(G)|-1. Thus |Z (G)|=1 or 3 which implies G$\cong$ S_3 then Γ the induced subgraph of $\Gamma_{S_3}^{(1,2,3)}=\Gamma_{S_3}^{(1,3,2)}$ is not 2-regular. If $g^2\neq 1$ or (g^2=1, g $\neq$ 1), then deg(x)=|G|-|Z(G)|- $\varepsilon|C_G$ (x)|-1, where ε=1 or 2. Since Γ is 2-regular, |Z (G)|=1 or 3. First, assume |Z (G)|=1 so $|C_G$ (x)|=2 or 4. If $|C_G$ (x)|=2, then G$\cong$ S_3. But it is not acceptable. Now suppose $|C_G$ (x)|=4, then G$\cong$ A_4 for which the associated graph is not 2-regular. Similarly, for |Z(G)|=3, G is a non-abelian group of order 18 and the degree of vertices in this case are 2 and 5.

(ii) Let Γ be a 3-regular subgraph of g-noncommuting graph Γ_G^g. Similar to the last part, for g=1 follows G$\cong$ S_3 which is not associated to a 3-regular graph. Clearly, it is not possible that deg(x) = |G|-|Z (G)|-1=3. In the case deg(x)=|G|-|Z (G)|-$\varepsilon|C_G$ (x)|-1 follows |Z (G)|=2 or 4, where ε=1 or 2. |Z (G)|=2 implies G$\cong$ D_8 or Q_8 , D_{12} or T for which clearly D_8 and Q_8 are not associated to 3-regular graph and also the degree of the vertices are 1 or 3 for the graph assigned to G$\cong$ D_{12} or T. Now, assume |Z(G)|=4. Consequently |G|=16 or 24. We have 4 non-abelian group of order 24 with center of order 4, but all the vertex of their associated graph have degree 3 or 7. There are 6 non-abelian groups of order 16 with center of size 4 and degree of all vertices are 3.

Conversely, if G is a group of order 16 and |Z (G)|=4, then the order of all centralizers of vertices of this graph is 8. Moreover, [G:Z(G)]=4 and since G is not abelian G/Z(G) is elementary abelian 2-group of order 4. By [7, Lemma 3.1.1] follows G'$\cong$ Z_2. Now, we claim deg(x)$\neq$ |G|-|Z(G)|-1 for every vertex x in Γ. Otherwise, [x,y] $\neq$ g and g^{-1} for all y$\in$ G. Therefore [x,y]=1, which implies x is a central element and a contradiction. Consequently, by Lemma 2.2 deg(x) = 3 and the graph associated to G is 3-regular.

(iii) Assume Γ is 4-regular. Thus G$\cong$ D_8 or Q_8. Γ is not 4-regular if g$\neq$ 1, although it is clear that Γ associated to D_8 or Q_8 is 4-regular for g=1.

(iv) Suppose Γ is 5-regular. Similar to the previous cases, we obtain possible orders for the group G such that among them |G|=27 and |Z(G)|=3 is acceptable. By an easy computation in GAP we can see that there exist 2 groups of order 27 with the center of order 3 whose associated graphs are 5-regular. ■

Remark 2.10. We deduce there are 6 non-abelian groups of order 16 which satisfy part (ii) of Theorem 2.9 and their graphs are the union of three K_4, by using the group theory package GAP. Obviously for g=1, the graphs associated to them are 8-regular.

If G is an abelian group and g$\neq$ 1, then clearly Γ_G^g is Hamiltonian. Furthermore, if G is not abelian and g$\notin$ K(G) then Γ_G^g is Hamiltonian. In [1] was proved that the non-commuting graph of G is Hamiltonian.

Recall from [1] that Γ_G is planar if and only if G $\cong$ S_3, D_8 or Q_8. Since it is a subgraph of Γ_G^1, we conclude that Γ_G^1 is planar whenever G$\cong$ S_3, D_8 or Q_8. Therefore, in the following theorem we consider Γ_G^g where g is a non-identity element of K(G).

Theorem 2.11. Let g be a non-identity element of a finite group G. Then Γ_G^g is a planar graph if and only if G is isomorphic to S_3, D_8 , Q_8 or an abelian group of order at most 4.

Proof. If G is an abelian group, then Γ_G^g is a complete graph, so we must have |G|$\leq$4. Now suppose that G is not abelian. If A is an abelian subgroup of G, then |A|$\leq$4. Therefore the order of every element of G is at most 4 and so |G|=2^n 3^m . Clearly |Z (G)|< 3. Thus G is not a 3-group. If m=0, then |Z(G)|=2 and there is an element x$\in$ G of order 4. Easily C_G (x) = < x > and < Z (G), x > is an abelian subgroup of G. So < Z (G), x > =< x >. Therefore $x^2\in$ Z (G) and G/Z(G) is elementary abelian 2-group which implies that G is an extra-special 2-group all its proper centralizers are maximal. Hence n=3 and G$\cong$ D_8 or Q_8 .

Now suppose m,n $\neq$ 0. If |Z(G)|= 2, then there are elements x$\in$ Z (G) of order 2 and y$\in$ G of order 3. Clearly, xy is an element of order 6, a contradiction. Therefore, G has a trivial center such that the order of its elements is at most 4. We claim that m=1 and n$\leq$3. If m>1, then the Sylow 3-subgroup of G has an abelian subgroup of order 9, a contradiction. Also maximal abelian subgroups of a Sylow 2-subgroup P of G are of order at most 4. As proved above, |P| divides 8. Therefore |G|=6, 12 or 24. The only groups of order 6, 12 or 24 with trivial center are S_3, A_4 and S_4. We have A_4'={(1),(1 2)(3 4),(1 3)(2 4),(1 4)(2 3)}. Clearly

{(1),(1 2)(3 4),(2 3 4),(2 4 3),(1 2 3)} is a complete subgraph of $\Gamma_{A_4}^{(12)(34)}$ which implies $\Gamma_{A_4}^{(12)(34)}$ is not planar. $\Gamma_{A_4}^{(13)(24)}$ and $\Gamma_{A_4}^{(14)(23)}$ are not also planar, because they are isomorphic to $\Gamma_{A_4}^{(12)(34)}$. Hence $\Gamma_{A_4}^{g}$ is not planar, for all $1 \neq g \in A_4'$. It is enough to verify the planarity of $\Gamma_{S_4}^{(12)(34)}$ and $\Gamma_{S_4}^{(123)}$. Since $\Gamma_{A_4}^{(12)(34)}$ is a subgraph of $\Gamma_{S_4}^{(12)(34)}$, that follows $\Gamma_{S_4}^{(12)(34)}$ is not planar. The complete graph with 5 vertices {(1),(3 4),(2 3),(2 4 3),(2 3 4)} is a subgraph of $\Gamma_{S_4}^{(123)}$, so it is not planar. Hence $\Gamma_{S_4}^{g}$ is not planar for all non-identity elements $g \in S_4' = A_4$. Finally, if $G \cong S_3$ then $\Gamma_{S_3}^{g}$ is planar for all non-identity elements $g \in A_3$. ■

Corollary 2.12. Let G be a non-abelian group such that $\Gamma_G^g \cong \Gamma_{s_3}^s$ ($s \in A_3$), then $G \cong S_3$.

One may ask if two graphs are isomorphic then which properties will be inherited from one to another. We answer to this question for the property of being an extra-special p-group of rank 2 in the following proposition.

Proposition 2.13. Let G be an extra-special p-group of rank 2 and $\Gamma_G^g \cong \Gamma_H^h$ for some finite group H, where $g \in G'$ and $g \neq 1$. If $|Z(G)|=|Z(H)|$ then H is an extra-special p-group of rank 2.

Proof. Clearly $|G|/|Z(G)|=|H|/|Z(H)|= p^2$. By [7, Lemma 3.1.1], $|H'| \leq p$. Since H is not abelian $|H'|=p$ and $H/Z(H) \cong Z_p \times Z_p$. Hence $H'=Z(H)$ and the result follows. ■

It is clear that if G and H are two groups such that $\Gamma_G^a \cong \Gamma_H^b$ and $a \in K(G)$, then $b \in K(H)$.

In [9] the probability $P_g(G)$ that the commutator of two randomly chosen elements in a finite group is equal to a given element g of that group was studied. Actually we have the following ratio

$$P_g(G) = |\{(x,y) \in G^2 : [x,y]=g\}| / |G|^2.$$

We use this probability and obtain a formula for the number of edges of Γ_G^g. Let us note that if we consider g = 1 then $P_1(G)$ is the probability that two randomly chosen elements of G commute and is known as the commutativity degree of G.

Proposition 2.14. Let G be a finite group. Then

(i) for a non-identity element $g \in G'$ such that $g^2 \neq 1$ we have

$$|E(\Gamma_G^g)| = \left(|G|^2 - |G| - 2|G|^2 P_g(G)\right)/2.$$

(ii) for a non-identity element $g \in G'$ such that $g^2 = 1$ we have

$$|E(\Gamma_G^g)| = \left(|G|^2 - |G| - |G|^2 P_g(G)\right)/2.$$

Moreover, if $g \notin G'$ then $|E(\Gamma_G^g)| = \left(|G|^2 - |G|\right)/2$.

Proof. The number of edges of the graph Γ_G^g is the number of pairs $(x,y) \in G^2$ such that $[x,y] \neq g$ and $[y,x] \neq g$. Suppose $A=\{(x,y) \in G^2 : [x,y]=g\}$. For the case (i) if $(x,y) \in A$, then $(y,x) \notin A$ also by definition of this graph there is no edge between x and y. Thus we put aside $2|G|^2 P_g(G)$ number of pairs out of total pair of elements. Furthermore, we lay aside |G| number of pairs of elements because this graph does not have any loop. For the second part, if $(x,y) \in A$ then $(y,x) \in A$. Therefore we must put aside $|G|^2 P_g(G)$ from total number of pairs of elements $|G|^2$. Hence, (ii) follows similarly. If $g \notin G'$ then $P_g(G)=0$ and the rest of assertion is clear. ■

Obviously, if g=1 then $|E(\Gamma_G^1)| = \left(|G|^2 - |G|^2 P_1(G)\right)/2$. Moreover, if $\Gamma_G^g \cong \Gamma_G^a$ and $|g|=|a|=2$ or $|g|,|a| \neq 2$ then $P_g(G) = P_a(G)$.

Proposition 2.15. Let G be an extra-special 2-group. If $1 \neq g \in G'$, then

$$|E(\Gamma_G^g)| = \left(|G|^2 - P_1(G)\right)/2.$$

Proof. Since G is an extra-special 2-group we have $G'=\{1,g\}$. Therefore, the commutator of every two elements of the group is g or 1. Thus, two vertices of the graph Γ_G^g join by an edge if they commute and the assertion follows. ■

A character theoretical formula for $P_g(G)$ was given in [10]. Pournaki et al. presented explicit formulas to compute $P_g(G)$ for some special groups. They also gave upper bounds for $P_g(G)$. Thus we can use all their results here to obtain formulas for the number of the edges of the graph just by substitution, when G is a certain group.

Theorem 2.16. Let G and H be two finite isoclinic groups with |Z(G)|=|Z(H)|. If (φ, ψ) is an isoclinism between G and H, then $\Gamma_G^g \cong \Gamma_H^{\psi(g)}$.

Proof. Suppose |G/Z(G)|=|H/Z (H)|=n and θ :Z(G) $\longrightarrow$Z(H) is a bijection. If isoclinism (φ, ψ) is defined by φ (g_i Z (G))= h_i Z (H) and ψ ([g_i z_1, g_j z_2])= [h_i z_1', h_j z_2'], 1≤i,j ≤ n where g_i and h_i are transversal of G/Z(G) and H/Z(H), respectively $z_1, z_2 \in$ Z (G) and z_1', z_2'∈ Z(H) . Clearly α: V(Γ_G^g) $\longrightarrow$V($\Gamma_H^{\psi(g)}$) which maps g_iz to h_i θ (z) is a bijection which preserves edges. ■

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