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Essentially Retractable Modules

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Abstract

We call a module M_R *essentially retractable* if $\mathrm{Hom}_R(M,N)\neq 0$ for all essential submodules N of M. For a right FBN ring R, it is shown that: (i) A non-zero module M_R is *retractable* (in the sense that $\mathrm{Hom}_R(M,N)\neq 0$ for all non-zero $N\leq M_R$) if and only if certain factor modules of M are essentially retractable nonsingular modules over R modulo their annihilators. (ii) A non-zero module M_R is essentially retractable if and only if there exists a prime ideal $P\in Ass(M_R)$ such that $\mathrm{Hom}_R(M,N)\neq 0$. Over semiprime right nonsingular rings, a nonsingular essentially retractable module is precisely a module with non-zero dual. Moreover, over certain rings R, including right FBN rings, it is shown that a nonsingular module M with enough uniforms is essentially retractable if and only if there exist uniform retractable R-modules $\{U_i\}_{i\in I}$ and R-homomorphisms $M\xrightarrow{\alpha}\oplus_{i\in I}U_i\xrightarrow{\beta}M$ with $\beta\alpha\neq 0$.

Keywords: Dual module; Essentially retractable; Homo-related

1. Introduction

Throughout all rings have non-zero identity elements and all modules are unital right R-modules. Any terminology not defined here may be found in [2,8]. S. M. Khuri [5] defined the concept of a *retractable* module: An R-module M is retractable if $\mathrm{Hom}_R(M,N)\neq 0$ for all non-zero submodules N of M. The endomorphism ring of a retractable module which also satisfies another condition, often and most notably nonsingularity, has been the subject of study in several articles [6,7,11]. More recently retractable modules have appeared in the study of modules whose endomorphism rings are prime [1], Baer or quasi-Baer [9], semiprime or nonsingular or finite uniform dimensional [4]; see also [3] and [10] where the terms "quotient-like" and "slightly compressible" were respectively used for retractable. We define *essential retractability* for a module M_R by requiring that $\mathrm{Hom}_R(M,N)\neq 0$ for all $N\leq_e M_R$ where the notation $N\leq_e M_R$ means N is an essential submodule of M_R. Clearly non-zero retractable implies essentially retractable, but the inverse implication is not true in general. According to Proposition 2.1 for an arbitrary module to be retractable, it suffices that all of its factor modules be essentially

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retractable. This simple observation is our primary motivation to characterize essentially retractable modules over certain rings. Characterization of retractable modules over right FBN rings is already found by Smith [10]. In our endeavor, we first observe that any nonsingular essentially retractable module has a non-zero dual. This yields the fact that for a nonsingular module over a prime ring, the conditions retractable, essentially retractable and having non-zero dual are all equivalent. We are then able to prove that over a right FBN ring R retractability of a module M is equivalent to essential retractability of factor modules of the form $N/Z(N)$ where $N = M/MP$ with $P \in Ass(M_R)$ (Theorem 2.8), and that M_R is essentially retractable if and only if $\mathrm{Hom}_R(M, R/P) \neq 0$ for some $P \in Ass(M_R)$. We note that each of the above factor modules is nonsingular over R modulo its annihilator. Thus over a right FBN ring the study of retractable modules reduces to that of nonsingular essentially retractable modules. Next we consider a ring R with the property that its non-zero right ideals contain non-zero retractable right ideals. The class of such rings is rather large as it contains all commutative rings, local rings with T-nilpotent Jacobson radical, right FBN rings and semiprime rings. For a nonsingular R-module M with enough uniforms, we prove that M is essentially retractable if and only if M is *home-related* to a direct sum of cyclic nonsingular uniform retractable right ideals (Theorem 2.14). The module M_R is said to be homo-related to L_R if there are $\alpha : M \to L$ and $\beta : L \to M$ such that $\beta\alpha \neq 0$.

2. Essentially Retractable Modules over Certain Rings

Proposition 2.1. Let M be any non-zero module over an arbitrary ring R. If any non-zero factor of M_R is essentially retractable then M_R is retractable.

Proof. Let $0 \neq N \leq M_R$. Because $E(N)$ is an injective R-module the inclusion map $N \to E(N)$ is extended to a non-zero R-module homomorphism $f : M \to E(N)$. Since $N \leq_e E(N)$, $(N \cap f(M))$ is an essential submodule of $f(M)$. Thus $\mathrm{Hom}_R(f(M), N \cap f(M))$ is non-zero by the essentially retractable condition on $f(M)$. It follows that $\mathrm{Hom}_R(M, N) \neq 0$, proving that M_R is retractable. □

Lemma 2.2. The following statements are equivalent for a non-zero R-module M.

(a) M_R is essentially retractable.

(b) There exists a non-zero $f \in End_R(M)$ such that Im (f) is an essentially retractable R-module.

Proof. (a) $\Rightarrow$ (b) This is clear.

(b) $\Rightarrow$ (a). Suppose that (b) holds. Thus there exists a non-zero homomorphism $h : f(M) \to K \cap f(M)$. It follows that $\mathrm{Hom}_R(M, K) \neq 0$. □

The following result is a consequence of Lemma 2.2 and gives some information about essentially retractable modules over a right self-injective right nonsingular ring. Then, in Theorem 2.5, we shall characterize essentially retractable modules over rings that belong to a larger class.

Propositin 2.3. If R is a right self-injective right nonsingular ring, then any R-module M is either singular or essentially retractable.

Proof. Suppose that M is a non-zero R-module. If $Z(M) \leq_e M_R$ then M_R is a singular module because R is a right nonsingular ring. Therefore assume that there is a non-zero $m \in M$ such that $Z(M) \cap mR = 0$. On the other hand, because R is right self-injective, it is easy to verify that any nonsingular cyclic right R-module is injective. Consequently, mR is an injective retractable right R-module. Thus M has a non-zero retractable direct summand. By Lemma 2.2, it follows that M_R is essentially retractable. □

We remark that a factor, or even a direct summand of a retractable module, need not be retractable. For example Z_{p^∞} is not retractable in Mod-Z, but $Z_{p^\infty} \oplus Z$ is clearly retractable where p is a prime number. However the class of essentially retractable modules is easily seen to be closed under direct sums, as is the case for the class of retractable modules [10, Proposition 1.4].

Lemma 2.4. Let M_R be retractable with a semiprime endomorphism ring S and let W be any direct sum of copies of M_R, then any non-zero submodule of W_R is retractable.

Proof. It is easy to verify that W_R is also retractable. Since S is a semiprime ring, so is any column finite matrix ring over S. Hence $T = End_R(W)$ is a semiprime ring. Now suppose that $0 \neq K \leq N \leq W_R$, I = $\mathrm{Hom}_R(W, N)$, and J = $\mathrm{Hom}_R(W, K)$. Clearly $J \subseteq I$ are right ideals in T. Since W_R is retractable,

$J \neq 0$. Thus $0 \neq J^2 \subset JI$ by semiprimness of *T*. Hence there exists $f \in J$ such that $fI \neq 0$. It follows that $f|_N : N \to K$ is non-zero, Proving that N_R is retractable. □

The following Theorem shows that nonsingular essentially retractable modules over a right nonsingular semiprime ring are exactly modules with non-zero duals. For an *R*-module *M*, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*.

Theorem 2.5. Let M be a nonsingular *R*-module. If *M* is essentially retractable then M^* is non-zero. The converse holds if *R* is a semiprime right nonsingular ring.

Proof. Let *m* be any non-zero element of *M*. Since $\mathrm{r.ann}(m)$ is not an essential right ideal of *R*, there exists a right ideal *I* in *R* such that $mI \cong I$. Thus any non-zero submodule of *M* contains a submodule isomorphic to a non-zero right ideal of *R*. Let A be a maximal independent family of non-zero submodules each of which is isomorphic to a non-zero right ideal. If $N = \oplus \mathrm{A}$, then *N* is an essential submodule of M_R. Thus, by our assumption, $\mathrm{Hom}_R(M, N) \neq 0$. It follows that $\mathrm{Hom}_R(M, R) \neq 0$.

Conversely, suppose that $Z_r(R) = 0$ and let *N* be as above. If there is a non-zero homomorphism $f : M \to R$, the Ker *f* is not an essential submodule of M_R because $Z_r(R) = 0$. Thus $f(N) \neq 0$. Hence $[f(N)]^2 \neq 0$ by the semiprime condition on *R*. It follows that $nf(M) \neq 0$ for some $n \in N$. Now $h = gof : M \to N$ is non-zero, where $g : f(M) \to N$ is defined by $g(x) = nx$. Since $h(M)$ can be embedded in a free right *R*-module, it is retractable by Lemma 2.4. Consequently, M_R is essentially retractable by Lemma 2.2. □

Corollary 2.6. Over a commutative semiprime ring, a nonsingular module is essentially retractable if and only if its dual module is non-zero.

Proof. By Theorem 2.5. □

Corollary 2.7. Let M be a nonsingular module over a prime ring *R*. Then the following statements are equivalent.

(a) $\mathrm{Hom}_R(M, R) \neq 0$.

(b) M_R is retractable.

(c) M_R is essentially retractable.

Proof. We first note that if *I* and *J* are right ideals in *R* with $IJ \neq 0$, then $\mathrm{Hom}_R(J, I) \neq 0$.

(a) $\Rightarrow$ (b). Let $f : M \to R$ be non-zero. For any non-zero $m \in M$, there is a right ideal *I* of *R* such that $mI \cong I$, and consequently $\mathrm{Hom}_R(f(M), I) \neq 0$. Thus $\mathrm{Hom}_R(M, mR) \neq 0$, proving that M_R is retractable.

That (b) $\Rightarrow$ (c) is clear, while (c) $\Rightarrow$ (a) follows by Theorem 2.5. □

Let *M* be a non-zero *R*-module. Following [2] an associated prime ideal of *M* means a prime ideal *P* of *R* such that, for some non-zero submodule *N* of *M*, $P = ann_R(L)$ for every non-zero submodule *L* of *N*. The set of associated prime ideals of M_R will be denoted by Ass(M_R). In [10], it is shown that if *R* is a right FBN ring then a non-zero right *R*-module *M* is retractable if and only if $\mathrm{Hom}_R(M, R/P) \neq 0$ for every associated prime ideal *P* of *M*. Using this and Theorem 2.5, we show in the following result that if *R* is a right FBN ring, then the retractability of a module is equivalent to essential retractability of certain factor modules.

Theorem 2.8. Let *M* be non-zero *R*-module. If *R* is a right FBN ring, then M_R is retractable if and only if for any $P \in Ass(M_R)$ the R/P-module $N/Z_{R/P}(N)$ is essentially retractable as a right *R*-module, where $N = M/MP$.

Proof. Let $P \in Ass(M_R)$, $N = M/MP$, $H = Z(N)$ the singular submodule of *N* as an R/P-module. If M_R is retractable then $\mathrm{Hom}_R(M, R/P)$ is non-zero. It follows that there exists a non-zero R/P-homomorphism $f : N \to R/P$. Since R/P is a right nonsingular ring, $f(H) = 0$ and so *f* induces a non-zero R/P-homomorphism $\bar{f} : N/H \to R/P$. Consequently, N/H is a nonsingular right R/P-module with non-zero dual. Thus N/H is an essentially retractable right R/P-module by Theorem 2.5. Now clearly, $(N/H)_R$ is essentially retractable. Conversely, let $(N/H)_R$ be essentially retractable. Then by Theorem 2.5, it is easily seen that $\mathrm{Hom}_R(N, R/P) \neq 0$ for any $P \in Ass(M_R)$ and so $\mathrm{Hom}_R(M, R/P) \neq 0$. Hence M_R is retractable. □

We note that each of the above *N*/*Z*(*N*) is nonsingular over *R* modulo its annihilator. Thus over right FBN ring the study of retractable modules reduces to that of nonsingular essentially retractable modules. Later, we shall investigate nonsingular essentially retractable modules with enough uniforms over more general rings including right FBN rings. But now, we are going to give a characterization of essentially retractable modules over a right FBN ring. The right R-module M is called *compressible* if for each non-zero submodule N of M, there exists a monomorphism $\theta : M \to N$.

Lemma 2.9. Let R be a right FBN ring. Then every non-zero cyclic right R-module N contains a uniform compressible submodule.

Proof. Let U be a uniform submodule of N and let $P \in Ass(U)$. Thus there exists a non-zero submodule V in U such that $P = ann_R(V)$. Now by [2, Corollary 8.3], $Z_{R/P}(V)$ is zero. It follows that there exists right ideal Y of R/P such that Y can be embedded in $V_{R/P}$. Since all non-zero right ideals in R/P are retractable and nonsingular, Y is compressible as an R/P-module. Clearly Y_R is also compressible. □

Theorem 2.10. Let M be a non-zero module over a right FBN ring R. Then M_R is essentially retractable if and only $\mathrm{Hom}_R(M, R/P) \neq 0$ for some $P \in Ass(M_R)$.

Proof. $(\Rightarrow)$. Let M_R be essentially retractable. By Lemma 2.9 and Zorn's Lemma, there exists an essential submodule N in M_R such that $N = \oplus_{i \in I} V_i$ where each V_i $(i \in I)$ is a uniform compressible submodule of M_R. Thus by our assumption, there is a non-zero $f : M \to V_i$ for some $i \in I$. Let $P \in Ass(f(M))$. Then there exists a non-zero clyclic submodule X of $f(M)$ such that $P = ann_R(X)$. Now by [2, Corollary 8.3], X is a nonsingular right R/P-module so that there exists a submodule Y of $X_{R/P}$ such that Y is isomorphic to a right ideal of R/P. Because $f(M)$ is a compressible right R-module, it can be embedded in Y. It follows that $\mathrm{Hom}_R(M, R/P) \neq 0$.

$(\Leftarrow)$. Assume that $f : M \to R/P$ is a non-zero homomorphism for some $P \in Ass(M_R)$. There exists a non-zero submodule N of M_R such that $P = ann_R(X)$ for every non-zero submodule X of N. By Lemma 2.9, suppose that X is some cyclic uniform compressible submodule of N. Again X is a torsionfree right R/P-module, hence a non-zero right ideal Y of R/P can be embedded in X. On the other hand, since R/P is a prime ring $Yf(M) \neq 0$. Consequently, $\mathrm{Hom}_R(f(M), Y) \neq 0$. It follows that $\mathrm{Hom}_R(M, X) \neq 0$. Now X_R is compressible and so is any of its submodules, hence M_R is essentially retractable by Lamma 2.2. □

Theorem 2.5 states, over certain rings, essentially retractable modules are precisely modules with non-zero dual. Now, in view of Lemma 2.9, we consider rings R in which *every non-zero cyclic right ideal contains a non-zero retractable right ideal.* For example if R is either commutative (or even right duo) or a local ring with T-nilpotent Jacobson radical then R has this property [note that if $x \in R \setminus I$ for some T-nilpotent right ideal I and $xI \not\subseteq I$ then there is $a_1 \in I$ such that $xa_1 \notin I$. Again if $xa_1 I \not\subseteq I$ then there exists $a_2 \in I$ such that $xa_1 a_2 \notin I$. If this process does not stop, we get a sequence $a_1, a_2, \ldots$ in I with $xa_1 \ldots a_k \notin I$ for all k. But I is T-nilpotent, hence there exists n such that $0 = a_1 \ldots a_n$, which is in contradiction with $xa_1 \ldots a_n \notin I$. It follows that there exists $r \in R$ such that $xr \notin I$ but $xrI \subseteq I$. Consequently, $\mathrm{Hom}_R(R/I, (xR + I)/I) \neq 0$. This shows that $(R/I)_R$ is retractable]. Other examples of such rings include right FBN rings as well as semiprime rings; see Lemmas 2.9 and 2.4.

In order to investigate essentially retractable modules over such rings, we introduce a condition for a pair of modules M and L, that is somewhat stronger then the condition $M_R^* \neq 0$ when $L = R$:

Let M and L be R-modules. We say that M is *homo-related* to L if there exist R-homomorphism $\alpha : M \to L$ and $\beta : L \to M$ such that $\beta\alpha \neq 0$.

Proposition 2.11. Let M be a non-zero R-module with $I = ann_R(M)$.

(a) M is homo-related to R/I if and only if $\mathrm{Hom}_R(M, R/I) \neq 0$.

(b) If R is a semiprime ring then M is homo-related to R if and only if $\mathrm{Hom}_R(M, R) \neq 0$.

Proof. We only prove(a) as (b) is proved by a similar method. Suppose that a non-zero homomorphism $\alpha : M \to R/I$ exists and let $\alpha(M) = K/I$. Thus $MK \neq 0$, so that there exists an element $m \in M$ such that $mK \neq 0$. Define $\beta : R/I \to M$ by $\beta(r + I) = mr$, which satisfies $\beta\alpha \neq 0$. The converse is obvious. □

The following two Lemmas are needed.

Lemma 2.12. Let $M = \oplus_{i \in I} M_i$ be a direct sum of uniform submodules M_i $(i \in I)$ and let N be any non-zero submodule of M. Then there exists a subset I' of I such that the canonical projection $\pi : M \to \oplus_{i \in I'} M_i$ is one to one on N and $\pi(N)$ is an essential submodule of $\oplus_{i\in I'} M_i$.

Proof. If $N_i = N \cap M_i \neq 0$ for all $i \in I$ then $\oplus N_i$ is an essential submodule of M and so $N \leq_e M_R$, in which case there remains nothing to prove. Suppose that $N \cap M_j = 0$ for some $j \in I$. By Zorn's Lemma there exists a maximal subset I'' of I such that $N \cap (\oplus_{i\in I''} M_i) = 0$. Note that I'' is a proper subset of I and hence $I' = I \setminus I''$ is a nonempty set. Let $\pi : M \to \oplus_{i\in I'} M_i$ denote the canonical projection. Then $\pi|_N : N \to \oplus_{i\in I'} M_i$ is a monomorphism because ker $(\pi|_N) = N \cap (\oplus_{i\in I''} M_i) = 0$. Let $k \in I'$. By the choice of I'', $N \cap \{M_k \oplus (\oplus_{i\in I''} M_i)\} \neq 0$ and hence $\pi(N) \cap M_k \neq 0$. It follows that $\pi(N)$ is an essential submodule of $\oplus_{i\in I'} M_i$. □

Lemma 2.13. Let M_R be retractable and $0 \neq N \leq M_R$. If $\mathrm{Hom_R}(M/N, N) = 0$ then N is retractable. Furthermore, if M_R is nonsingular then any essential submodule of M_R is retractable.

Proof. Let $0 \neq K \leq N_R$. There is $0 \neq f \in S$ such that $f(M) \subseteq K$. If $f(N) = 0$, then the rule $m + n \to m + Kerf$ yields a non-zero homomorphism $M/N \to M \ \ker f \cong \mathrm{Im} f$ which is in contradiction with our assumption $\mathrm{Hom_R}(M/N, N) = 0$. Thus $f(N) \neq 0$, hence $f|_N$ is a non-zero endomorphism of N with image in K. The last statement is now clear. □

Recall that a right R-module M is said to have *enough uniforms* if every non-zero submodule of M contains a uniform submodule.

Theorem 2.14. Let M be a nonsingular module over a ring R whose non-zero right ideals contain non-zero retractable right ideals. If M_R has enough uniforms, then the following statements are equivalent.

(a) M_R is essentially retractable.

(b) M_R is homo-related to a direct sum $\oplus_{i\in I} U_i$ of cyclic nonsingular uniform retractable right ideals U_i of R.

(c) M_R is homo-related to a direct sum of uniform retractable right R-modules.

Proof. (a)$\Rightarrow$(b). By our assumption and Zorn's Lemma there exists an essential submodule N in M_R such that $N = \oplus_{i\in I} V_i$ where each V_i $(i \in I)$ is a uniform submodule. Since each V_i is a nonsingular right R-module, there is a right ideal U_i in R such that U_i can be embedded in V_i. By our assumption, we may suppose that U_i is cyclic and retractable. Now $\oplus_{i\in I} U_i$ is isomorphic to an essential submodule of M_R and so by (a), there exists a non-zero homomorphism $f : M \to \oplus_{i\in I} U_i =: L$. It follows that M is homo-related to L.

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (a). Generally, if W is a non-zero nonsingular right R-module and L is a direct sum of uniform retractable right R-modules and there exist homomorphisms $f : W \to L$ and $g : L \to W$ such that $f \neq 0$ and g is one to one, then W_R is essentially retractable. This is because, by Lemmas 2.12 and 2.13, $f(W)$ is non-zero retractable and then W_R is essentially retractable by Proposition 2.2.

Now suppose that $f : M \to L =: \oplus_{i\in I} U_i$ and $g : L \to M$ such that $gof \neq 0$ where each U_i $(i \in I)$ is a uniform retractable module.

Case 1. Assume that for each $i \in I$, $g(U_i) \neq 0$. Because M_R is nonsingular, g is one to one on U_i $(i \in I)$. It follows that there exists a monomorphism $h : L \to W := M^{(I)}$. Also $0 \neq fo\pi :$ $W \to L$ where $\pi : W \to M$ is any epimorphism. Thus, by the first part, W_R and so M_R is essentially retractable.

Case 2. Assume that there exists $i \in I$ such that $g(U_i) = 0$. By Zorn's Lemma, choose a maximal subset I' of I such that $g(\oplus_{i\in I'} U_i) = 0$ and set $L' = \oplus_{j\in J} U_j$ where $J = I \setminus I'$. Again, as in case 1, L' can be embedded in $M^{(J)}$ and because $gof \neq 0$, there exists a non-zero homomorphism $h : M^{(J)} \to L'$. Thus M_R is essentially retractable. □

Corollary 2.15. Let R be a finite uniform dimensional commutative ring. Then a nonsingular R-module is essentially retractable if and only if it is homo-related to a direct sum of uniform retractable modules.

Proof. If the cyclic R-module R/I is nonsingular, then it is easy to verify that I is an essentially closed right ideal of R. Hence the uniform dimension of R is bigger than the uniform dimension of $(R/I)_R$. This shows that any nonsingular R-module has enough uniforms. The result is now clear by Theorem 2.14. □

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