Journal of Sciences, Islamic Republic of Iran 24(3): 275-285 (2013) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

$\Re$-torsion free Acts Over Monoids

A. Zare, A. Golchin* and H. Mohammadzadeh

Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Islamic Republic of Iran

Received: 7 February 2013 / Revised: 25 June 2013 / Accepted: 15 July 2013

Abstract

In this paper firt of all we introduce a generalization of torsion freeness of acts over monoids, called $\Re$-torsion freeness. Then in section 1 of results we give some general properties and in sections 2, 3 and 4 we give a characterization of monoids for which this property of their right Rees factor, cyclic and acts in general implies some other properties, respectively.

Keywords: $\Re$-torsion free; Rees factor act; cyclic act

Introduction

Throughout this paper S will denote a monoid with identity element 1. We refer the reader to [11] and [12] for basic definitions and terminology relating to semigroups and acts over monoids and to [1], [13] and [14] for definitions and results on flatness which are used here.

A monoid S is called *left (right) collapsible* if for any $s, s' \in S$ there exists $z \in S$ such that $zs = zs'$ ($sz = s'z$). A submonoid P of S is called *weakly left collapsible* if for any $s, s' \in P$, $z \in S$, $sz = s'z$ implies the existence of $u \in P$ such that $us = us'$. It is obvious that every left collapsible submonoid is weakly left collapsible, but not the converse. A monoid S is called *right (left) reversible*, if for any $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$ ($su = s'v$). A submonoid P of S is called *weakly right reversible*, if for any $s, s' \in P$, $z \in S$, $sz = s'z$ implies the existence of $u, v \in P$ such that $us = vs'$. A right ideal K_S of a monoid S is called *left stabilizing*, if for any $k \in K_S$, there exists $l \in K_S$ such that $lk = k$. K_S is called *left annihilating*, if for any $t \in S$, $x, y \in S \setminus K_S$, $xt, yt \in K_S$ implies that $xt = yt$. K_S is called *strongly left annihilating*, if for all $s, t \in S \setminus K_S$ and for all homomorphisms $f : {}_S(St \cup Ss) \to {}_S S$ $f(s), f(t) \in K_S$ implies that $f(s) = f(t)$. K_S is called *completely left annihilating*, if for all $x, y, z, t, t' \in S$,

$$[(xt \neq yt') \wedge (tz = t'z)] \Rightarrow [(xt \notin K_S) \vee (yt' \notin K_S) \vee (x \in K_S) \vee (y \in K_S)]$$

K_S is called P_E*-left annihilating*, if for all $x, y, t, t' \in S$,

$$(xt \neq yt') \Rightarrow [(xt \notin K_S) \vee (yt' \notin K_S) \vee (x \in K_S) \vee (y \in K_S) \vee (\exists u, v \in S, e, f \in E(S), et = t, ft' = t', ut = vt', xe \neq ue \Rightarrow xe, ue \in K_S, yf \neq vf \Rightarrow yf, vf \in K_S)]$$

K_S is called *E-left annihilating*, if for all $x, y, t \in S$,

* Corresponding author, Tel.: +98(541)2447166, Fax: +98(541)2447166, E-mail: agdm@math.usb.ac.ir

$$(xt \neq yt) \Rightarrow [(xt \notin K_S) \vee (yt \notin K_S) \vee$$
$$(x \in K_S) \vee (y \in K_S) \vee$$
$$(\exists u,v \in S, e,f \in E(S), et = t = ft$$
$$, ut = vt, xe \neq ue \Rightarrow xe, ue \in K_S,$$
$$yf \neq vf \Rightarrow yf, vf \in K_S)]$$

A nonempty set A is called a *right S-act*, usually denoted A_S , if S acts on A unitarily from the right; that is, there exists a mapping $A \times S \to A$, $(a,s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s,t \in S$. Left S-acts ${}_SA$ are defined dually. If A_S be an act, then we define Green's equivalence relation $\Re$ on A_S by the following rule:

$$(a,b) \in \Re \Leftrightarrow aS = bS$$

for all $a,b \in A$.

A right S-act A satisfies Condition (P), if for all $a,a' \in A$, $s,s' \in S$, $as = a's'$ implies that there exist $b \in A$, $u,v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A monoid S is called *right PCP*, if all principal right ideals of S satisfy Condition (P). A right S-act A satisfies Condition (P'), if for all $a,a' \in A$, $s,s',z \in S$, $as = a's'$, $sz = s'z$ imply that there exist $b \in A$, $u,v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A right S-act A satisfies Condition (P_E), if for all $a,a' \in A$, $s,s' \in S$, $as = a's'$ implies that there exist $b \in A$, $u,v,e^2 = e, f^2 = f \in S$ such that $ae = bue$, $a'f = bvf$, $es = s$, $fs' = s'$ and $us = vs'$. It is obvious that Condition (P) implies Condition (P_E), but not the converse, for this see [2]. A satisfies Condition (E), if for all $a \in A$, $s,s' \in S$, $as = as'$ implies that there exist $b \in A$, $u \in S$ such that $a = bu$ and $us = us'$. A satisfies Condition (EP), if for all $a \in A$, $s,s' \in S$, $as = as'$ implies that there exist $b \in A$, $u,v \in S$ such that $a = bu = bv$ and $us = vs'$. A satisfies Condition (E'), if for all $a \in A$, $s,s',z \in S$, $as = as'$, $sz = s'z$ imply that there exist $b \in A$, $u \in S$ such that $a = bu$ and $us = us'$. A satisfies Condition ($E'P$), if for all $a \in A$, $s,s',z \in S$, $as = as'$, $sz = s'z$ imply that there exist $b \in A$, $u,v \in S$ such that $a = bu = bv$ and $us = vs'$. It is obvious that Condition (E) $\Rightarrow$ Condition (EP) $\Rightarrow$ Condition ($E'P$) and Condition (E) $\Rightarrow$ Condition (E') $\Rightarrow$ Condition ($E'P$). In [3] and [4] we gave a characterization of monoids by Conditions (EP) and ($E'P$) of their acts. A right S-act A satisfies Condition (PWP), if for all $a,a' \in A$, $s \in S$, $as = a's$ implies that there exist $b \in A$ and $u,v \in S$ such that $a = bu$, $a' = bv$ and $us = vs$. A right S-act A satisfies Condition (PWP_E), if for all $a,a' \in A$, $s \in S$, $as = a's$ implies that there exist $b \in A$ and $u,v,e^2 = e, f^2 = f \in S$ such that $ae = bue$, $a'f = bvf$, $es = fs = s$ and $us = vs$. In [7] we gave a characterization of monoids by Condition (PWP_E) of their acts. A is called regular, if all cyclic subacts of A are projective. A is called *faithful*, if for $s,t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$. A is called *strongly faithful*, if for $s,t \in S$ the equality $as = at$ for some $a \in A$ implies that $s = t$. A is called *P-regular*, if all cyclic subacts of A satisfy Condition (P). In [9] we gave a characterization of monoids by P-regularity of their acts. A is called *strongly* (P) *-cyclic* if for any $a \in A$ there exists $z \in S$ such that $\ker \lambda_Z = \ker \lambda_a$ and zS satisfies Condition (P). In [8] we gave a characterization of monoids by strong (P)-cyclic of their acts.

Let S be a monoid and I be a proper right ideal of S. Let x, y and z denote elements not belonging to S. If $A = ((S \setminus I) \times \{x,y\}) \bigcup (I \times \{z\})$ and S acts on A from the right as follows:

$$(u,x)s = \begin{cases} (us,x), \ if \ \ us \notin I \\ (us,z), \ if \ \ us \in I \end{cases}$$
$$(u,y)s = \begin{cases} (us,y), \ if \ \ us \notin I \\ (us,z), \ if \ \ us \in I \end{cases}$$
$$(u,z)s = (us,z),$$

then the right S-act A is called *amalgam* of S by I and is denoted by $S \coprod^{I} S$.

Results

1. General properties

Definition 1.1. An act A_S is called $\Re$ *-torsion free* if for any $a,b \in A$ and $c \in S$, c right cancellabe, $ac = bc$ and $a \Re b$ imply that $a = b$.

We use the abbreviation $\Re$TF for $\Re$-torsion freeness. It is clear that torsion freeness implies $\Re$-torsion freeness, but not the converse, see the following example.

Example 1.1. Let $S = (\mathbf{N}, \cdot)$, and consider the amalgam $A_S = \mathbf{N} \coprod^{\mathbf{N}\setminus\{1\}} \mathbf{N}$. Then $(1, x) \neq (1, y)$, but $(1, x)2 = (1, y)2$. Hence A_S is not torsion free. It can easily be seen that A_S is $\Re$ -torsion free.

Proposition 1.1. *Let S be a monoid. Then:*

(1) *The one-element act* Θ_S *is* $\Re$ *-torsion free.*

(2) S_S *is* $\Re$ *-torsion free.*

(3) *If an act is* $\Re$ *-torsion free, then all its subacts are* $\Re$ *-torsion free.*

(4) A_i, $i \in I$, *are* $\Re$ *-torsion free if and only if* $A_S = \coprod_{i \in I} A_i$ *is* $\Re$ *-torsion free.*

(5) *If* A_i, $i \in I$, *are* $\Re$ *-torsion free right S-acts, then* $A_S = \prod_{i \in I} A_i$ *is* $\Re$ *-torsion free.*

Proof. It is clear from definitions.□

Proposition 1.2. *Let S be a monoid. Then:*

(1) *All right S-acts satisfying Condition (EP) are* $\Re$ *-torsion free.*

(2) *All right S-acts satisfying Condition (E) are* $\Re$ *-torsion free.*

Proof. (1). Suppose the right *S*-act A_S satisfies Condition (*EP*) and let $ac = a'c$, aRa', for $a, a' \in A_S$ and right cancellable $c \in S$. Since aRa', there exist $s, t \in S$ such that $a = a's$ and $a' = at$. Since A_S satisfies Condition (*EP*), the equality $ac = atc$ implies that there exist $b \in A_S$ and $u, v \in S$ such that $a = bu = bv$ and $uc = vtc$. Then the right cancellability of c implies $u = vt$, and so $a' = at = bvt = bu = a$, as required.

(2). Since Condition (*E*) $\Rightarrow$ Condition (*EP*), it is obvious. □

Proposition 1.3. *Let S be a monoid. Then:*

(1) *All P-regular right S-acts are* $\Re$ *-torsion free.*

(2) *All strongly (P)-cyclic right S-acts are* $\Re$ *-torsion free.*

(3) *All regular right S-acts are* $\Re$ *-torsion free.*

(4) *All strongly faithful right S-acts are* $\Re$ *-torsion free.*

Proof. (1). It follows from [9, Theorem 2.2] and using the same argument as in the proof of (1) of Proposition 1.2.

Since strong faithfulness $\Rightarrow$ regularity $\Rightarrow$ strong (*P*)-cyclic $\Rightarrow$ *P*-regularity, (2), (3) and (4) are obvious. □

Notice that it is not yet known if the faithfulness implies $\Re$ -torsion freeness.

2. Characterization by $\Re$ -torsion freeness of right Rees factor acts

In this section we characterize monoids by $\Re$ -torsion freeness of right Rees factor acts. We recall that if K_S is a right ideal of *S*, the Rees congruence ρ_K is defined by $(a, b) \in \rho_K$ if $a, b \in K$ or $a = b$ and the resulting factor act is called the Rees factor act and is denoted by S / K_S. We say an ideal K_S of *S* satisfies Condition (*), if $xc, yc \in K_S$ and $x\Re y$, $x, y \in S \setminus K_S$, $c \in S$ right cancellable, imply $x = y$.

Lemma 2.1. *Let S be a monoid and* K_S *be a right ideal of S. Then:*

(1) $[x]_{\rho_K} \Re [y]_{\rho_K}$ *implies either* $x, y \in S \setminus K_S$ *or* $x, y \in K_S$, *for all* $x, y \in S$.

(2) $x\Re y$ *implies* $[x]_{\rho_K} \Re [y]_{\rho_K}$, *for all* $x, y \in S$.

(3) $x\Re y$ *if and only if* $[x]_{\rho_K} \Re [y]_{\rho_K}$, *for all* $x, y \in S \setminus K_S$.

Proof. (1). If $[x]_{\rho_K} \Re [y]_{\rho_K}$, then there exist $s, t \in S$ such that $[x]_{\rho_K} = [y]_{\rho_K} s = [ys]_{\rho_K}$ and $[y]_{\rho_K} = [x]_{\rho_K} t = [xt]_{\rho_K}$. Thus either $x = ys$ or $x, ys \in K_S$ and either $y = xt$ or $y, xt \in K_S$. If $x \notin K_S$, then $x = ys$, and so $y \notin K_S$. If $x \in K_S$, then $y \in K_S$, since $y = xt$ or $y, xt \in K_S$. Thus $x \in K_S$ if and only if $y \in K_S$.

(2). It is obvious.

(3). Let $x, y \in S \setminus K_S$. If $x\Re y$, then $[x]_{\rho_K} \Re [y]_{\rho_K}$. If $[x]_{\rho_K} \Re [y]_{\rho_K}$, then there exist $s, t \in S$ such that either $x = ys$ or $x, ys \in K_S$ and either $y = xt$ or $y, xt \in K_S$. Since $x, y \in S \setminus K_S$, we have $x = ys$ and $y = xt$, by (1), and so $x\Re y$. □

Theorem 2.1. *Let S be a monoid and K_S be a right ideal of S. Then the right Rees factor S-act S/K_S is $\Re$-torsion free if and only if K_S satisfies Condition (*).*

Proof. Necessity. Suppose the right Rees factor *S*-act S/K_S is $\Re$-torsion free, and let $xc, yc \in K_S$, $x\Re y$, for $x, y \in S \setminus K_S$, $c \in S$ right cancellable. Then $[x]_{\rho_K} c = [y]_{\rho_K} c$ and $[x]_{\rho_K} \Re [y]_{\rho_K}$, by (2) of Lemma 2.1. Hence, $[x]_{\rho_K} = [y]_{\rho_K}$, and so $x = y$ or $x, y \in K_S$. But $x, y \in S \setminus K_S$, and so $x = y$, as required.

Sufficiency. Suppose $[x]_{\rho_K} c = [y]_{\rho_K} c$ and $[x]_{\rho_K} \Re [y]_{\rho_K}$, for $x, y \in S$, $c \in S$ right cancellable. Then $xc = yc$ or $xc, yc \in K_S$. If $xc = yc$, then $x = y$, and so $[x]_{\rho_K} = [y]_{\rho_K}$, as required. Thus we suppose $xc, yc \in K_S$. Since $[x]_{\rho_K} \Re [y]_{\rho_K}$, either $x, y \in K_S$ or $x, y \in S \setminus K_S$, by (1) of Lemma 2.1. If $x, y \in K_S$, then $[x]_{\rho_K} = [y]_{\rho_K}$, as required. If $x, y \in S \setminus K_S$, then $x\Re y$, by (3) of Lemma 2.1. Thus by the assumption $x = y$, and so $[x]_{\rho_K} = [y]_{\rho_K}$, as required. □

Remark 2.1. If K_S is a left annihilating right ideal of a monoid *S*, then K_S satisfies Condition (*), but not the converse, otherwise, by Theorem 2.1, [12, III, 10.11] and that principal weak flatness implies $\Re$-torsion freeness, all left stabilizing right ideals are left annihilating, and so by [14, Theorem 10], all principally weakly flat right Rees factor *S*-acts satisfy Condition (*PWP*), which is not true. By [6, Lemma 3.4], all P_E-left annihilating right ideals are left stabilizing, thus every P_E-left annihilating right ideal satisfies Condition (*), but not the converse, otherwise, all torsion free right Rees factor *S*-acts are principally weakly flat, which is not true.

The following example shows that there are monoids *S* and right Rees factor *S*-acts which are not $\Re$-torsion free.

Example 2.1. Let $S = \mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$. Then *S* with multiplication is a commutative and cancellative monoid. If $K_S = (1+i)S = \{a + bi \mid a, b \in \mathbf{Z}, 2 \mid a + b\}$, then $5, -5 \in S \setminus K_S$, $-5 \times 2 = -10 \in K_S$, $5 \times 2 = 10 \in K_S$, and $5\Re -5$, but $5 \neq -5$, thus the right Rees factor *S*-act S/K_S is not $\Re$-torsion free, by Theorem 2.1.

As we saw in Example 1.1, the following example shows also that for Rees factor acts, $\Re$-torsion freeness does not imply torsion freeness.

Example 2.2. Let $S = (\mathbf{N}, \cdot)$. If $K_S = 2S$, then S/K_S is not torsion free, but it is $\Re$-torsion free. Thus for Rees factor acts $\Re$-torsion freeness does not imply torsion freeness and all properties which imply torsion freeness.

Now, it is natural to ask for monoids over which $\Re$-torsion freeness of Rees factor acts implies torsion freeness and all properties which imply torsion freeness.

Theorem 2.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts are torsion free.*

(2) *If a proper right ideal K_S of S satisfies Condition (*), then K_S satisfies the following condition:*

$xc \in K_S$, $x, c \in S$, c right cancellable, implies $x \in K_S$.

Proof. It follows from Theorem 2.1, and [12, III, 8.10]. □

Theorem 2.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts are principally weakly flat.*

(2) *If a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing.*

Proof. It follows from Theorem 2.1, and [12, III, 10.11]. □

Theorem 2.4. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts*

satisfy Condition (PWP).

(2) *If a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing and left annihilating.*

Proof. It follows from Theorem 2.1, and [14, Theorem 10]. □

Theorem 2.5. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (PWP_E).*

(2) *If a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing and E-left annihilating.*

Proof. It follows from Theorem 2.1, and [7, Theorem 4.2]. □

Theorem 2.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are flat.*

(2) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are weakly flat.*

(3) *S is right reversible and if a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing.*

Proof. It follows from Theorem 2.1, [12, III, 12.2], and [12, III, 12.17]. □

Theorem 2.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (WP).*

(2) *S is right reversible and if a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing and strongly left annihilating.*

Proof. It follows from Theorem 2.1, [14, Theorem 17], and [14, Corollary 18]. □

Theorem 2.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (P).*

(2) *S is right reversible and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S|=1$.*

Proof. It follows from Theorem 2.1, [12, III, 13.7], and [12, III, 13.9]. □

Theorem 2.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (P_E).*

(2) *S is right reversible and if a proper right ideal K_S of S satisfies Condition (*), then K_S is P_E-left annihilating.*

Proof. It follows from Theorem 2.1, and [6, Theorem 3.5]. □

Theorem 2.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (P').*

(2) *S is weakly right reversible and if a proper right ideal K_S of S satisfies Condition (*), then K_S is left stabilizing and completely left annihilating.*

Proof. It follows from Theorem 2.1, and [10, Theorem 4.3]. □

Theorem 2.11. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts satisfy Condition (E).*

(2) *S is left collapsible and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S|=1$.*

Proof. It follows from Theorem 2.1, [12, III, 14.3], and [12, III, 14.10]. □

Theorem 2.12. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are projective.*

(2) *S contains a left zero and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S|=1$.*

Proof. It follows from Theorem 2.1, [12, III, 17.2], and [12, III, 17.15]. □

Theorem 2.13. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are free.*

(2) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are projective generators.*

(3) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are generators.*

(4) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are faithful.*

(5) *All $\mathfrak{R}$-torsion free right Rees factor S-acts are*

strongly faithful.

(6) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). It follows from [12, III, 18.1].

Since $\Theta_S \cong S/S_S$ is an $\Re$-torsion free cyclic right Rees factor S-act, and Θ_S is faithful (strongly faithful) if and only if $S = \{1\}$, implications (4) $\Rightarrow$ (6) and (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1), (5). If $S = \{1\}$, then all right S-acts are free (strongly faithful). □

Theorem 2.14. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts are P-regular.*

(2) *S is right reversible, if S contains a left zero, then S is right PCP, and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S| = 1$.*

Proof. It follows from Theorem 2.1, and [9, Theorem 3.1]. □

Theorem 2.15. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts are strongly (P)-cyclic.*

(2) *S is right PCP, contains a left zero and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S| = 1$.*

Proof. It follows from Theorem 2.1, and [8, Theorem 3.1]. □

Theorem 2.16. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right Rees factor S-acts are are regular.*

(2) *S is right PP, contains a left zero and if a proper right ideal K_S of S satisfies Condition (*), then $|K_S| = 1$.*

Proof. It follows from Theorem 2.1, [12, III, 19.4], and [12, III, 19.6] . □

3. Characterization by $\Re$-torsion freeness of cyclic right acts

In this section we characterize monoids by $\Re$-torsion freeness of cyclic right acts.

Let S be a monoid, $s,t \in S$ and C_r be the set of all right cancellable elements of S. Set

$$F_1 = \{(x,y) \in S \times S \mid \exists c \in C_r, (xc, yc) \in \rho(s,t), [x]_{\rho(s,t)} \Re [y]_{\rho(s,t)}\},$$

$$F_{i+1} = \{(x,y) \in S \times S \mid \exists c \in C_r, (xc, yc) \in \rho(F_i), [x]_{\rho(F_i)} \Re [y]_{\rho(F_i)}\}$$

for $i \in \mathbf{N}$. It can easily be seen that F_i is reflexive and symmetric, for every $i \in \mathbf{N}$. Also,

$$\rho(s,t) \subseteq F_1 \subseteq \rho(F_1) \subseteq F_2 \subseteq \rho(F_2) \subseteq .....\rho(F_i) \subseteq F_{i+1} \subseteq .....$$

It is clear that $\rho_{\Re TF}(s,t) = \bigcup_{i \in \mathbf{N}} \rho(F_i)$ is a right congruence on S containing (s,t).

Theorem 3.1. *Let S be a monoid and $s,t \in S$. Then $\rho_{\Re TF}(s,t)$ is the smallest right congruence containing (s,t), where $S/\rho_{\Re TF}(s,t)$ is $\Re$-torsion free.*

Proof. If $[x]_{\rho_{\Re TF}(s,t)} c = [y]_{\rho_{\Re TF}(s,t)} c$ and $[x]_{\rho_{\Re TF}(s,t)} \Re [y]_{\rho_{\Re TF}(s,t)}$, for $x,y \in S$ and $c \in C_r$, then there exist $l_1, l_2 \in S$ such that $(x, yl_1), (y, xl_2) \in \rho_{\Re TF}(s,t)$. Thus there exist $i,j,k \in \mathbf{N}$ such that $(xc, yc) \in \rho(F_i)$, $(x, yl_1) \in \rho(F_j)$ and $(y, xl_2) \in \rho(F_k)$.

If $h = \max\{i,j,k\}$, then $(xc, yc), (x, yl_1), (y, xl_2) \in \rho(F_h)$, and so $(xc, yc) \in \rho(F_h)$ and $[x]_{\rho(F_h)} \Re [y]_{\rho(F_h)}$. By definition, $(x,y) \in F_{h+1}$, and so $(x,y) \in \rho(F_{h+1}) \subseteq \rho_{\Re TF}(s,t)$.

Thus $[x]_{\rho_{\Re TF}(s,t)} = [y]_{\rho_{\Re TF}(s,t)}$, as required. Let τ be a right congruence on S containing (s,t), where S/τ is $\Re$-torsion free. We show that $\rho_{\Re TF}(s,t) \subseteq \tau$. Since $(s,t) \in \tau$, we have $\rho(s,t) \subseteq \tau$. If $(x,y) \in F_1$, then there exists $c \in C_r$ such that $(xc, yc) \in \rho(s,t)$ and $[x]_{\rho(s,t)} \Re [y]_{\rho(s,t)}$, and so $(xc, yc) \in \tau$ and $[x]_\tau \Re [y]_\tau$. Since S/τ is $\Re$-

torsion free, $(x,y)\in\tau$. Thus $F_1\subseteq\tau$, and so $\rho(F_1)\subseteq\tau$. Suppose then that $\rho(F_i)\subseteq\tau$, $i\in\mathbf{N}$. If $(x,y)\in F_{i+1}$, then there exists $c\in C_r$ such that $(xc,yc)\in\rho(F_i)$ and $[x]_{\rho(F_i)}\Re[y]_{\rho(F_i)}$. Since $\rho(F_i)\subseteq\tau$ and S/τ is $\Re$-torsion free, $(x,y)\in\tau$. Hence $F_{i+1}\subseteq\tau$, and so $\rho(F_{i+1})\subseteq\tau$. Thus $\rho(F_i)\subseteq\tau$, for all $i\in\mathbf{N}$, and so $\rho_{\Re TF}(s,t)\subseteq\tau$. □

Theorem 3.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (P).*

(2) *For any $t,t'\in S$, there exist $u,v\in S$ such that $ut=vt'$ and $(u,1),(v,1)\in\rho_{\Re TF}(t,t')$.*

(3) *For any $s,t,t'\in S$, there exist $u,v\in S$ such that $ut=vt'$ and $(u,s),(v,s)\in\rho_{\Re TF}(st,st')$.*

Proof. (1) $\Rightarrow$ (2). The cyclic right *S*-act $S/\rho_{\Re TF}(t,t')$ is $\Re$-torsion free, and so it satisfies Condition (*P*). Thus by [12, III, 13.4], there exist $u,v\in S$ such that $ut=vt'$ and $(u,1),(v,1)\in\rho_{\Re TF}(t,t')$.

(2) $\Rightarrow$ (3). Suppose $s,t,t'\in S$. Then there exist $u',v'\in S$ such that $u'st=v'st'$ and $(u',1),(v',1)\in\rho_{\Re TF}(st,st')$. If $u:=u's$ and $v:=v's$, then $ut=vt'$ and $(u,s),(v,s)\in\rho_{\Re TF}(st,st')$.

(3) $\Rightarrow$ (1). Suppose τ is a right congruence on *S*, where S/τ is $\Re$-torsion free and let $(t,t')\in\tau$. Then by assumption, there exist $u,v\in S$ such that $ut=vt'$ and $(u,1),(v,1)\in\rho_{\Re TF}(t,t')$. By Theorem 3.1, $\rho_{\Re TF}(t,t')\subseteq\tau$, and so $(u,1),(v,1)\in\tau$. Thus S/τ satisfies Condition (*P*), by [12, III, 13.4]. □

Theorem 3.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (P_E).*

(2) *For any $x,y,t,t'\in S$, there exist $u,v\in S$ and $e,f\in E(S)$ such that $ut=vt'$, $et=t$, $ft'=t'$, $(xe,ue),(yf,vf)\in\rho_{\Re TF}(xt,yt')$.*

Proof. Using [6, Theorem 2.5] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (P').*

(2) *For any $x,y,t,t',z\in S$, the equality $tz=t'z$ implies that there exist $u,v\in S$ such that $ut=vt'$, $(x,u),(y,v)\in\rho_{\Re TF}(xt,yt')$.*

Proof. Using [10, Theorem 3.1] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.5. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (E).*

(2) *For any $s,t\in S$, there exists $u\in S$ such that $ut=us$ and $(u,1)\in\rho_{\Re TF}(s,t)$.*

Proof. Using [12, III, 14.8] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (E').*

(2) *For any $s,t,z\in S$, the equality $tz=sz$ implies that there exists $u\in S$ such that $ut=us$ and $(u,1)\in\rho_{\Re TF}(s,t)$.*

Proof. It follows from Theorem 3.1, definition of Condition (*E'*) and using the same argument as in the proof of Theorem 3.2. □

Theorem 3.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free cyclic right S-acts satisfy Condition (E'P).*

(2) *For any $x,y,z\in S$, the equality $xz=yz$ implies that there exist $u,v\in S$ such that $ux=vy$ and $(u,1),(v,1)\in\rho_{\Re TF}(x,y)$.*

(3) *For any $x,t,t',z\in S$, the equality $tz=t'z$ implies that there exist $u,v\in S$ such that $ut=vt'$ and $(u,x),(v,x)\in\rho_{\Re TF}(xt,xt')$.*

Proof. Using [3, Theorem 2.10] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.8. *Let S be a monoid. If all $\Re$-torsion free cyclic right S-acts are flat, then for any left* *congruence* λ *on S and any* $s,t \in S$, *there exist* $u,v \in S$ *such that* $(us,vt) \in \lambda$, $(u,1) \in \rho_{\Re TF}(s,t) \vee s\lambda$ *and* $(v,1) \in \rho_{\Re TF}(s,t) \vee t\lambda$.

Proof. Suppose λ is a left congruence on *S* and let $s,t \in S$. Then the cyclic right *S*-act $S / \rho_{\Re TF}(s,t)$ is $\Re$-torsion free, and so it is flat. Thus by [12, III, 12.11], there exist $u,v \in S$ such that $(us,vt) \in \lambda$, $(u,1) \in \rho_{\Re TF}(s,t) \vee s\lambda$ and $(v,1) \in \rho_{\Re TF}(s,t) \vee t\lambda$. □

Theorem 3.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$*-torsion free cyclic right S-acts are weakly flat.*

(2) *For any* $s,t \in S$, *there exist* $u,v \in S$ *such that* $us = vt$, $(u,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_s$ *and* $(v,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_t$.

Proof. (1) $\Rightarrow$ (2). The cyclic right *S*-act $S / \rho_{\Re TF}(s,t)$ is $\Re$-torsion free, and so it is weakly flat. Thus by [12, III, 11.5], there exist $u,v \in S$ such that $us = vt$, $(u,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_s$ and $(v,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_t$.

(2) $\Rightarrow$ (1). Suppose τ is a right congruence on *S*, where S / τ is $\Re$-torsion free and let $(s,t) \in \tau$. By Theorem 3.1, $\rho_{\Re TF}(s,t) \subseteq \tau$ and by assumption, there exist $u,v \in S$ such that $us = vt$, $(u,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_s$ and $(v,1) \in \rho_{\Re TF}(s,t) \vee \ker \rho_t$. Thus $(u,1) \in \tau \vee \ker \rho_s$ and $(v,1) \in \tau \vee \ker \rho_t$, and so S / τ is weakly flat, by [12, III, 11.5]. □

Theorem 3.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$*-torsion free cyclic right S-acts satisfy Condition (PWP).*

(2) *For any* $x,y,t \in S$, *there exist* $u,v \in S$ *such that* $ut = vt$ *and* $(u,x),(v,y) \in \rho_{\Re TF}(xt,yt)$.

Proof. Using [13, Lemma 2.7] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.11. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$*-torsion free cyclic right S-acts satisfy Condition (*PWP_E*).*

(2) *For any* $x,y,t \in S$, *there exist* $u,v \in S$ *and* $e,f \in E(S)$ *such that* $ut = vt$ *and* $(ue,xe),(vf,yf) \in \rho_{\Re TF}(xt,yt)$.

Proof. Using [7, Theorem 3.7] and Theorem 3.1, it is similar to the proof of Theorem 3.2. □

Theorem 3.12. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$*-torsion free cyclic right S-acts are principally weakly flat.*

(2) *For any* $u,v,s \in S$, $(u,v) \in \rho_{\Re TF}(us,vs) \vee \ker \rho_s$.

Proof. (1) $\Rightarrow$ (2). Suppose $u,v,s \in S$. The cyclic right *S*-act $S / \rho_{\Re TF}(us,vs)$ is $\Re$-torsion free, and so it is principally weakly flat. Since $(us,vs) \in \rho_{\Re TF}(us,vs)$ we have $(u,v) \in \rho_{\Re TF}(us,vs) \vee \ker \rho_s$, by [12, III, 10.7].

(2) $\Rightarrow$ (1). Suppose τ is a right congruence on *S*, where S / τ is $\Re$-torsion free and let $(us,vs) \in \tau$. Then by Theorem 3.1, $\rho_{\Re TF}(us,vs) \subseteq \tau$. By assumption, $(u,v) \in \rho_{\Re TF}(us,vs) \vee \ker \rho_s$, and so $(u,v) \in \tau \vee \ker \rho_s$. Thus S / τ is principally weakly flat, by [12, III, 10.7]. □

Theorem 3.13. *Let S be a monoid. Then:*

(1) $\rho_{\Re TF}(s,t) \subseteq \rho_{TF}(s,t)$.

(2) *All* $\Re$*-torsion free cyclic right S-acts are torsion free if and only if* $\rho_{\Re TF}(s,t) = \rho_{TF}(s,t)$.

Proof. (1) . $\rho_{TF}(s,t)$ is the right congruence containing (s,t), where $S / \rho_{TF}(s,t)$ is torsion free. Thus by Theorem 3.1, $\rho_{\Re TF}(s,t) \subseteq \rho_{TF}(s,t)$, since torsion freeness implies $\Re$-torsion freeness.

(2). Using [12, III, 8.4], Theorem 3.1, and (1), it is similar to the proof of Theorem 3.2. □

Theorem 3.14. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$*-torsion free cyclic right S-acts are free.*

(2) *All* $\Re$*-torsion free cyclic right S-acts are are projective generators.*

(3) *All* $\Re$*-torsion free cyclic right S-acts are generators.*

(4) *All $\Re$-torsion free cyclic right S-acts are faithful.*

(5) *All $\Re$-torsion free cyclic right S-acts are strongly faithful.*

(6) $S = \{1\}$.

Proof. It follow from Theorem 2.13. □

4. Characterization by $\Re$-torsion freeness of right acts

In this section we characterize monoids by $\Re$-torsion freeness of right acts.

Lemma 4.1. *Let S be a monoid and (U) be a property of S-acts which implies torsion freeness. Then the following statements are equivalent:*

(1) *All right S-acts satisfy (U).*

(2) *All $\Re$-torsion free right S-acts satisfy (U).*

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). We claim that $cS = S$, for any right cancellable $c \in S$. Otherwise, $cS \neq S$, for some right cancellable $c \in S$. Then the right *S*-act $S_S \coprod^{cS} S_S$ satisfies Condition (*E*), and so by (2) of Proposition 1.2, it is $\Re$-torsion free. Thus by assumption, $S_S \coprod^{cS} S_S$ is torsion free, and so the equality $(1,x)c = (1,y)c$, implies $(1,x) = (1,y)$, which is a contradiction. Thus $cS = S$, and so all right cancellable elements of *S* are right invertible. Thus all right *S*-acts are torsion free, by [12, IV, 6.1], and so all right *S*-acts satisfy (*U*), as required. □

Theorem 4.1. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right S-acts are free.*

(2) *All $\Re$-torsion free right S-acts are projective generators.*

(3) *All $\Re$-torsion free right S-acts are projective.*

(4) *All $\Re$-torsion free right S-acts are strongly flat.*

(5) *All $\Re$-torsion free right S-acts are generators.*

(6) *All $\Re$-torsion free right S-acts are faithful.*

(7) *All $\Re$-torsion free right S-acts are strongly faithful.*

(8) $S = \{1\}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (8) $\Rightarrow$ (1) are obvious.

(4) $\Rightarrow$ (8). Since strong flatness and pullback flatness coincide, it follows from Lemma 4.1 and [15, Theorem 3.4].

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8). The same argument can be used as in the proof of Theorem 2.13. □

Theorem 4.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right S-acts are weakly pullback flat.*

(2) *All $\Re$-torsion free right S-acts are weakly kernel flat.*

(3) *All $\Re$-torsion free right S-acts are principally weakly kernel flat.*

(4) *All $\Re$-torsion free right S-acts are translation kernel flat.*

(5) *All $\Re$-torsion free right S-acts satisfy Condition (P).*

(6) *All $\Re$-torsion free right S-acts satisfy Condition (WP).*

(7) *All $\Re$-torsion free right S-acts satisfy Condition (PWP).*

(8) *All $\Re$-torsion free right S-acts satisfy Condition (P').*

(9) *S is a group.*

Proof. Implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (9) follow from Lemma 4.1, and [1, Proposition 9].

(8) $\Leftrightarrow$ (9). It follows from Lemma 4.1, and [10, Theorem 2.5]. □

Theorem 4.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right S-acts are flat.*

(2) *All $\Re$-torsion free right S-acts are flat.*

Proof. Since flatness implies torsion freeness, it follow from Lemma 4.1. □

Theorem 4.4. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All $\Re$-torsion free right S-acts satisfy Condition (P_E).*

(2) *All $\Re$-torsion free right S-acts are weakly flat.*

(3) *S is regular and satisfies Condition: (R): for all $s,t \in S$ there exists $w \in Ss \cap St$ such that $(w,s) \in \rho(s,t)$.*

Proof. (1) $\Rightarrow$ (2). It follows from [2, Theorem 2.3].

(2) $\Rightarrow$ (3). It follows from Lemma 4.1, and [12, IV, 7.5].

(3) $\Rightarrow$ (1). It follows from [6, Theorem 2.1]. □

Theorem 4.5. *Let S be a monoid. Then the following* *statements are equivalent:*

(1) *All* $\Re$ *-torsion free right S-acts are principally weakly flat.*

(2) *All* $\Re$ *-torsion free right S-acts satisfy Condition (* PWP_E *).*

(3) *S is regular.*

Proof. (1) $\Leftrightarrow$ (3). It follows from Lemma 4.1, and [12, IV, 6.6].

(2) $\Leftrightarrow$ (3). It follows from Lemma 4.1, and [7, Theorem 3.1]. □

Theorem 4.6. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$ *-torsion free right S-acts are torsion free.*

(2) *Every right cancellable element of S is right invertible.*

Proof. It follows from Lemma 4.1, and [12, IV, 6.1]. □

Theorem 4.7. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$ *-torsion free right S-acts are regular.*

(2) *All* $\Re$ *-torsion free finitely generated right S-acts are regular.*

(3) *All* $\Re$ *-torsion free cyclic right S-acts are regular.*

(4) *All* $\Re$ *-torsion free monocyclic right S-acts are regular.*

(5) $S=\{1\}$ *or* $S=\{0,1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). It follows from [5, Theorem 1.8].

(5) $\Rightarrow$ (1). It follows from [12, IV, 14.4]. □

Theorem 4.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$ *-torsion free right S-acts are divisible.*

(2) *All* $\Re$ *-torsion free finitely generated right S-acts are divisible.*

(3) *All* $\Re$ *-torsion free cyclic right S-acts are divisible.*

(4) S_S *is divisible.*

(5) *Every left cancellabe element of S is left invertible.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Since S_S is $\Re$ -torsion free, it is clear.

(4) $\Rightarrow$ (5). It follows from [12, III, 2.2].

(5) $\Rightarrow$ (1). It follows from [12, III, 2.2]. □

Theorem 4.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All* $\Re$ *-torsion free right S-acts are principally weakly injective.*

(2) *All* $\Re$ *-torsion free finitely generated right S-acts are principally weakly injective.*

(3) *All* $\Re$ *-torsion free cyclic right S-acts are principally weakly injective.*

(4) *S is regular.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). All principal right ideals of *S* are $\Re$ -torsion free, by (2) and (3) of Proposition 1.1. Thus all principal right ideals of *S* are principally weakly injective, and so *S* is regular, by [12, IV, 1.6].

(4) $\Rightarrow$ (1). By [12, IV, 1.6], it is obvious. □

It is not yet known that when all right (Rees factor, cyclic) acts are $\Re$ -torsion free, but here we give some equivalents of that.

Theorem 4.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right S-acts are* $\Re$ *-torsion free.*

(2) *All divisible right S-acts are* $\Re$ *-torsion free.*

(3) *All principally weakly injective right S-acts are* $\Re$ *-torsion free.*

(4) *All fg-weakly injective right S-acts are* $\Re$ *-torsion free.*

(5) *All weakly injective right S-acts are* $\Re$ *-torsion free.*

(6) *All injective right S-acts are* $\Re$ *-torsion free.*

(7) *All cofree right S-acts are* $\Re$ *-torsion free.*

Proof. (1) $\Rightarrow$ (2). It is obvious.

Since cofreeness $\Rightarrow$ injectivity $\Rightarrow$ weak injectivity $\Rightarrow$ fg-weak injectivity $\Rightarrow$ principal weak injectivity $\Rightarrow$ divisibility, implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) follow.

(7) $\Rightarrow$ (1). Every right *S*-act can be embeded into a cofree right *S*-act. Thus by (3) of Proposition 1.1, all right *S*-acts are $\Re$ -torsion free. □

References

1. Bulman-Fleming S., Kilp M. and Laan V. Pullbacks and Flatness Properties of Acts *II*, *Comm. Algebra*, **29** (2): 851-878 (2001).
2. Golchin A. and Renshaw J. A flatness property of acts over monoids, Conference on Semigroups, University of St. Andrews: 72-77(1998).
3. Golchin A. and Mohammadzadeh H. On Condition (*E'P*) , *J. Sci. Islam. Repub. Iran* **17**(4): 343-349, (2006).
4. Golchin A. and Mohammadzadeh H. On Condition (*EP), Int. Math. Forum* **2** (19): 911-918 (2007).

5. Golchin A. and Mohammadzadeh H. On Regularity of Acts, *J. Sci. Islam. Repub. Iran* **19** (4): 339-345 (2008).
6. Golchin A. and Mohammadzadeh H. On Homological Classification of Monoids by Condition (P_E) of Right Acts, IJPAM **25**: 175-186 (2009).
7. Golchin A. and Mohammadzadeh H. On Condition (PWP_E), Southeast Asian Bull. *Math* **33**: 245-256 (2009).
8. Golchin A., Rezaei P. and Mohammadzadeh H. On Strongly (*P*) -cyclic Acts, *Czech. Math.* J, **59** (134): 595-611 (2009).
9. Golchin A., Mohammadzadeh H. and Rezaei P. On *P*-regularity of Acts, *APM* **2**: 104-108 (2012).
10. Golchin A. and Mohammadzadeh H. On Condition (P'), Semigroup Forum, Published online: 09 October(2012).
11. Howie J. M. Fundamentals of Semigroup Theory, London Mathematical Society Monographs, Oxford University Press, London (1995).
12. Kilp M., Knauer U. and Mikhalev A. Monoids, Acts and Categories. Walter de Gruyter, Berlin, New York (2000).
13. Laan V. Pullback and Flatness Properties of Acts. Ph.D Thesis, Tartu (1999).
14. Laan V. Pullbacks and Flatness Properties of Acts I, *Comm. Algebra*, **29** (2): 829-850 (2001).
15. Normak P. On Equalizer-flat and Pullback-flat Acts, Semigroup Forum, **36** (2): 293-313 (1987).