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Double Derivations, Higher Double Derivations and Automatic Continuity

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Abstract

Let $\mathcal{A}$ be a Banach algebra. Let δ, ε be linear mappings on $\mathcal{A}$. First we demonstrate a theorem concerning the continuity of double derivations; especially that all of (δ, ε)-double derivations are continuous on semi-simple Banach algebras, in certain case. Afterwards we define a new vocabulary called "(δ, ε)-**higher double derivation**" and present a relation between this subject and derivations and finally give some theorems and corollaries about the automatic continuity of this notion.

Keywords: Derivation; (σ, τ)-Derivation; (δ, ε)-Double derivation; Higher double derivation

Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping d on $\mathcal{A}$ is called a derivation if that satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. It is known that if d is a derivation, then

$$\mathrm{d}^{\mathrm{n}}(\mathrm{ab}) = \sum_{\mathrm{k}=0}^{\mathrm{k}=\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}} \mathrm{d}^{\mathrm{k}}(\mathrm{a}) \mathrm{d}^{\mathrm{n}-\mathrm{k}}(\mathrm{b})$$

for each $a, b \in \mathcal{A}$ and for all non-negative integers n. A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if

$$d_n(ab) = \sum_{k=0}^{k=n} d_k(a) d_{n-k}(b)$$

for all $a, b \in \mathcal{A}$ and each non-negative integer n. A typical example of a higher derivation is $\{\frac{\delta^n}{n!}\}$ where $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation. It seems that this example has been the first motivation to define higher derivations. Higher derivations were introduced by Hasse and Schmidt in [5]. In [8, 16], higher derivations were characterized in different aspects.

Now let σ, τ be two linear mappings on $\mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a (σ, τ)-derivation if it satisfies the generalized Leibniz rule $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for each $x, y \in \mathcal{A}$. By a σ-derivation we mean a (σ, σ)-derivation. Specially if $\delta : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $d = \delta\sigma$ is a σ-derivation. In [7], (σ, τ)-higher derivations were defined. A simple example of a (σ, τ)-higher derivation is $\{\frac{\delta^n}{n!}\}$ in which

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$\delta:\mathcal{A}\to\mathcal{A}$ is a (σ,τ)-derivation with the property that $\delta\sigma=\sigma\delta$ and $\delta\tau=\tau\delta$.

Let δ,ε be two ordinary derivations; we see that $d=\delta\varepsilon$ satisfies

$$d(ab)=d(a)b+ad(b)+\delta(a)\varepsilon(b)+\varepsilon(a)\delta(b)\qquad (a,b\in\mathcal{A}). \tag{1}$$

This can be assumed as a generalization of the notion of a derivation. Now suppose that $\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ are two linear mappings. A linear mapping $d:\mathcal{A}\to\mathcal{A}$ is said to be a (δ,ε)-double derivation if it satisfies (1). By a δ-double derivation we mean a (δ,δ)-double derivation. For example, it is clear that each σ-derivation, is a $(\sigma-I,d)$-double derivation. Moreover, every homomorphism $\varphi:\mathcal{A}\to\mathcal{A}$ is a $(\frac{\varphi}{2}-I,\varphi)$-double derivation. For more details concerning σ-derivations, (σ,τ)-derivations and double derivations see [6, 7, 8, 9, 10, 17, 18].

In 1958, I. Kaplansky stated a conjecture that any derivation of a C*-algebra would be automatically continuous [13]. In 1960, S. Sakai gave a affirmative solution to the conjecture by Kaplansky [20]; a simple proof of the theorem was shown by Kishimoto [14]. In 1968, B. E. Johnson and A. M. Sinclair proved that each derivation on semi-simple Banach algebra is automatically continuous [12]. In 1969, B. E. Johnson, proved that all of derivations on commutative Banach algebra are automatically continuous [11]. In [10], N. P. Jewell and A. M. Sinclair showed a very important theorem about the automatic continuity of derivations and in [9], N. P. Jewell proved some very significant theorems about the automatic continuity of module valued derivations and higher derivations. Also F. Gulick [4] and R. J. Loy [15] have shown that the automatic continuity of derivations on semi-simple Banach algebras can be extended to higher derivations. In [6], there is a result about the automatic continuity of double derivations on semi-simple Banach algebras.

In this paper, first we prove a theorem concerning the automatic continuity of double derivations; actually we extend N. P. Jewell and A. M. Sinclair Theorem [10] for double derivations. After that, we define a new theme called (δ,ε)-higher double derivation and present a relation between this subject and derivations and subsequently produce some theorems and corollaries about the continuity of (δ,ε)-higher double derivations based on this relation. At the end of the paper, again we state some theorems concerning the automatic continuity of higher double derivations.

Results

1. Double Derivation

Let $\mathcal{A}$ be an algebra and $\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ be two linear mappings. A linear mapping d on $\mathcal{A}$ is said to be a (δ,ε)-double derivation if

$$d(ab)=d(a)b+ad(b)+\delta(a)\varepsilon(b)+\varepsilon(a)\delta(b),\quad (a,b\in\mathcal{A}).$$

Prior to anything, we recall a few concepts that those were needed both in this section and the next section.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear operator $f:\mathcal{A}\to\mathcal{X}$ is said to be left (resp. right)-intertwining, if the map $g:\mathcal{A}\to\mathcal{X}$ defined by $g(b)=af(b)-f(ab)$ (resp. $g(a)=f(a)b-f(ab)$) is continuous. A linear operator $g:\mathcal{A}\to\mathcal{X}$ is said to be intertwining, if it is both left-intertwining and right-intertwining. For more about this fact, we refer the reader to [2].

Remark 1.1. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, $\mathfrak{A}_1$ be the Banach algebra unitization of $\mathcal{A}$ and $B(\mathfrak{A}_1,\mathcal{X})$ be the Banach space of all bounded linear operators from $\mathfrak{A}_1$ to $\mathcal{X}$.

i. Let $a\in\mathcal{A}$ and $T\in B(\mathfrak{A}_1,\mathcal{X})$ be arbitrary; we define

$$(aT)(w)=aT(w),(Ta)(w)=T(aw)$$

for all $w\in\mathfrak{A}_1$. Thus, with this definition, $B(\mathfrak{A}_1,\mathcal{X})$ is a Banach $\mathcal{A}$-bimodule.

ii. The map $U:B(\mathfrak{A}_1,\mathcal{X})\to\mathcal{X}$ by $U(T)=T(1)$ is continuous $\mathcal{A}$-bimodule homomorphism.

iii. Let $S:\mathcal{A}\to\mathcal{X}$ be a left-intertwining map. Then there exist a module valued derivation D from $\mathcal{A}$ to $B(\mathfrak{A}_1,\mathcal{X})$ so that $S=UoD$.

To see the facts of this Remark and their proof, we refer the reader to [3]. Similarly, the result will be obtained with right-intertwining in stead of left-intertwining by considering the module operations in (*i*), on the other side.

The following theorem is an extension of N. P. Jewell and A. M. Sinclair Theorem [10].

Theorem 1.1. Let $\mathcal{A}$ be a Banach algebra with the property that for each infinite dimensional closed bi-ideal K in $\mathcal{A}$ there is a sequence $\{a_n\}$ in $\mathcal{A}$ such that $(Ka_n a_{n-1}...a_1)^- \supset (Ka_{n+1}a_n...a_1)^-$ (strictly subset) for all positive integers n. Let $\mathcal{A}$ contains no non-zero finite dimensional nilpotent ideal. Let d be a (δ,ε)-double derivation on A with continuous δ,ε. Then d is automatically continuous.

Proof. Suppose $x \in S(d)$, separating space of d, and $a \in \mathcal{A}$ is arbitrary; then there exist a sequence $\{x_n\}$ in $\mathcal{A}$ such that $x_n \to 0$ and $d(x_n) \to x$. Since $d(ax_n) = d(a)x_n + ad(x_n) + \delta(a)\varepsilon(x_n) + \varepsilon(a)\delta(x_n)$ and since δ,ε are continuous, one can shows that $d(ax_n) \to ax$, while $ax_n \to 0$. Consequently, $S(d)$ is a closed bi-ideal. Consider $\mathcal{A}$ as a Banach $\mathcal{A}$-bimodule with its own product. Obviously, since δ,ε are continuous, the mapping d is left-intertwining. Then by [2, Theorem 5.2.24], $S(d)$ is separating ideal. Thus by the definition of separating ideal and the assumption, $S(d)$ is finite dimentional and also $rad(S(d))$, the radical of $S(d)$, is finite dimensional and from [2, Theorem 1.5.6] it is nilpotent and according to the hypothesis, $rad(S(d)) = 0$; i.e. $S(d)$ is semi-simple. If $S(d) \neq 0$, in view of [1, p. 135] it will has an identity element, e say. Then there is a sequence $\{y_n\}$ in $\mathcal{A}$ such that $y_n \to 0$ and $d(y_n) \to e$. Writing $d(ey_n) = ed(y_n) + d(y_n)e + \delta(e)\varepsilon(y_n) + \varepsilon(y_n)\delta(e)$, since δ,ε are continuous, since $ey_n \in S(d)$, since $S(d)$ is finite dimensional and since d is continuous on $S(d)$, we get $e = 0$. But it is a contradiction. It means that $S(d) = 0$.

For example, all (δ,ε)-double derivations with continuous δ,ε on semi-simple Banach algebras are automatically continuous; because in view of the proof of [10, Corollary 9], these spaces have all of the conditions of Theorem 1.1.

In the next section, we are going to define a new subject named "higher double derivation"; but first we introduce a notation. Let δ_1,δ_2 be two linear mappings on a space. We construct a family of linear mappings $\{\phi_{n,k}^{\delta_1,\delta_2}\}$, $(n \in \mathbb{N}, 0 \le k \le 2^n - 1)$, called the binary family for the ordered pair of (δ_1,δ_2), as follows: write k in base 2 with exactly n digits, and put δ_1,δ_2 in place of 1, 0 respectively; for example, if $n = 4$ then $6 = (0110)_2$, $10 = (1010)_2$, $\phi_{4,6}^{\delta_1,\delta_2} = \delta_2\delta_1\delta_1\delta_2 = \delta_2\delta_1^2\delta_1$ and $\phi_{4,10}^{\delta_1,\delta_2} = \delta_1\delta_2\delta_1\delta_2$. When no confusion can occur, we simply write $\phi_{n,k}$ instead of $\phi_{n,k}^{\delta_1,\delta_2}$. Also we define $\phi_{0,0}^{\delta_1,\delta_2} = I$, the identity mapping.

Lemma 1.1. Let m, n be natural numbers and $0 \le k, l \le 2^n - 1$. Then

i) $\delta_1\phi_{n,k} = \phi_{n+1,k+2^n}$;

ii) $\phi_{n,k}\delta_1 = \phi_{n+1,2k+1}$;

iii) $\delta_2\phi_{n,k} = \phi_{n+1,k}$;

iv) $\phi_{n,k}\delta_2 = \phi_{n+1,2k}$;

v) $\phi_{n,k}\phi_{m,l} = \phi_{m+n,\,l+\sum_{i=0}^{u} 2^{m+i}}$,

where u is the number of δ_1 `s in the mappings $\phi_{n,k}$.

Proof. The reader is referred to [6] and [17].

Let δ,ε be two linear mappings on a linear space $\mathcal{A}$. We define two linear mappings α,β on $\mathcal{A} \otimes \mathcal{A}$ as follows:

$$\alpha(a \otimes b) = d(a) \otimes b + a \otimes d(b)$$

$$\beta(a \otimes b) = \delta(a) \otimes \varepsilon(b) + \varepsilon(a) \otimes \delta(b)$$

for each $a,b \in \mathcal{A}$. Now by supposing that $\mathcal{A}$ is an algebra, consider the mapping $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defined by $\tau(a \otimes b) = ab$. If d be a (δ,ε)-double derivation on $\mathcal{A}$, by defining $\psi_{n,k} := \phi_{n,k}^{\alpha,\beta}$, and according to [6, Theorem 2.7], we have

$$d^n(ab) = d^n(\tau(a \otimes b)) = \tau\Big(\sum_{j=0}^{2^n-1} \psi_{n,j}(a \otimes b)\Big). \qquad (2)$$

Spacially if d commutes with δ,ε, by considering $\varphi_{n,k} := \phi_{n,k}^{\delta,\varepsilon}$, applying the relation (2) and using the induction on n, it can be routinely concluded that

$$d^n(ab) = \sum_{m=0}^{n} \sum_{i=0}^{2^m-1} \sum_{k=0}^{n-m} \binom{n}{m}\binom{n-m}{k} \varphi_{m,i}(d^k(a))\varphi_{m,2^m-1-i}(d^{n-m-k}(b)). \qquad (3)$$

The relation (3) is very important and we will use it in the next section for presenting a significant example.

2. Higher Double Derivation

In this section we are going to extend the concept higher derivations to the notion double derivations. Let $\mathcal{A}$ be an algebra and $\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ be two linear mappings. Also suppose that $\varphi_{n,k}:=\phi_{n,k}^{\delta,\varepsilon}$; that is, the same notation stated in the last section.

Definition 2.1. A sequence $\{d_n\}_{n=0}^{p}$ (p might be ∞) on $\mathcal{A}$ is called a "(δ,ε) **-higher double derivation**" of rank p if for each $a,b\in\mathcal{A}$ and each $n=0,1,2,...,p$

$$d_n(ab)=\sum_{m=0}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\frac{1}{m!}\varphi_{m,i}\left(d_k(a)\right)\varphi_{m,2^m-1-i}\left(d_{n-m-k}(b)\right).\quad(4)$$

Note that if $n=0$, then

$$\begin{aligned}d_0(ab)&=\sum\nolimits_{m=0}^{0}\sum\nolimits_{i=0}^{0}\sum\nolimits_{k=0}^{0}\frac{1}{m!}\varphi_{m,i}\left(d_k(a)\right)\varphi_{m,2^m-1-i}\left(d_{n-m-k}(b)\right)\\&=\frac{1}{0!}\varphi_{0,0}\left(d_0(a)\right)\varphi_{0,0}\left(d_0(b)\right)\\&=d_0(a)d_0(b)\end{aligned}$$

i.e. the mapping d_0 is a homomorphism. Also if $n=1$, then

$$\begin{aligned}d_1(ab)&=\sum_{m=0}^{1}\sum_{i=0}^{2^m-1}\sum_{k=0}^{1-m}\frac{1}{m!}\varphi_{m,i}\left(d_k(a)\right)\varphi_{m,2^m-1-i}\left(d_{1-m-k}(b)\right)\\&=\frac{1}{0!}[\varphi_{0,0}\left(d_0(a)\right)\varphi_{0,0}\left(d_1(b)\right)+\varphi_{0,0}\left(d_1(a)\right)\varphi_{0,0}\left(d_0(b)\right)]\\&+\frac{1}{1!}[\varphi_{1,0}\left(d_0(a)\right)\varphi_{1,1}\left(d_0(b)\right)+\varphi_{1,1}\left(d_0(a)\right)\varphi_{1,0}\left(d_1(b)\right)]\\&=d_0(a)d_1(b)+d_1(a)d_0(b)+\varepsilon\left(d_0(a)\right)\delta\left(d_0(b)\right)\\&\quad+\delta\left(d_0(a)\right)\varepsilon(d_0(b)).\end{aligned}$$

It is clear that if $d_0=I$, then the mapping d_1 is a (δ,ε) -double derivation.

Also if $\delta=0$ or $\varepsilon=0$, then $\{d_n\}$ is a higher derivation; because, on the one hand, according to the relation (4), we have

$$d_n(ab)=\sum_{k=0}^{n}d_k(a)d_{n-k}(b)+\sum_{m=1}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\frac{1}{m!}\varphi_{m,i}\left(d_k(a)\right)\varphi_{m,2^m-1-i}\left(d_{n-m-k}(b)\right).$$

On the other hand, it is plain that if $\delta=0$ or $\varepsilon=0$, then for each $m>0$, $\varphi_{m,i}=0$ or $\varphi_{m,2^m-1-i}=0$ (or the both). Consequently, in this case,

$$d_n(ab)=\sum_{k=0}^{n}d_k(a)d_{n-k}(b).$$

It means that if $\delta=0$ or $\varepsilon=0$, then $\{d_n\}$ is a higher derivation.

A higher derivation or higher double derivation of rank p is said to be continuous if d_n is continuous for each $n=0,1,2,...,p$. It is said to be onto if d_0 is onto.

As we know, a standard example of a higher derivation is $\{\frac{\delta^n}{n!}\}$ where $\delta:\mathcal{A}\to\mathcal{A}$ is a derivation. Also an ordinary example of a (σ,τ) -higher derivation is $\{\frac{\delta^n}{n!}\}$ in which $\delta:\mathcal{A}\to\mathcal{A}$ is a (σ,τ) -derivation with the property that $\delta\sigma=\sigma\delta$ and $\delta\tau=\tau\delta$ [7]. Now we want to give an analogous example concerning the concept higher double derivation by helping of the relation (3) which stated in the late of the last section.

Example 2.1. Let D be a (δ,ε)-double derivation commutes with δ,ε. For $n\in\mathbb{N}$, define $d_n:=\frac{D^n}{n!}$. Since D is a (δ,ε) -double derivation, using the relation (3), we can write

$$\begin{aligned}d_n(ab)&=\frac{D^n}{n!}(ab)\\&=\frac{1}{n!}\sum_{m=0}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\binom{n}{m}\binom{n-m}{k}\varphi_{m,i}\left(D^k(a)\right)\varphi_{m,2^m-1-i}\left(D^{n-m-k}(b)\right)\\&=\sum_{m=0}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\frac{1}{m!}\varphi_{m,i}\left(\frac{D^k}{k!}(a)\right)\varphi_{m,2^m-1-i}\left(\frac{D^{n-m-k}}{(n-m-k)!}(b)\right)\end{aligned}$$

$$= \sum_{m=0}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\frac{1}{m!}\varphi_{m,i}\left(d_k\left(a\right)\right)\varphi_{m,2^m-1-i}\left(d_{n-m-k}\left(b\right)\right).$$

It means that $\{d_n\}$ is a (δ,ε)-higher double derivation.

Theorem 2.1. Let $\{d_n\}$ be a (δ,ε)-higher double derivation on a Banach algebra $\mathcal{A}$ with continuous δ,ε,d_0. Also let every module valued derivation from $\mathcal{A}$ be continuous. Then $\{d_n\}$ is continuous.

Proof. Because d_0 is homomorphism and continuous, with the following definition, $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule:

$$a.x = d_0(a)x \text{ and } x.a = xd_0(a), \ (a,x \in \mathcal{A}).$$

Clearly, d_1 is a left- intertwining map, and accordingly from Remark 1.1, there exist a continuous $\mathcal{A}$-bimodule homomorphism $U : B(\mathfrak{A}_1,\mathcal{A}) \to \mathcal{A}$ and a module valued derivation D_1 from $\mathcal{A}$ to $B(\mathfrak{A}_1,\mathcal{A})$ so that $d_1 = UoD_1$. Thus d_1 is continuous. Now by induction, suppose that $d_2,d_3,..d_{n-1}$, are continuous. Since

$$d_n(ab) = d_0(a)d_n(b) + \sum\nolimits_{j=1}^{n} d_j(a)d_{n-j}(b)$$

$$+\sum_{m=1}^{n}\sum_{i=0}^{2^m-1}\sum_{k=0}^{n-m}\frac{1}{m!}\varphi_{m,i}\left(d_k\left(a\right)\right)\varphi_{m,2^m-1-i}\left(d_{n-m-k}\left(b\right)\right)$$

and since $d_0,d_1,d_2,..d_{n-1},\delta,\varepsilon$ are continuous, it follows that $d_n(ab) - d_0(a)d_n(b)$ is continuous; it means that $d_n(ab) - ad_n(b)$ is continuous. Thus d_n is left-intertwining and consequently, similar to the proof used for d_1, the linear mapping d_n is continuous.

J. R. Ringrose in [19], proved that all module valued derivations from a C^*-algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is continuous. So by Theorem 2.1, all of (δ,ε)-higher double derivations on C*-algebras with continuous δ,ε,d_0 are continuous.

Corollary 2.1. Let $\{d_n\}$ be a (δ,ε)-higher double derivation with continuous δ,ε,d_0 on a Banach algebra $\mathcal{A}$ which satisfies the following conditions:

i. If K is a closed ideal of infinite codimension in $\mathcal{A}$, then there exist sequences $\{a_n\}$, $\{b_n\}$ in $\mathcal{A}$ satisfying $a_nb_1\dots b_{n-1} \notin K$ and $a_nb_1\dots b_n \in K$ for all $n \geq 2$.

ii. Every closed ideal having finite codimension in $\mathcal{A}$ has a bounded left (or right) approximate identity.

Then $\{d_n\}$ is automatically continuous.

Proof. From [9, Theorem 2], every module valued derivation from $\mathcal{A}$ into each Banach $\mathcal{A}$-bimodule is continuous. Thus from Theorem 2.1, $\{d_n\}$ is continuous.

For example, from [9], all of C*-algebras and the group algebra $L^1(G)$ in which G is locally compact abelian group satisfy the hypotheses of Corollary 2.1. Then all of (δ,ε)-higher double derivations with continuous δ,ε,d_0 on these spaces are continuous

It is a well known result due to B. E. Johnson and A. M. Sinclair [12] that every derivation on a semi-simple Banach algebra is continuous. Here we give a similar result for higher double derivations. The next theorem is an extend of [6, Theorem 3.3] for higher double derivations. The proof is almost similar to the proof of [6, Theorem 3.3].

Theorem 2.2. Let $\{d_n\}$ be a (δ,ε)-higher double derivation on a semi-simple Banach algebra $\mathcal{A}$ with continuous δ,ε and that $d_0 = I$, identity mapping. Then $\{d_n\}$ is continuous.

The following theorem is an extend of Theorem 2.2.

Theorem 2.3. Let $\mathcal{A}$ be as in Theorem 1.1. Let $\{d_n\}$ be a (δ,ε)-higher double derivation on $\mathcal{A}$ with continuous δ,ε,d_0 and that the restriction of d_0 to $S(d_n)$ is identity, for all n. Then $\{d_n\}$ is continuous.

Proof. By the hypothesis, d_0 is continuous. By induction on n we show that d_n is continuous. Similar to what stated in the proof of Theorem 2.1, the mapping d_n is left-intertwining. Thus from [2, Theorem 5.2.24], $S(d_n)$ is a separating ideal and consequently on account of the definition of separating ideal and the assumption, $S(d_n)$ is finite dimentional, therefore $rad(S(d_n))$ is finite dimensional and from [2, Theorem 1.5.6] is nilpotent and according to the hypothesis, $rad(S(d)) = 0$; i.e. $S(d)$ is semi-simple. We claim that $S(d_n) = 0$; assume, on the contrary, $S(d_n) \neq 0$. Thus from [1, p. 135], $S(d_n)$ has to has an identity element, e say. So there is a sequence $\{x_m\}$ in $\mathcal{A}$ such that $x_m \to 0$ and $d_n(x_m) \to e$. By applying the definition of (δ,ε)-higher double derivation, and that $\delta,\varepsilon,d_0,\cdots,d_{n-1}$ are continuous, and since d_0 is identity on $S(d_n)$, similar working to that in the proof of Theorem 1.1, we get $e=0$; but this is a contradiction. It means $S(d_n)=0$; i.e. d_n is continuous.

Recall that if $\mathcal{A}$ is a semi-simple Banach algebra, the above theorem holds.

Theorem 2.4. Let $\mathcal{A}$ be as in Theorem 1.1. Let $\{d_n\}$ be a (δ,ε)-higher double derivation from $\mathcal{A}$ onto $\mathcal{A}$ with continuous δ,ε that $\mathrm{kerld_0} \subset \mathrm{kerld_n}$ and also that for each n at least one of the linear mappings δ or ε is equal to zero on $S(d_n)$. Then $\{d_n\}$ is continuous.

Proof. Since d_0 is onto and homomorphism, on account of [10, Theorem 2], it is continuous. Consequently, similar to the proof of Theorem 2.1, $S(d_n)$ is semi-simple, finite dimentional and also has to has an identity element. Therefore, similar to the proof used in [9, Theorem 7], it follows that $S(d_n)=0$.

Conclusion

In certain case, all of double derivations on semi-simple Banach algebras are continuous. By the way, there is a relation between module valued derivations and higher double derivations on Banach algebras and based on this relation, all of higher double derivations on C*-algebras and also on $L^1(G)$ are continuous, in spacial conditions. Meanwhile, under more spacial conditions, higher double derivations on semi-simple Banach algebras are continuous.

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