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Nonharmonic Gabor Expansions

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Abstract

We consider Gabor systems generated by a Gaussian function and prove certain classical results of Paley and Wiener on nonharmonic Fourier series of complex exponentials for the Gabor expansion. In particular, we prove a version of Plancherel-Pólya theorem for entire functions with finite order of growth and use the Hadamard factorization theorem to study regularity, exactness and deficiency of Gabor systems.

Keywords: Gabor system; Nonharmonic Fourier series; Deficiency; Entire functions; Order of growth.

Introduction

The notion of Gabor transform, named after Dennis Gabor [7], is a special case of the short-time Fourier transform. Gabor transform and Gabor expansion are useful tools for signal processing, and recently are heavily studied in Engineering and Applied Mathematics)see for instance [12],]18]). The structure of Gabor spaces and Gabor systems in $L^2(\mathbb{R}^{2d})$ are studied by Abreu [1] and Christensen [4]. Borichev et al. [3] studied the stability problem for the Gabor expansions generated by a Gaussian function. In [2], Ascensi and Bruna proved that the unique Gabor atom with analytical Gabor space is the Gaussian function.

On the other hand, the theory of nonharmonic Fourier series is concerned with the completeness and expansion properties of the sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p[-\pi,\pi]$. In 1952, Duffin and Schaeffer [5] used frames to study this theory, and later Young put together many results in his book [19]. Reid [15] proved that if $\{\lambda_n\}$ is a sequence of real numbers whose differences are nondecreasing, then the set of complex exponentials $\{e^{i\lambda_n t}\}$ is a Riesz-Fischer sequence in $L^2[-A,A]$, for every $A>0$. Jaffard [9] investigated how the regularity of nonharmonic Fourier series is related to the spacing of their frequencies. This is obtained by using "a transformation which simultaneously captures the advantages of the Gabor and wavelet transforms" [9].

In this paper, we study nonharmonic Gabor expansions. We restate and prove some of the results on nonharmonic Fourier series (as reported in the book by Young [19]) for Gabor systems instead of sets of complex exponentials. Most of the work can be fully understood with elementary knowledge of functional and complex analysis.

Let us introduce the notions and basic results, needed later in the paper.

Definition 2.1 A sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$ of real or complex numbers is said to be *separated* if for some positive number ε, $|\lambda_n-\lambda_m|\geq\varepsilon$, whenever $n\neq m$, and is said to be *symmetric* if $\lambda_{-n}=-\lambda_n$ for $n\geq 0$.

Definition 2.2 A sequence of vectors $\{x_n\}$ in a normed space $\mathcal{X}$ is said to be *complete* if its linear span

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is dense in $\mathcal{X}$, that is, if for each vector x and each $\varepsilon > 0$ there is a finite linear combination $c_1x_1 + \cdots + c_nx_n$ such that

$$\|x - (c_1x_1 + \cdots + c_nx_n)\| < \varepsilon.$$

Definition 2.3 A sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is said to be a *Bessel* sequence if

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 < \infty,$$

for every element $f \in \mathcal{H}$. It is called a *Riesz-Fischer* sequence if the moment problem

$$\langle f, f_n\rangle = c_n \qquad (n = 1, 2, 3, \cdots)$$

admits at least one solution $f \in \mathcal{H}$, whenever $\{c_n\} \in l^2$.

Proposition 2.4 Let $\{f_n\}$ be a sequence in a Hilbert space $\mathcal{H}$. Then

(i) $\{f_n\}$ is a Bessel sequence with bound M if and only if the inequality

$$\left\|\sum c_n f_n\right\|^2 \le M \sum |c_n|^2$$

holds for every finite sequence of scalars $\{c_n\}$,

(ii) $\{f_n\}$ is a Riesz-Fischer sequence with bound m if and only if the inequality

$$m \sum |c_n|^2 \le \left\|\sum c_n f_n\right\|^2$$

holds for every finite sequence of scalars $\{c_n\}$.

Remark 2.5 For a sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$, the moment problem

$$\langle f, f_n\rangle = c_n \qquad (n \ge 1)$$

has at most one solution for every choice of the scalars c_n, if and only if $\{f_n\}$ is complete.

Definition 2.6 For $c \in \mathbb{R}^d$ the unitary operators T_c and E_c on $L^2(\mathbb{R}^d)$ defined by

$$T_cf(t) = f(t - c), \quad E_cf(t) = e^{2\pi ict}f(t)$$

are called the *translation* and *modulation* operators, respectively. Given a non-zero window $h \in L^2(\mathbb{R}^d)$ and lattice parameters $a, b > 0$, the subset $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}^d}$ of $L^2(\mathbb{R}^d)$ is called a *Gabor system*. The system $\{E_{\mu_n}T_{\lambda_n}h\}_{n\in\mathbb{Z}}$ with $(\lambda_n, \mu_n) \in \mathbb{R}^{2d}$, for $n \ge 1$, is called an *irregular* Gabor system. A Gabor system is *exact* in $L^2(\mathbb{R}^d)$ if it is complete, but fails to be complete if we remove any term.

These systems was first introduced by Gabor in 1946 for signal processing [7], and they are still widely used.

Definition 2.7 [17, p.138] Let f be an entire function. If there exist a positive number ρ and constants $A, B > 0$ such that

$$|f(z)| \le Ae^{B|z|^\rho} \qquad (z \in \mathbb{C}),$$

then we say that f has an order of growth $\le \rho$. We define the *order of growth* of f as

$$\rho_f = \inf \rho,$$

the infimum being over all $\rho > 0$ such that f has an order of growth $\le \rho$. For example, the order of growth of the entire function e^{z^2} is 2.

The following is the fundamental factorization theorem for entire functions of finite order of growth. It is due to Hadamard, who used the result in his celebrated proof of the Prime Number Theorem. It is one of the classical theorems in analytic function theory. The *canonical factors* are defined by $E_0(z) = 1 - z$ and $E_k(z) = (1 - z)e^{z+z^2/2+\cdots+z^k/k}$, for $k \ge 1$. The integer k is called the *degree* of the corresponding canonical factor.

Theorem 2.8 (Hadamard Factorization Theorem) [17, Th. 5.5.1] Suppose f is entire and has order of growth ρ_0. Let k be the integer with $k \le \rho_0 \le k + 1$. If $z_1, z_2, \cdots$ denote the (non-zero) roots of f, then

$$f(z) = e^{P(z)}z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{z_n}\right),$$

where P is a polynomial of degree $\le k$, and m is the order of the zero of f at $z = 0$.

Definition 2.9 The *(Bargmann-)Fock space* $\mathcal{F}(\mathbb{C}^d)$ is the Hilbert space of all entire functions f on $\mathbb{C}^d$ for which

$$\|f\|_{\mathcal{F}}^2 = \int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi|z|^2} dz$$

is finite. The canonical inner product on $\mathcal{F}(\mathbb{C}^d)$ is defined by

$$\langle f, g\rangle_{\mathcal{F}} = \int_{\mathbb{C}^d} f(z)\overline{g(z)}e^{-\pi|z|^2} dz$$

The *Bargmann transform* of a function $f \in L^2(\mathbb{R}^d)$ is the function Bf on $\mathbb{C}^d$ defined by

$$Bf(z) = 2^{d/4} \int_{\mathbb{R}^d} f(\xi)e^{-\pi\xi^2}e^{-\frac{\pi}{2}z^2}e^{2\pi\xi z}\, d\xi.$$

Definition 2.10 Fix a function $h \in L^2(\mathbb{R}^d)$ (called the window function). The *Gabor transform* with respect to the window h is the isomorphic inclusion

$$V_h\colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{C}^d),$$

defined by

$$V_hf(x + iy) = 2^{d/4} \int_{\mathbb{R}^d} f(\xi)\overline{h(\xi - x)}e^{-2\pi i\xi y}\, d\xi$$

for every $f \in L^2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$. The following subspace of $L^2(\mathbb{C}^d)$, which is the image of $L^2(\mathbb{R}^d)$ under the Gabor transform with the window h,

$$\mathcal{G}_h = \{V_h f : f \in L^2(\mathbb{R}^d)\},$$

is called the *Gabor* or *model space*. A simple calculation shows that the Bargmann transform is the Gabor transform with the *Gaussian window* $g(\xi) = 2^{d/4}e^{-\pi\xi^2}$ in $L^2(\mathbb{R}^d)$, defined by

$$(2.1)\quad V_g f(x - iy) = e^{i\pi xy} e^{-\pi\frac{|x+iy|^2}{2}}(Bf)(x + iy).$$

For more details, we refer the reader to [1, 2, 6, 8].

Results

In this section we discuss the fundamental completeness properties of the Gabor systems to lay the ground for a more penetrating investigation of nonharmonic Gabor expansions in $L^2(\mathbb{R}^2)$. Our guidelines are the fundamental ideas of Paley and Wiener [14], applied to the case of complex exponentials $\{e^{i\lambda_n t}\}$ over a finite interval of the real axis.

Let $\{(\lambda_n, \mu_n)\}_{n\in\mathbb{Z}}$be an arbitrary countable subset of $\mathbb{R}^2$ and

$$(3.1)\quad \{\varphi_n(\xi)\}_{n\in\mathbb{Z}} = \{E_{\mu_n}T_{\lambda_n}g(\xi)\}_{n\in\mathbb{Z}} = \{\sqrt{2}e^{2\pi i\mu_n\xi - \pi(\xi-\lambda_n)^2}\}_{n\in\mathbb{Z}},$$

where $\xi \in \mathbb{R}^2$ or $\mathbb{C}$, be the corresponding Gabor system with respect to the Gaussian window g in $L^2(\mathbb{R}^2)$. If $\{\varphi_n\}_{n\in\mathbb{Z}}$ is incomplete in $L^2(\mathbb{R}^2)$, then the closed linear span $\mathcal{M}$ of $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a proper subspace of $L^2(\mathbb{R}^2)$. By Hahn-Banach Theorem, there exists a function F in $L^2(\mathbb{R}^2)$ such that $F|_{\mathcal{M}} = 0$ and $F \neq 0$. Riesz Representation Theorem implies that $F = F_\varphi$, for some φ in $L^2(\mathbb{R}^2)$ and

$$F(h) = F_\varphi(h) = \int_{\mathbb{R}^2} h\,\varphi\, d\xi.$$

For $(z, w) \in \mathbb{C}^2$, take

$$(3.2)\quad f(z, w) = \sqrt{2}\int_{\mathbb{R}^2} e^{2\pi i w\xi - \pi(\xi - z)^2}\varphi(\xi)d\xi,$$

then $f(\lambda_n, \mu_n) = F(\varphi_n) = 0$, since $F|_{\mathcal{M}} = 0$.

Remark 3.1 The system (3.1) is incomplete in $L^2(\mathbb{R}^2)$ if and only if there exists a nontrivial entire function of the form (3.2) in the Gabor space $\mathcal{G}_g$, which is zero for every (λ_n, μ_n). Furthermore, since

$$f(z, w) = V_g\varphi(z, -w) = e^{i\pi zw}e^{-\pi\frac{|z|^2+|w|^2}{2}}(B\varphi)(z, w),$$

we have

$$|f(z, w)| \leq \|\varphi\|_2 e^{\frac{\pi}{2}|(z,w)|^2}.$$

Theorem 3.2 Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a symmetric sequence of real numbers. If the Gabor system

$$(3.3)\quad \{\sqrt[4]{2}e^{2\pi i\lambda_n t - \pi(t-\lambda_n)^2}\}_{n\in\mathbb{Z}}$$

is exact in $L^2(\mathbb{R})$, then for $A \in \mathbb{R}$, the product

$$e^{Az^2}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)e^{\frac{z^2}{\lambda_n^2}}$$

converges to an entire function which belongs to the Gabor space with Gaussian window in $L^2(\mathbb{R})$.

Proof. By Remark 3.1, if the system (3.3) is exact, then there exists an entire function f in the Gabor space $\mathcal{G}_g$ such that $f(\lambda_n) = 0$, for $n \neq 0$, and

$$f(z) = \sqrt[4]{2}\int_{\mathbb{R}} e^{2\pi izt - \pi(t-z)^2}\varphi(t)dt.$$

Since $f(\lambda_n) = 0$, for $n \neq 0$, and $\{\lambda_n\}$ is symmetric, $\varphi(-t)$ has the same orthogonality properties as $\varphi(t)$. But by Remark 2.5, $\varphi(t)$ is unique, so $\varphi(t)$ must be even. Hence $f(z)$ is even. Now $f(z)$ vanishs only at λ_n with $n \neq 0$. ,Indeed if $f(z)$ vanishes at $z = \gamma$, then the function

$$\tilde{f}(z) = \frac{zf(z)}{z - \gamma}$$

would also belong to $\mathcal{G}_g$ and would vanish at every λ_n. The system (3.3) would then be incomplete in $L^2(\mathbb{R})$, contrary to the hypothesis.

Let us observe that the function $\tilde{f}$ belongs to $\mathcal{G}_g$. Since the Bargmann transform is related to the Gabor transform by formula (2.1), it is sufficient to show that the function $e^{i\pi xy}e^{\frac{\pi}{2}(|x|^2+|y|^2)}\tilde{f}(z)$; $z = x + iy$,

belongs to the Fock space $\mathcal{F}(\mathbb{C})$. In other words, we must show that the integral

$$\int_{\mathbb{C}} \frac{|z|^2}{|z-\gamma|^2}\left|f(z)e^{i\pi xy}e^{\frac{\pi}{2}(|x|^2+|y|^2)}\right|^2 e^{-\pi(|x|^2+|y|^2)}dxdy; \quad z = x + iy,$$

is finite. Since $\lim_{z\to\infty}\left|\frac{z}{z-\gamma}\right| = 1$, we have $\left|\frac{z}{z-\gamma}\right| \leq 3/2$ outside a square T with complement T^c. Thus, for

$$g(z) = f(z)e^{i\pi xy}e^{\frac{\pi}{2}(|x|^2+|y|^2)}$$

the above integral is no larger than

$$\int_T \frac{|z|^2}{|z-\gamma|^2}|g(z)|^2 e^{-\pi(|x|^2+|y|^2)}dxdy$$

$$+9/4 \int_{T^c} |g(z)|^2 e^{-\pi(|x|^2+|y|^2)}\, dx\, dy$$

$$\leq \int_T \frac{|z|^2}{|z-\gamma|^2} |g(z)|^2 e^{-\pi(|x|^2+|y|^2)}\, dx\, dy$$

$$+9/4 \int_{\mathbb{C}} |g(z)|^2 e^{-\pi(|x|^2+|y|^2)}\, dx\, dy.$$

In the last expression, since T is compact, the first integral is finite, and since the function $g(z)$ is in the Fock space $\mathcal{F}(\mathbb{C})$, so is the second integral. Next, since

$$|f(z)| \leq \|\varphi\|_2 e^{\frac{\pi}{2}|z|^2},$$

the order of growth of f is 2, and by Hadamard's factorization theorem,

$$f(z) = e^{Az^2+Bz} \prod_{n=1}^{\infty} E_2\left(\frac{z}{\lambda_n}\right)$$

$$= e^{Az^2+Bz} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) e^{\frac{z^2}{\lambda_n^2}}.$$

Since $f(z)$ and the canonical product are both even, $B = 0$ and

$$f(z) = e^{Az^2} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) e^{\frac{z^2}{\lambda_n^2}}.$$

This proves the claim and completes the proof. ■

We have the following version of Plancherel-Pólya theorem. We give the proof which is similar to [19, Th. 2.16] for the sake of completeness.

Theorem 3.3 (Plancherel-Pólya). If $f(z)$ is an entire function of order of growth $\leq \tau$ and for some positive number p,

$$\int_{-\infty}^{\infty} |f(x)|^p\, dx < \infty,$$

then

$$\int_{-\infty}^{\infty} |f(x+iy)|^p\, dx \leq e^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p\, dx.$$

The proof will require two preliminary lemmas. Suppose that $q(z)$ is a non constant, continuous function in the closed upper half-plane, $Im\, z \geq 0$, and analytic in its interior. Let a and p be positive real numbers and put

$$Q(z) = \int_{-a}^{a} |q(z+t)|^p\, dt.$$

It is clear that $Q(z)$ is continuous for $Im\, z \geq 0$. Since $|q(z)|^p$ is subharmonic for $Im\, z > 0$ (see [10, p.83]), so is $Q(z)$.

Lemma 3.4 Let $q(z)$ be a function of order of growth $\leq \tau$ in the half-plane $Im\, z \geq 0$ and suppose that the following quantities are both finite:

$$M = \sup_{-\infty<x<\infty} Q(x) \text{ and } N = \sup_{y>0} Q(iy).$$

Then on this half-plane,

$$Q(z) \leq \max(M, N).$$

Proof. Since $q(z)$ is of order of growth $\leq \tau$, there exist positive numbers A and B such that

(3.4) $\quad |q(z)| \leq Ae^{B|z|^\tau} \quad (Im\, z \geq 0).$

For each positive number ε, define the auxiliary function

(3.5) $\quad q_\varepsilon(z) = q(z) e^{-\varepsilon(\lambda(z+a))^{\frac{3}{2}}},$

where $\lambda = e^{-i\pi/4}$. The exponent of e in (3.5) has two possible determinations in the half-plane $Im\, z > 0$; we choose the one whose real part is negative in the quarter-plane $x > -a,\ y \geq 0$. Put

$$Q_\varepsilon(z) = \int_{-a}^{a} |q_\varepsilon(z+t)|^p\, dt,$$

which is defined and continuous in the upper half-plane $Im\, z \geq 0$, and subharmonic in its interior. A simple calculation involving (3.4) and (3.5) shows that in the quarter plane $x > -a,\ y \geq 0$,

(3.6) $\quad |q_\varepsilon(z)| \leq Ae^{B|z|^\tau - \varepsilon\gamma|z+a|^{\frac{3}{2}}},$

where $\gamma = \cos 3\pi/8$, and $|q_\varepsilon(z)| \leq |q(z)|$. Hence

$$Q_\varepsilon(z) \leq Q(z) \quad (x \geq 0, y \geq 0),$$

and in particular

$Q_\varepsilon(x) \leq M \quad$ for $\quad x \geq 0$

and

$Q_\varepsilon(iy) \leq N \quad$ for $\quad y \geq 0.$

Let z_0 be a fixed but arbitrary point in the first quadrant. We shall apply the maximum principle to $Q_\varepsilon(z)$ in the region $\Omega = \{z: Re\, z \geq 0,\ Im\, z \geq 0, |z| \leq R\}$, choosing R large enough so that (i) $z_0 \in \Omega$, and (ii) the maximum value of $Q_\varepsilon(z)$ on Ω is not attained on the circular arc $|z| = R$ (this is possible by virtue of (3.6)). Since $Q_\varepsilon(z)$ does not reduce to a constant, the maximum value of $Q_\varepsilon(z)$ on Ω must be attained on one of the coordinate axes, and in particular,

$$Q_\varepsilon(z_0) \leq \max(M, N).$$

Now let $\varepsilon \to 0$. This establishes the result for the first quadrant, the proof for the second quadrant is similar. ■

Lemma 3.5 In addition to the hypotheses of Lemma 3.4, suppose that

(3.7) $\quad \lim_{y\to\infty} q(x+iy) = 0,$

uniformly in x, for $-a \leq x \leq a$. Then

$Q(z) \le M(Im\, z \ge 0)$.

Proof. It is sufficient to show that $N \le M$. By virtue of (3.7), the function $Q(iy)$ tends to zero as $y \to \infty$, and so must attain its least upper bound N for some finite value of y, say $y = y_0$. If $y_0 = 0$, then

$$N = Q(iy_0) = Q(0) \le M.$$

If $y_0 > 0$, then the maximum principle shows that the least upper bound of $Q(z)$ in the half-plane $Im\, z \ge 0$ cannot be attained at the interior point $z = iy_0$. Therefore, by Lemma 3.4,

$$N = Q(iy_0) < \max(M, N),$$

and again $N < M$. ■

Theorem 3.3 now follows.

Proof of Theorem 3.3 It is sufficient to prove the theorem when $y > 0$ and $f(z)$ is not identically zero. Let ε be a fixed positive number and consider the function

$$q(z) = f(z)e^{i(\tau+\varepsilon)z}.$$

It is easy to see that, for each positive number a, the functions $q(z)$ and $Q(z)$ satisfy the conditions of Lemmas 3.4 and 3.5. Consequently, for $y > 0$,

$$Q(iy) \le M < \int_{-\infty}^{\infty} |q(x)|^p\, dx.$$

This together with the definitions of $q(z)$ and $Q(z)$ implies

$$e^{-p(\tau+\varepsilon)y} \int_{-a}^{a} |f(x+iy)|^p\, dx < \int_{-\infty}^{\infty} |f(x)|^p\, dx.$$

To get the result, first let $a \to \infty$, then let $\varepsilon \to 0$. ■

The order of entire function $f(z)$, $z \in \mathbb{C}^d$, is defined as

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

where $M(r) = \sup_{|z|=r} |f(z)|$ is the maximum modulus function [11]. The Definition 2.7, given by Stein and Shakarchiin in [17] for entire functions of one complex variable, could be used for entire functions of several complex variables.

Proposition 3.6 Let $f(z, w)$ be an entire function of order of growth $\le \tau$ and suppose that $\{\lambda_n\}$, $\{\mu_n\}$ are increasing sequences of real numbers such that

$$\lambda_{n+1} - \lambda_n \ge \varepsilon_1 > 0$$

and

$$\mu_{n+1} - \mu_n \ge \varepsilon_2 > 0.$$

If for some positive number p,

$$\text{(3.8)} \qquad \sup_n \int_{-\infty}^{\infty} |f(x_z, \mu_n)|^p\, dx_z < \infty$$

$$\sup_n \int_{-\infty}^{\infty} |f(\lambda_n, x_w)|^p\, dx_w < \infty,$$

then

$$\sum_n |f(\lambda_n, \mu_n)|^p < \infty.$$

Proof. First, using the Plancherel- Pólya Theorem, observe that the conditions (3.8) imply that

$$\sup_n \int_{-\infty}^{\infty} |f(z, \mu_n)|^p\, dx_z \le e^{p\tau|y_z|} \int_{-\infty}^{\infty} \sup_n |f(x_z, \mu_n)|^p\, dx_z,$$

and

$$\sup_n \int_{-\infty}^{\infty} |f(\lambda_n, w)|^p\, dx_w \le e^{p\tau|y_w|} \int_{-\infty}^{\infty} \sup_n |f(\lambda_n, x_w)|^p\, dx_w.$$

Now, since $|f|^p$ is pluri-subharmonic, the inequality

$$\text{(3.9)} \qquad |f(z_0, w_0)|^p \le \frac{1}{2\pi} \int_0^{2\pi} \left|f((z_0, w_0) + (\zeta, \eta)e^{i\theta}\right|^p\, d\theta$$

holds for all values of (ζ, η) [10, 16]. Fix $\eta = 0$, multiply both sides of (3.9) by ζ and integrate between 0 and δ_1,

$$\int_0^{\delta_1} |f(z_0, w_0)|^p \zeta\, d\zeta \le \frac{1}{2\pi} \int_0^{\delta_1} \int_0^{2\pi} \left|f\left(z_0 + \zeta e^{i\theta}, w_0\right)\right|^p d\theta\, \zeta\, d\zeta.$$

Thus

$$|f(z_0, w_0)|^p \le \frac{1}{\pi\delta_1^2} \iint_{\Omega_1} |f(z, w_0)|^p\, dx_z\, dy_z,$$

where $\Omega_1 = \{(z, w_0): |z - z_0| \le \delta_1\}$. Similarly fix $\zeta = 0$, multiply both sides of (3.9) by η and integrate between 0 and δ_2,

$$|f(z_0, w_0)|^p \le \frac{1}{\pi\delta_2^2} \iint_{\Omega_2} |f(z_0, w)|^p\, dx_w\, dy_w,$$

where $\Omega_2 = \{(z_0, w): |w - w_0| \le \delta_2\}$. Then

$$2|f(z_0,w_0)|^p \le \frac{1}{\pi\delta_1^2}\iint_{\Omega_1} |f(z,w_0)|^p \, dx_z \, dy_z + \frac{1}{\pi\delta_2^2}\iint_{\Omega_2} |f(z_0,w)|^p \, dx_w \, dy_w.$$

Now let

$$\Omega_1^n = \{(\lambda_n + z, \mu_n) : |z| \le \delta_1\}$$

and

$$\Omega_2^n = \{(\lambda_n, \mu_n + w) : |w| \le \delta_2\},$$

then

$$\begin{aligned} 2\sum_n |f(\lambda_n,\mu_n)|^p &\le \sum_n \left(\frac{1}{\pi\delta_1^2}\iint_{\Omega_1^n} |f(\lambda_n + z,\mu_n)|^p \, dx_z \, dy_z + \frac{1}{\pi\delta_2^2}\iint_{\Omega_2^n} |f(\lambda_n,\mu_n + w)|^p \, dx_w \, dy_w\right) \\ &\le \sum_n \left(\frac{1}{\pi\delta_1^2}\int_{-\delta_1}^{\delta_1}\int_{-\delta_1}^{\delta_1} |f(\lambda_n + z,\mu_n)|^p dx_z \, dy_z + \frac{1}{\pi\delta_2^2}\int_{-\delta_2}^{\delta_2}\int_{-\delta_2}^{\delta_2} |f(\lambda_n,\mu_n + w)|^p dx_w \, dy_w\right) \\ &= \sum_n \left(\frac{1}{\pi\delta_1^2}\int_{-\delta_1}^{\delta_1}\int_{\lambda_n-\delta_1}^{\lambda_n+\delta_1} |f(z,\mu_n)|^p dx_z \, dy_z + \frac{1}{\pi\delta_2^2}\int_{-\delta_2}^{\delta_2}\int_{\mu_n-\delta_2}^{\mu_n+\delta_2} |f(\lambda_n,w)|^p dx_w \, dy_w\right). \end{aligned}$$

It is clear that the last expression above is no larger than

$$\sum_n \left(\frac{1}{\pi\delta_1^2}\int_{-\delta_1}^{\delta_1}\int_{\lambda_n-\delta_1}^{\lambda_n+\delta_1} \sup_n |f(z,\mu_n)|^p dx_z \, dy_z + \frac{1}{\pi\delta_2^2}\int_{-\delta_2}^{\delta_2}\int_{\mu_n-\delta_2}^{\mu_n+\delta_2} \sup_n |f(\lambda_n,w)|^p dx_w \, dy_w\right).$$

Now for $\delta_1 = \frac{\varepsilon_1}{2}$ and $\delta_2 = \frac{\varepsilon_2}{2}$, the intervals $(\lambda_n - \delta_1, \lambda_n + \delta_1)$ are pairwise disjoint, and the same holds for intervals $(\mu_n - \delta_2, \mu_n + \delta_2)$, thus

$$\begin{aligned} 2\sum_n |f(\lambda_n,\mu_n)|^p &\le \frac{1}{\pi\delta_1^2}\int_{-\delta_1}^{\delta_1}\int_{-\infty}^{\infty} \sup_n |f(z,\mu_n)|^p dx_z \, dy_z \\ &+ \frac{1}{\pi\delta_2^2}\int_{-\delta_2}^{\delta_2}\int_{-\infty}^{\infty} \sup_n |f(\lambda_n,w)|^p dx_w \, dy_w. \end{aligned}$$

We conclude that

$$\begin{aligned} 2\sum_n |f(\lambda_n,\mu_n)|^p &\le \frac{1}{\pi\delta_1^2}\int_{-\delta_1}^{\delta_1}\left(e^{p\tau|y_z|}\int_{-\infty}^{\infty} \sup_n |f(x_z,\mu_n)|^p dx_z\right) dy_z \\ &+ \frac{1}{\pi\delta_2^2}\int_{-\delta_2}^{\delta_2}\left(e^{p\tau|y_w|}\int_{-\infty}^{\infty} \sup_n |f(\lambda_n,x_w)|^p dx_w\right) dy_w \\ &= B_1 \sup_n \int_{-\infty}^{\infty} |f(x_z,\mu_n)|^p dx_z \\ &+ B_2 \sup_n \int_{-\infty}^{\infty} |f(\lambda_n,x_w)|^p dx_w < \infty, \end{aligned}$$

where

$B_1 = B_1(p,\tau,\varepsilon_1)$ and $B_2 = B_2(p,\tau,\varepsilon_2)$. ■

Remark 3.7 In the above proposition, if we replace the conditions (3.8) by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x_z,x_w)|^p \, dx_z \, dx_w < \infty$$

the interior integral is finite everywhere except on a null set. If we use Fubini to change the order of integration, we get a null set for the integral against the second variable. If we know that none of λ_n's and μ_n's lie in these null sets, the conclusion still holds.

Theorem 3.8 If $\{\lambda_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\in\mathbb{Z}}$ are separated sequences of real numbers such that $0 \le \lambda_n \le 1$ and $0 \le \mu_n \le 1$, for each n, then the Gabor system (3.1) forms a Bessel sequence in $L^2(\mathbb{R}^2)$. If $\sum_n |c_n|^2 < \infty$, then the Gabor expansion

$$\sum_n c_n e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2}$$

converges in mean to an element of $L^2(\mathbb{R}^2)$.

Proof. If $\phi \in L^2(\mathbb{R}^2)$ then the inner product

$$a_n = \langle \sqrt{2} e^{2\pi i \mu_n \xi - \pi(\xi - \lambda_n)^2}, \phi(\xi)\rangle;$$

is just the value $f(\lambda_n,\mu_n)$ of the entire function

$$f(z,w) = \sqrt{2}\int_{\mathbb{R}^2} \varphi(\xi) \, e^{2\pi i w \xi - \pi(\xi - z)^2} d\xi; \quad \varphi(\xi) = \overline{\phi(\xi)},$$

in the Gabor space $\mathcal{G}_g$ and f is of order of growth 2. We have

$$\sup_n \int_{-\infty}^{\infty} |f(x_z,\mu_n)|^p \, dx_z \le 2^{\frac{p}{2}} M^p e^{\pi}$$

$$\int_{-\infty}^{\infty}\left[\int_{\mathbb{R}} e^{-2\pi(x_\xi-x_z)^2}\,dx_\xi \int_{\mathbb{R}} e^{-2\pi(|y_\xi|-1)^2}\,dy_\xi\right]^{\frac{p}{2}} dx_z < \infty,$$

and similarly

$$\sup_n \int_{-\infty}^{\infty} |f(\lambda_n, x_w)|^p\,dx_w < \infty.$$

Therefore f satisfies the conditions (3.8) and by Proposition 3.6, we have

$$\sum_n \left|\langle \sqrt{2}e^{2\pi i\mu_n\xi-\pi(\xi-\lambda_n)^2}, \phi(\xi)\rangle\right|^2 = \sum_n |a_n|^2 = \sum_n |f(\lambda_n,\mu_n)|^2 < \infty.$$

Thus the Gabor system (3.1) forms a Bessel sequence in $L^2(\mathbb{R}^2)$. The second part follows from the first one by Proposition 2.4. ■

Paley and Wiener showed in Theorem XLII of [14] that whenever

$\lim_{n\to\pm\infty}(\lambda_{n+1}-\lambda_n)=\infty$,

for a sequence of real numbers $\{\lambda_n\}$, the exponentials are weakly independent over an arbitrarily short interval: $\sum a_n e^{i\lambda_n t}=0$ only when all coefficients a_n are zero. The next lemma states a similar statement for the set of complex exponentials replaced by the system (3.3). Here l.i.m. is used to show the limit in mean-square in L^2. The proof is almost identical to that of Paley and Wiener.

Lemma 3.9 Let no a_n vanish, $\sum_{-\infty}^{\infty}|a_n|^2$ converges, and let

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$$

such that

$$\lim_{n\to\pm\infty}(\lambda_{n+1}-\lambda_n)=\infty.$$

Let

$$f(t) = \underset{N\to\infty}{l.i.m.}\sum_{-N}^{N} a_n e^{2\pi i\lambda_n t-\pi(t-\lambda_n)^2};$$

over every bounded range. If $f(t)$ is equivalent to zero over any interval (a,b) then $f(t)$ is equivalent to zero over every interval, and all the coefficients a_n vanish.

Now we want to show that if the separation of the λ_n's is big enough then the system (3.3) is a Riesz-Fischer sequence.

Theorem 3.10 Let $\{\lambda_n\}$ be a sequence of real numbers whose differences are non-decreasing and satisfy

$$\sum \frac{1}{(\lambda_{k+1}-\lambda_k)^2} < \infty.$$

Then the Gabor system (3.3) is a Riesz-Fischer sequence in $L^2(\mathbb{R})$.

Proof. We adapt the proof of [15, Th. 1]. By the second part of the Proposition 2.4, we have to show that for all finite sequences of scalars $\{c_n\}$ and some constant $m>0$,

$$(3.10)\quad m\sum |c_n|^2 \le \left\|\sum c_n \sqrt[4]{2}\, e^{2\pi i\lambda_n t-\pi(t-\lambda_n)^2}\right\|^2.$$

Using c to denote an l^2sequence $\{c_1,c_2,\cdots\}$, the above inequality (3.10) is the same as

$$\frac{\langle Gc,c\rangle_{l^2}}{\langle c,c\rangle_{l^2}} \ge m,$$

where the l^2 operator G is the Gram matrix of the members of the Gabor system (3.3). It is to be shown that the eigenvalues of finite subsections of G are bounded away from zero, which in turn follows from these two conditions:

(1) $Gv=0$ implies $v=0$, for every l^2 sequence v.
(2) $G=I+M$, where M is a compact operator.

The first condition is satisfied by Lemma 3.9. To verify condition (2), observe that the entries of $G=I+M$ are

$$g_{nm} = \sqrt{2}\int_{-\infty}^{\infty} e^{2\pi i(\lambda_n-\lambda_m)t-\pi(t-\lambda_n)^2-\pi(t-\lambda_m)^2}\,dt.$$

Now M can be shown to be compact by showing that its Schmidt norm is finite. Since G is symmetric, it suffices to show that

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} g_{nm}^2 < \infty.$$

The sum is bounded above,

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} g_{nm}^2 = 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left(\int_{-\infty}^{\infty} e^{2\pi i(\lambda_n-\lambda_m)t-\pi(t-\lambda_n)^2-\pi(t-\lambda_m)^2}\,dt\right)^2$$

$$\le 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left(\int_{-\infty}^{\infty} e^{2\pi i(\lambda_n-\lambda_m)t}\,dt\right)^2$$

$$= 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\lim_{A\to\infty}\left(\int_{-A}^{A} e^{2\pi i(\lambda_n-\lambda_m)t}\,dt\right)^2$$

$$\leq \frac{2}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{(\lambda_n-\lambda_m)^2}$$
$$< \frac{2}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{(\lambda_{n+1}-\lambda_n)^2(m-n)^2},$$

where $(\lambda_m-\lambda_n) \leq (\lambda_{n+1}-\lambda_n)(m-n)$ follows from the assumption that differences are non-decreasing. Letting $k = m + n$, one concludes that

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} g_{nm}^2 < \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{(\lambda_{n+1}-\lambda_n)^2} < \infty,$$

establishing the result. ■

Theorem 3.11 Let

$$f(z) = \int_{-\infty}^{\infty} \alpha(t)e^{2\pi izt-\pi(t-z)^2}\,dt,$$

where $\alpha \in L^2(\mathbb{R})$. If $f(\mu) = 0$ and $g(z) = \frac{z-\lambda}{z-\mu}f(z)$,

then there exists a function β in $L^2(\mathbb{R})$ such that

$$g(z) = \int_{-\infty}^{\infty} \beta(t)e^{2\pi izt-\pi(t-z)^2}\,dt. \tag{3.11}$$

Moreover,

$$\beta(t) = \alpha(t) + 2\pi(i+1)(\lambda-\mu) e^{-2\pi i\mu t+\pi(t-\mu)^2}\int_{-\infty}^{t}\alpha(s)e^{2\pi i\mu s-\pi(s-\mu)^2}ds, \tag{3.12}$$

almost everywhere on $\mathbb{R}$.

Proof. To motivate the proof, let us suppose that $g(z)$ is in fact representable in the form (3.11), and try to deduce (3.12). If (3.11) holds, then

$$\frac{1}{z-\mu}\int_{-\infty}^{\infty}\alpha(t)e^{2\pi izt-\pi(t-z)^2}\,dt = \frac{1}{z-\lambda}\int_{-\infty}^{\infty}\beta(t)e^{2\pi izt-\pi(t-z)^2}\,dt.$$

The trick in solving for $\beta(t)$ is to transform each of these integrals by first rewriting

$$e^{2\pi izt-\pi(t-z)^2}$$

as

$$e^{2\pi izt-\pi(t-z)^2} = e^{2\pi i(z-\mu)t+2\pi i\mu t-\pi(t-\mu)^2+\pi(z-\mu)(2t-z-\mu)}.$$

and then integrating by parts. When this is done, the result is

$$\frac{1}{z-\mu}\int_{-\infty}^{\infty}\alpha(t)e^{2\pi izt-\pi(t-z)^2}\,dt = \int_{-\infty}^{\infty}\alpha_1(t)e^{2\pi izt-\pi(t-z)^2}\,dt,$$

with

$$\alpha_1(t) = = -2(i+1)\pi e^{-2\pi i\mu t+\pi(t-\mu)^2}\int_{-\infty}^{t}\alpha(s)e^{2\pi i\mu s-\pi(s-\mu)^2}ds,$$

and

$$\frac{1}{z-\lambda}\int_{-\infty}^{\infty}\beta(t)e^{2\pi izt-\pi(t-z)^2}\,dt = \int_{-\infty}^{\infty}\beta_1(t)e^{2\pi izt-\pi(t-z)^2}\,dt,$$

with

$$\beta_1(t) = -2(i+1)\pi e^{-2\pi i\lambda t+\pi(t-\lambda)^2}\int_{-\infty}^{t}\beta(s)e^{2\pi i\lambda s-\pi(s-\lambda)^2}ds.$$

It follows that $\alpha_1(t) = \beta_1(t)$ almost everywhere on $\mathbb{R}$, and so

$$e^{2\pi i(\lambda-\mu)t+\pi(\lambda-\mu)(2t-(\lambda+\mu))}\int_{-\infty}^{t}\alpha(s)e^{2\pi i\mu s-\pi(s-\mu)^2}ds = \int_{-\infty}^{t}\beta(s)e^{2\pi i\lambda s-\pi(s-\lambda)^2}ds.$$

To obtain (3.12), differentiate both sides of this equation with respect to t. Now simply observe that all of the above steps are reversible, that is, $\beta \in L^2(\mathbb{R})$. ■

Remark 3.12 A similar result holds when f is of the form

$$f(z) = \int_{-\infty}^{\infty} e^{2\pi izt-\pi(t-z)^2}\,d\alpha(t),$$

and α is of bounded variation on $\mathbb{R}$, only now

$$g(z) = \int_{-\infty}^{\infty} e^{2\pi izt-\pi(t-z)^2}\,d\beta(t),$$

with

$$d\beta(t) = d\alpha(t) + 2\pi i(\lambda-\mu) e^{-2\pi i\mu t+\pi(t-\mu)^2}\int_{-\infty}^{t} e^{2\pi i\mu s-\pi(s-\mu)^2}d\alpha(s).$$

Corollary 3.13 The completeness of system (3.3) is unaffected if one λ_n is replaced by another number.

Nowak [13] showed that the deficit of the regular Gabor system generated by $h \in L^2(\mathbb{R}^d)$ and $a, b > 0$ is either zero or infinite, if the system is a Bessel sequence in $L^2(\mathbb{R}^d)$. The next result on the deficit of the irregular Gabor system (3.3) is proved as in [19, Th. 4.6]. Here we give the proof for the sake of completeness.

Theorem 3.14 If $\{\lambda_n\}$ is a separated sequence of real numbers such that

$$\lambda_{n+1} - \lambda_n > 1; \quad (n = 0, \pm 1, \pm 2, \cdots)$$

then the Gabor system (3.3) has infinite deficiency in $L^2(\mathbb{R})$.

Proof. Let N be a fixed but arbitrary positive integer. If K is large enough, then we can replace

$$\lambda_0, \lambda_1, \cdots, \lambda_K$$

by

$$\mu_0, \mu_1, \cdots, \mu_{K+N+1},$$

so that the resulting sequence, relabeled $\{\mu_n\}$, satisfies

$$\inf_n (\mu_{n+1} - \mu_n) > 1.$$

By Theorem 3.10 there is a function $\varphi \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi(t) \sqrt[4]{2}\, e^{-2\pi i \mu_n t - \pi (t-\mu_n)^2} dt = \begin{cases} 1 & if \quad n = 0, \\ 0 & if \quad n \neq 0. \end{cases}$$

Thus the system

$$\left\{ \sqrt[4]{2} e^{2\pi i \mu_n t - \pi (t-\mu_n)^2} : n \neq 0 \right\}$$

is incomplete in $L^2(\mathbb{R})$, and we conclude by the above corollary that the deficiency of the system in $L^2(\mathbb{R})$ is at least N. ■

Conclusion

In this paper, certain known results on nonharmonic Fourier expansions are proved for Gabor systems. Hadamard Factorization and Plancherel-Pólya theorems are used to get results about the entire functions of finite order of growth in the Bargmann-Fock space. Completeness and deficiency of Gabor systems are studied.

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