On the *U-WPF* Acts over Monoids

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Abstract

Valdis Laan in [5] introduced an extension of strong flatness which is called weak pullback flatness. In this paper we introduce a new property of acts over monoids, called *U-WPF* which is an extension of weak pullback flatness and give a classification of monoids by this property of their acts and also a classification of monoids when this property of acts implies others. We also show that regularity and strong faithfulness of acts both imply *U-WPF*. An equivalent of that over monoids for which torsion freeness implies *U-WPF* is given too.

Keywords: U-WPF; Strongly faithful; Regular

1. Introduction and Preliminaries

We deal in this paper with what is generally referred to as homological classification of monoids by flatness properties. Many papers have appeared recently investigating the conditions on a monoid which are necessary and sufficient to make a variety of the flatness properties coincide, either for all right acts, or for all right acts of certain type. Conditions (*P*) and (*E*) of acts over monoids have appeared in many papers and monoids are classified when these properties of acts imply other flatness properties and vice versa. A right act is strongly flat if it satisfies conditions (*P*) and (*E*). Valdis Laan in [5] introduced an extension of strong flatness which is called weak pullback flatness. In this paper we introduce a new property of acts over monoids, called *U-WPF* which is an extension of weak pullback flatness and give a classification of monoids by this property of their acts and also a classification of monoids when this property of acts implies others. We also show that regularity and strong faithfulness of acts both imply *U-WPF*. An equivalent of that over monoids for which torsion freeness implies *U-WPF* is given too.

Throughout this paper S will denote a monoid. We refer the reader to [3], [4], for basic definitions and terminology relating to semigroups and acts over monoids and to [6], [7] for definitions and results on flatness which are used here.

A monoid S is left collapsible if for every $s, t \in S$ there exists $u \in S$ such that $us = ut$. A monoid S satisfies Condition (*K*) if every left collapsible submonoid of S contains a left zero. A monoid S is right *PP* if every principal right ideal of S is projective. If for every $s \in S$ there exists $x \in S$ such that $sxs = s$, then S is called a regular monoid, for example for any field K and any $n \in N$ the monoid $(Mat_n(K), .)$ of all $n \times n$ matrices with entries in K is a regular monoid.

A right ideal K of S is left stabilizing if for every $k \in K$ there exists $l \in K$ such that $lk = k$. If I is a proper right ideal of S, then

$$B_S = S \coprod^I S$$

$$= \{(a, x) \mid a \in S \setminus I\} \dot{\cup} I \dot{\cup} \{(a, y) \mid a \in S \setminus I\}$$

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with

$$(a,z)s = \begin{cases} (as,z), & as \notin I \\ \\ as & \text{otherwise} \end{cases}$$

for every $a \in S \setminus I, s \in S$ and $z \in \{x, y\}$ is a right S-act. (Note that by $\dot{\bigcup}$ we mean disjoint union.)

A right S-act A satisfies Condition (P) if for all $a, a' \in A, s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A, u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs'$. It satisfies Condition (E) if for all $a \in A, s, s' \in S$, $as = as'$ implies that there exist $a' \in A, u \in S$ such that $a = a'u$ and $us = us'$. If for all $a \in A, s, s', z \in S$, $as = as'$ and $sz = s'z$ imply that there exist $a' \in A, u \in S$ such that $a = a'u$ and $us = us'$, then it satisfies Condition (E'). A right S-act is weakly pullback flat if it satisfies both Conditions (P) and (E'). A right S-act A satisfies Condition (P_E) if whenever $a, a' \in A, s, s' \in S$, and $as = a's'$ there exist $a'' \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = a''ue, a'f = a''vf, es = s, fs' = s'$ and $us = vs'$. It is shown in [2] that Condition (P_E) implies weak flatness, but the converse is not true.

A right S-act A is strongly faithful if for $s, t \in S$ the equality $as = at$ for some $a \in A$ implies that $s = t$.

If S is a left cancellative monoid, then S as a right S-act is strongly faithful.

We use the following abbreviations; strong flatness = *SF*, weak pullback flatness = *WPF*, weak kernel flatness = *WKF*, principal weak kernel flatness = *PWKF*, translation kernel flatness = *TKF*, weak homoflatness = *WP*, principal weak homoflatness = *PWP*, weak flatness = *WF*, principal weak flatness = *PWF*.

2. General Properties

Definition 2.1. Let S be a monoid. A right S-act A is *U-WPF* if there exists a family $\{B_i \mid i \in I\}$ of subacts of A such that $A = \bigcup_{i \in I} B_i$ and $B_i, i \in I$ is *WPF*. An example of a *U-WPF* right S-act is given in section 3 before theorem 3.1.

Theorem 2.2. Let S be a monoid. Then
(1) *Every WPF right S-act is U-WPF.*
(2) *A right S-act A is U-WPF if and only if for every $a \in A$ there exists a subact B of A such that $a \in B$ and B is WPF.*
(3) *If $\{B_i \mid i \in I\}$ is a family of subacts of a right*

S-act A such that for every $i \in I$, B_i is U-WPF, then $\bigcup_{i \in I} B_i$ is U-WPF.

(4) *For every proper right ideal I of $B_S = S \amalg^I S$ is U-WPF.*

(5) *A cyclic right S-act is WPF if and only if it is U-WPF.*

Proof. The proof of (1), (2), (3), (5) is straightforward.

Now let I be a proper right ideal of S and let $B_S = S \amalg^I S$. Then

$$C_S = \{(a,x) \mid a \in S \setminus I\} \dot{\bigcup} I$$

$$\cong S_S \cong \{(a,y) \mid a \in S \setminus I\} \dot{\bigcup} I = D_S$$

Since S_S is weakly pullback flat, then C_S and D_S are weakly pullback flat and so $B_S = C_S \bigcup D_S$ is *U-WPF*, thus we have (4). □

Note that Condition (P) does not imply *U-WPF*, otherwise by Theorem 2.2, for cyclic right S-acts, *WPF* and Condition (P) coincide, which by ([6, Example 6]), is not true in general.

3. Classification of Monoids by *U-WPF* of Right Acts

Although weak pullback flatness implies torsion freeness, but note that *U-WPF* does not imply torsion freeness in general, for if $S = (N, .)$, where N is the set of natural numbers, and if $A_S = N \amalg^{2N} N$, then by Theorem 2.2, A_S is *U-WPF*, but A_S is not torsion free, otherwise $2 = (1, x)2 = (1, y)2$ implies that $(1, x) = (1, y)$, which is a contradiction. Now it is obvious that *U-weak pullback flatness* does not imply weak pullback flatness and so it is natural to ask for monoids which *U-weak pullback flatness* of their acts implies other properties such as torsion freeness.

Theorem 3.1. *For any monoid S the following statements are equivalent:*
(1) *All right S-acts are torsion free.*
(2) *All U-WPF right S-acts are torsion free.*
(3) *All right cancellable elements of S are right invertible.*

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Suppose that c is a right cancelable element of S. If $cS = S$, then obviously c is right invertible. Thus we suppose that $cS \neq S$ and that

$A_S = S \coprod^{cS} S$. Then by Theorem 2.2, A_S is *U-WPF*. Thus by the assumption A_S is torsion free and so $c = (1, x)\, c = (1, y)c$ implies that $(1, x) = (1, y)$, which is a contradiction. Hence for every right cancelable element c of $S, cS = S$ that is, every right cancelable element of S is right invertible as required.

$(3) \Rightarrow (1)$. By ([4, IV, 6.1]), It is obvious. □

As Theorem 3.1 shows, to see that all acts are torsion free, it suffices to show that all *U-WPF* acts are torsion free.

Theorem 3.2. *For any monoid S the following statements are equivalent:*

    (1) *All U-WPF right S-acts are flat.*
    (2) *All U-WPF right S-acts satisfy Condition (P_E).*
    (3) *All U-WPF right S-acts are WF.*
    (4) *All U-WPF right S-acts are PWF.*
    (5) *S is regular.*

Proof. Implications $(1) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(2) \Rightarrow (3)$. Since by ([2, Theorem 2.3]), Condition (P_E) implies weak flatness, then it is obvious.

$(4) \Rightarrow (5)$. Let $s \in S$. If $sS = S$, then it is obvious that s is regular.

Thus we suppose that $sS \neq S$, and that $A_S = S \coprod^{sS} S$. By Theorem 2.2, A_S is *U-WPF* and so by the assumption A_S is principally weakly flat. Thus by ([4, III, 12.19]), sS is left stabilizing and so there exists $l \in sS$ such that $s = ls$. Hence there exists $x \in S$ such that $l = sx$ and so $s = ls = sxs$ that is, s is regular.

$(5) \Rightarrow (1)$. Suppose A is a *U-WPF* right S-act. Then there exists a family $\{B_i \,/\, i \in I\}$ of subacts of A such that B_i is weakly pullback flat and $A = \bigcup_{i \in I} B_i$. We show that A is flat. Thus we suppose that for the left S-act ${}_S M$, $a \otimes m = a' \otimes m'$ in $A_S \otimes_S M$ for $a, a' \in A_S$ and $m, m' \in {}_S M$. We show that the same equality holds also in $A_S \otimes_S (Sm \cup Sm')$. Since $a \otimes m = a' \otimes m'$ in $A_S \otimes_S M$, then we have a tossing

$$s_1 m_1 = m$$
$$as_1 = a_1 t_1$$
$$s_2 m_2 = t_1 m_1$$
$$a_1 s_2 = a_2 t_2$$
$$s_3 m_3 = t_2 m_2$$
$$\dots$$
$$a_{k-1} s_k = a' t_k$$
$$\dots$$
$$m' = t_k m_k .$$

of length k, where $s_1, \dots, s_k, t_1, \dots t_k \in S$, $a_1, \dots, a_{k-1} \in A_S$, $m_1, \dots, m_k \in {}_S M$.

If $k = 1$ then we have

$$s_1 m_1 = m$$
$$as_1 = a' t_1$$
$$m' = t_1 m_1 .$$

Since S is regular the equality $as_1 = a' t_1$ implies that $a' t_1 = a' t_1 s_1' s_1$ for $s_1' \in V(s_1)$. Since $A = \bigcup_{i \in I} B_i$, then there exists $i_0 \in I$ such that, $a' \in B_{i_0}$. Since B_{i_0} is *WPF*, then B_{i_0} satisfies Condition (*P*) and so there exist $a'' \in A_S$ and $u, v \in S$ such that $a' = a'' u = a'' v$ and $ut_1 = vt_1 s_1' s_1$ From the last equality we obtain $um' = ut_1 m_1 = vt_1 s_1' s_1 m_1 = vt_1 s_1' m$. Since $m = s_1 m_1$, then $s_1 s_1' m = m$ and so we get

$$a \otimes m = a \otimes s_1 s_1' m = as_1 \otimes s_1' m = a' t_1 \otimes s_1' m =$$
$$a'' ut_1 \otimes s_1' m = a'' \otimes ut_1 s_1' m = a'' \otimes vt_1 s_1' s_1 s_1' m =$$
$$a'' \otimes vt_1 s_1' m = a'' \otimes um' = a'' u \otimes m' = a' \otimes m'$$

in $A_S \otimes_S (Sm \cup Sm')$.

Suppose that $k \geq 2$ and that the required equality holds for tossing of length less than k. From $as_1 = a_1 t_1$ we obtain equalities $a_1 t_1 = a_1 t_1 s_1' s_1$ for $s_1' \in V(s_1)$ and $as_1 = as_1 t_1' t_1$ for $t_1' \in V(t_1)$. Since $A = \cup_{i \in I} B_i$, then there exist $i_1, i_2 \in I$ such that $a_1 \in B_{i_1}$ and $a \in B_{i_2}$. Since B_{i_1} and B_{i_2} are *WPF*, then B_{i_1} and B_{i_2} satisfy Condition (*P*) and so there exist $a_1'' \in B_{i_1}$, $a_2'' \in B_{i_2}$ and $u_1, u_2, v_1, v_2 \in S$ such that $a_1 = a_1'' u_1 = a_1'' u_2$, $u_1 t_1 = u_2 t_1 s_1' s_1$ and $a = a_2'' v_1 = a_2'' v_2$, $v_1 s_1 = v_2 s_1 t_1' t_1$. Therefore, we have the following tossings

$$v_1 s_1 m_1 = v_1 m$$
$$a_2'' v_1 s_1 = a_1'' u_1 t_1$$
$$u_1 s_2 m_2 = u_1 t_1 m_1$$

of length 1 and

$$u_1 s_2 m_2 = u_1 t_1 m_1$$
$$a_1'' u_1 s_2 = a_2 t_2$$
$$s_3 m_3 = t_2 m_2$$
$$\dots$$
$$a_{k-1} s_k = a' t_k$$
$$m' = t_k m_k .$$

of length $k-1$.

From the tossing of length 1, we have $a_2'' \otimes v_1 m = a_1'' \otimes u_1 s_2 m_2$ in $A_S \otimes_S M$. By inductive hypothesis we have $a_2'' \otimes v_1 m = a_1'' \otimes u_1 s_2 m_2$ in $A_S \otimes_S (S v_1 m \cup S u_1 s_2 m_2)$. Since $u_1 s_2 m_2 = u_1 t_1 m_1 = u_2 t_1 s_1' s_1 m_1 = u_2 t_1 s_1' m \in Sm$, then we have $a_2'' \otimes v_1 m = a_1'' \otimes u_1 s_2 m_2$ in $A_S \otimes_S (Sm \cup Sm')$.

Also from tossing of the length $k-1$, we have $a_1'' \otimes u_1 t_1 m_1 = a' \otimes m'$ in $A_S \otimes_S M$. By inductive hypothesis we have $a_1'' \otimes u_1 t_1 m_1 = a' \otimes m'$ in $A_S \otimes_S (S u_1 t_1 m_1 \cup Sm')$. Since $u_1 t_1 m_1 = u_2 t_1 s_1' m \in Sm$, then $a_1'' \otimes u_1 t_1 m_1 = a' \otimes m'$ in $A_S \otimes_S (Sm \cup Sm')$. Thus we have $a \otimes m = a_2'' v_1 \otimes m = a_2'' \otimes v_1 m = a_1'' \otimes u_1 s_2 m_2 = a_1'' \otimes u_1 t_1 m_1 = a' \otimes m'$ in $A_S \otimes_S (Sm \cup Sm')$.

$(1) \Rightarrow (2)$. Since flatness implies principal weak flatness, then by the assumption *U-WPF* implies principal weak flatness and so by the proof of $(4) \Rightarrow (5)$, S is regular. Thus S is left *PP* and so by ([2, Theorem 2.5]), weak flatness and Condition (P_E) coincide and so we are done. □

Theorem 3.3. *For any monoid S the following statements are equivalent:*

(1) *All U-WPF right S-acts are WPF.*
(2) *All U-WPF right S-acts satisfy Condition (P).*
(3) *All U-WPF right S-acts are WKF.*
(4) *All U-WPF right S-acts are PWKF.*
(5) *All U-WPF right S-acts are TKF.*
(6) *All U-WPF right S-acts are (WP).*
(7) *All U-WPF right S-acts are (PWP).*
(8) *S is a group.*

Proof. Implications $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (7)$ and $(1) \Rightarrow (2) \Rightarrow (6) \Rightarrow (7)$ are obvious.

$(7) \Rightarrow (8)$. Suppose that for $s \in S$, $sS \neq S$ and that $A_S = S \coprod^{sS} S$. Then by Theorem 2.2, A_S is a *U-WPF* right S-act and so by the assumption A_S is *(PWP)*. Thus $s = (1, x)s = (1, y)s$ implies that there exist $a \in A_S$ and $u, v \in S$ such that $(1, x) = au$, $(1, y) = av$ and $us = vs$. Equality $(1, x) = au$ implies the existence of $t \in S \setminus sS$ such that $a = (t, x)$. Similarly, $a = (t', y)$ for some $t' \in S \setminus sS$ and so we have a contradiction. Hence for every $s \in S$, $sS = S$ and so S is a group.

$(8) \Rightarrow (1)$. Since S is a group, then by ([1, Proposition 9]), every right S-act is weakly pullback flat and so

every *U-WPF* is weakly pullback flat. □

Corollary 3.4. *For any monoid S the following statements are equivalent:*

(1) *All U-WPF right S-acts are free.*
(2) *All U-WPF right S-acts are projective generators.*
(3) *All U-WPF right S-acts are projective.*
(4) *All U-WPF right S-acts are strongly flat.*
(5) *$S = \{1\}$.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. Since every *U-WPF* right S-act is strongly flat, then every *U-WPF* right S-act is *WPF* and so by Theorem 3.3, S is a group. Thus by ([1, Proposition 9]), every right S-act is *WPF* and so by Theorem 2.2, every right S-act is *U-WPF*. Thus by the assumption every right S-act is strongly flat and so by ([4, IV, 10.5]), $S = \{1\}$.

$(5) \Rightarrow (1)$. Since $S = \{1\}$, then all right S-acts are free and so all *U-WPF* right S-acts are free as required. □

Note that *U-WPF* does not imply strong faithfulness, since every monoid S as a right S-act is *U-WPF*, but S is not strongly faithful, otherwise S is left cancellative which is not true in general. Now see the following theorem.

Theorem 3.5. *For any monoid S the following statements are equivalent:*

(1) *All U-WPF right S-acts are strongly faithful.*
(2) *All U-WPF right S-acts satisfy Condition (E) and S is left cancellative.*

Proof. $(1) \Rightarrow (2)$. Since S_S is *U-WPF*, then by the assumption S_S is strongly faithful and so S is left cancellative. Also it is obvious that every strongly faithful right S-act satisfies Condition (E) and so by the assumption every *U-WPF* right S-act satisfies Condition (E) .

$(2) \Rightarrow (1)$. Suppose that A is a *U-WPF* right S-act and $as = at$ for $a \in A$ and $s, t \in S$. Since by the assumption A satisfies Condition (E) , then there exist $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = ut$. But S is left cancellative and so $s = t$, hence A is strongly faithful. □

Note that the above theorem is also true, when right acts in general replaced by finitely generated or cyclic right acts.

Lemma 3.6. *Let S be a monoid and A a right S-act. If A is regular, then for all $a \in A$ and $s, t \in S$, $as = at$ implies that there exists $u \in S$ such that $a = au$ and $us = ut$.*

Proof. Suppose that A is a regular right S-act. Then by

([4, III, 19.2]), for every $a \in A$ there exists $e \in E(S)$ such that $\ker \lambda_a = \ker \lambda_e$ and so $aS \cong eS$. By ([4, III, 17.8]), eS is projective and so eS satisfies Condition (E). Thus aS satisfies Condition (E) and so $as = at$ implies that there exist $w_1, w_2 \in S$ such that $a = (aw_1)w_2$ and $w_2 s = w_2 t$. Thus $a = a(w_1 w_2)$ and $(w_1 w_2)s = (w_1 w_2)t$. If $w_1 w_2 = u$, then $a = au$ and $us = ut$ as required. □

Now from Lemma 3.6, we have

Corollary 3.7. *Let S be a monoid and A a right S-act. If A is regular, then A satisfies Condition (E).*

Lemma 3.8. *Let S be a right PP monoid and A a right S-act. If A satisfies Condition (E), then for every $a \in A$, aS satisfies Condition (E).*
Proof. Suppose that A satisfies Condition (E) and let $as = at$ for $a \in A$, $s, t \in S$. Then there exist $a' \in A$, $u \in S$ such that $a = a'u$, $us = ut$. Since S is right *PP*, then there exists $e \in E(S)$ such that $\ker \lambda_e = \ker \lambda_u$. Thus $es = et,\ u = ue$ and so $a = ae$. Hence aS satisfies Condition (E) as required. □

Note that *U-WPF* does not imply regularity, for if $S = \{0, 1, x\}$ with $x^2 = 0$, then S is not right *PP* and so by ([4, III, 19.3]), and ([4, IV, 11.15]), S_S is not regular, but it is obvious that S_S is a *U-WPF* cyclic right act. Now it is natural to ask for monoids S over which *every U-WPF* right S-act is regular.

Theorem 3.9. *Let S be a monoid. Then every U-WPF right S-act is regular if and only if S is right PP, it satisfies Condition (K) and every WPF right S-act is strongly flat.*
Proof. Since S_S is *U-WPF*, then by the assumption S_S is regular and so by ([4, III, 19.3]), all principal right ideal of S are projective. Thus by ([4, IV, 11.15]), S is right *PP*. Also by the assumption all strongly flat cyclic right *S-act* are regular. Thus by ([4, III, 19.3]), all strongly flat cyclic right *S*-acts are projective and so by ([4, IV, 11.2]), S satisfies Condition (K). Since by Corollary 3.7, every regular right *S*-act satisfies Condition (*E*), then by the assumption and Theorem 2.2, every *WPF* right *S*-act satisfies Condition (*E*) and so every *WPF* right *S*-act is strongly flat.

Conversely, suppose that A is a *U-WPF* right S-act. Then there exists a family $\{B_i \mid i \in I\}$ of subacts of A such that $A = \cup_{i \in I} B_i$ and B_i, $i \in I$ is *WPF*.

Thus by the assumption every B_i is *SF* and so satisfies Condition (E). Now let $a \in A$. Then there exists $i_0 \in I$ such that $a \in B_{i_0}$. Since S is right *PP* and B_{i_0} satisfies condition (E), then by Lemma 3.8, aS satisfies Condition (E) and so aS is *SF*. Since S satisfies Condition (K), then by ([4, IV, 11.2]), aS is projective and so by ([4, III, 19.3]), A is regular as required. □

Note that if $S = (N, .)$, where N is the set of natural numbers, then N is left cancellative. It can easily be seen that $A_S = N \coprod^{2N} N$ is strongly faithful. Since the right ideal $2N$ of N is not left stabilizing, then by ([4, III, 12.19]), A_S is not *PWF* and so A_S is not *WPF*. Hence strong faithfulness does not imply weak pullback flatness in general, but by the following theorem it can be seen that strong faithfulness implies *U-WPF*.

Theorem 3.10. *Let S be a monoid and A a right S-act. If A is strongly faithful, then A is U-WPF.*
Proof. Suppose that $a \in A$ and let $\psi_a : aS \to S$ be such that $as \mapsto s$. Since A is strongly faithful, then ψ_a is well-defined and so it is an isomorphism that is, $aS \cong S$ for every $a \in A$. Thus all cyclic subacts of A are *WPF*. Since $A = \cup_{a \in A} aS$, then A is *U-WPF*. □

Theorem 3.11. *Let S be a monoid and A a right S-act. If A is regular, then A is U-WPF.*
Proof. Suppose that A is a regular right *S*-act. Then by ([4, III, 19.3]), all cyclic subacts of A are projective and so all cyclic subacts of A are *WPF*. Since $A = \bigcup_{a \in A} aS$, then A is *U-WPF* as required. □

By the argument after Theorem 2.2, we saw that Condition (*P*) does not imply *U-WPF*, thus torsion freeness does not imply *U-WPF* either. Now we show that monoids for which all torsion free right acts are *U-WPF* are the same as those for which all torsion free cyclic right acts are *WPF*.

Theorem 3.12. *For any monoid S the following statements are equivalent:*
   (1) *All torsion free right S-acts are U-WPF.*
   (2) *All torsion free finitely generated right S-acts are U-WPF.*
   (3) *All torsion free cyclic right S-acts are WPF.*
Proof. (1) $\Rightarrow$ (2). It is obvious.
   (2) $\Rightarrow$ (3). Since by Theorem 2.2, for cyclic acts *WPF* and *U-WPF* coincide, then we are done.

(3) $\Rightarrow$ (1). Since all subacts of every torsion free right act is torsion free, then for a torsion free right S-act A, aS, $a \in A$ is also torsion free. Thus by the assumption for every $a \in A$, aS is *WPF*. Now since $A = \cup_{a \in A} aS$, then A is *U-WPF* as required. $\square$

Now we give a classification of monoids by *U-WPF* of right acts.

Theorem 3.13. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are U-WPF.*
(2) *All finitely generated right S-acts are U-WPF.*
(3) *All cyclic right S-acts are U-WPF.*
(4) *S is a group or a group with a zero adjoined.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). By Theorem 2.2, and ([1, Proposition 25]), it is obvious.

(4) $\Rightarrow$ (1). Let A be a right S-act. Since by ([1, Proposition 25]), all cyclic right S-acts are *WPF*, then

for every $a \in A$, aS is *WPF*. Since $A = \cup_{a \in A} aS$, then A is *U-WPF* as required. $\square$

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