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An Approximate Method for System of Nonlinear Volterra Integro-Differential Equations with Variable Coefficients

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Abstract

In this paper, we apply the differential transform (DT) method for finding approximate solution of the system of linear and nonlinear Volterra integro-differential equations with variable coefficients, especially of higher order. We also obtain an error bound for the approximate solution. Since, in this method the coefficients of Taylor series expansion of solution is obtained by a recurrence relation, thus we can use arbitrary number of Taylor series terms to obtain solutions with desired accuracy. Here we give some preliminary results of the differential transform and show that the DT method can be easily applied to a wide class of linear and nonlinear systems. Finally, the accuracy and simplicity of this method will be verified by solving some examples.

Keywords: Volterra integro-differential equations; Taylor series; Differential transform

Introduction

The DT method is an iterative procedure to obtain Taylor series solutions of differential and integral equations (see [1]). This method was first introduced by Zhou [2] in 1986 for solving linear and nonlinear initial value problems in electric analysis (see also [3]).

Up to now, the differential transform method has been developed for solving various types of differential and integral equations. In [4,5], Ayaz presented extension of DT for solving system of differential equations and differential-algebraic equations. In [3] and [6] this method applied to partial differential equations and in [7] and [8] to the one dimensional Voltrra integral and integro-differential equations. Also in [9] the DT method has been developed for solving two dimensional Volterra integral equations.

On the other hand, the Volterra integral and integro-differential equation systems (such as system of model describing biological species living together) have many interesting applications in applied sciences (for example see [10,11]).

Although many methods available for solving the system of integral and integro-differential equations (for example see [12,15]), but DT method is simple and need not much computational works and we can solve systems by high accuracy. Recently, Biazar and Eslami developed the DT method for systems of Volterra integral equations of the second kind [16].

The subject of presented paper is to apply the DT method for solving system of linear and nonlinear Volterra integro-differential equations of the second

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kind with variable coefficients. Here we consider a system of the form

$$\begin{cases} \sum_{j=1}^{m} a_{1j}(x) D_{1j}^{(1)} y_j(x) \\ \quad -\lambda_1 \int_{x_0}^{x} k_1(x,t) F_1\left(D_{11}^{(2)} y_1(t), \cdots, D_{1m}^{(2)} y_m(t)\right) dt \\ \quad = f_1(x) \\ \sum_{j=1}^{m} a_{2j}(x) D_{2j}^{(1)} y_j(x) \\ \quad -\lambda_2 \int_{x_0}^{x} k_2(x,t) F_2\left(D_{21}^{(2)} y_1(t), \cdots, D_{2m}^{(2)} y_m(t)\right) dt \\ \quad = f_2(x) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \sum_{j=1}^{m} a_{mj}(x) D_{mj}^{(1)} y_j(x) \\ \quad -\lambda_m \int_{x_0}^{x} k_m(x,t) F_m\left(D_{m1}^{(2)} y_1(t), \cdots, D_{mm}^{(2)} y_m(t)\right) dt \\ \quad = f_m(x) \end{cases} \tag{1.1}$$

with the supplementary conditions

$$y_j^{(i)}(x_0) = c_{ij}, \ j = 1, \ldots, m, \ i = 0, 1, \ldots, m_j - 1, \tag{1.2}$$

where $m_j = max\{\alpha_{1j}^{(1)}, \ldots, \alpha_{mj}^{(1)}, \alpha_{1j}^{(2)}, \ldots, \alpha_{mj}^{(2)}\}$ and $\alpha_{ij}^{(k)}$ denotes the order of differential operator $D_{ij}^{(k)}$ for $i = 1, 2$. We also assume that $x \in [x_0, b]$, where $x_0, b \in R$ are finite. Finally we assume that the problem (1.1)-(1.2) has a unique solution.

Materials and Methods

2.1. Some Results of the Differential Transform

The basic definition of DT and fundamental theorems about it can be found in [1-8], however for convenience in this section we review the DT. Differential transform of order n for the function $f(x)$ at x_0 is defined as (see [8])

$$F(n) = \frac{1}{n!}\left[\frac{d^n f(x)}{dx^n}\right]_{x=x_0}, \tag{2.1}$$

and its inverse transform is defined as

$$f(x) = \sum_{n=0}^{\infty} F(n)(x - x_0)^n. \tag{2.2}$$

The relations (2.1) and (2.2) imply that

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{d^n f(x)}{dx^n}\right]_{x=x_0} (x - x_0)^n, \tag{2.3}$$

which is the Taylor series of function $f(x)$.

In the following theorem, we summarize some fundamental properties of the differential transform (see [8]).

Theorem 2.1 Let $F(n)$, $U(n)$ and $V(n)$ be the differential transforms of the functions $f(x)$, $u(x)$ and $v(x)$ at $x_0 = 0$ respectively, then we have

a. If $f(x) = x^k$, then

$$F(n) = \delta_{n,k}.$$

b. If $f(x) = sin(ax + b)$, then

$$F(n) = \frac{a^n}{n!} sin\left(\frac{n\pi}{2} + b\right).$$

c. If $f(x) = cos(ax + b)$ then

$$F(n) = \frac{a^n}{n!} cos\left(\frac{n\pi}{2} + b\right).$$

d. If $f(x) = e^{\alpha x}$, then

$$F(n) = \frac{a^n}{n!}.$$

e. If $f(x) = u(x) \pm v(x)$, then

$$F(n) = U(n) \pm V(n).$$

f. If $f(x) = au(x)$, then

$$F(n) = aU(n).$$

g. If $f(x) = u(x)v(x)$, then

$$F(n) = \sum_{k=0}^{n} U(k)V(n-k). \qquad \square$$

We also recall the following theorems respectively from [3] and [8] to apply the DT for the differential and integral parts of (1.1).

Theorem 2.2 Let $F(n)$, $U(n)$ and $V(n)$ be the differential transforms of the functions $f(x)$, $u(x)$ and $v(x)$ in $x_0 = 0$ respectively, then we have

a. If $f(x) = \frac{d^r u(x)}{dx^r}$, $r = 1, 2, \cdots$ then

$$F(n) = (n+1)(n+2)\cdots(n+r)U(n+r)$$

b. If $f(x)=\frac{du(x)}{dx}\frac{dv(x)}{dx}$, then

$$F(n)=\sum_{k=0}^{n}(k+1)(n-k+1)U(k+1)V(n-k+1). \quad \square$$

Theorem 2.3 Assume that $U(n)$, $V(n)$, $H(n)$ and $G(n)$ are the differential transforms of the functions $u(x)$, $v(x)$, $h(x)$ and $g(x)$ respectively, then we have

a. If $g(x)=\int_{x_0}^{x}u(t)v(t)dt$, then

$$G(n)=\frac{1}{n}\sum_{k=0}^{n-1}U(k)V(n-k-1), \qquad n=1,2,\ldots, \quad G(0)=0. \tag{2.4}$$

b. If $g(x)=h(x)\int_{x_0}^{x}u(t)dt$, then

$$G(n)=\sum_{k=0}^{n-1}H(k)\frac{U(n-k-1)}{(n-k)}, \qquad n=1,2,\ldots, \quad G(0)=0. \ \square \tag{2.5}$$

2.2. Error Bound

In this section, we obtain an error bound for the approximate solution. To this end, we define the error function of the i-th component of $y(x)$ as

$$e_i(x)=y_i(x)-y_{iN}(x), \tag{3.1}$$

where $y_i(x)$ and $y_{iN}(x)$ are the i-th components of the exact and approximate solutions of system (1.1), respectively.

Then the error bound is given by the following theorem.

Theorem 3.1 For the error function $e_i(x)$ defined by (3.1) we have

$$|e_i(x)|=|y_i(x)-y_{iN}(x)|\le\frac{M_i\,|x|^{N+1}}{(N+1)!}, \qquad i=1,2,...,m, \tag{3.2}$$

where M_i is a nonnegative constants such that

$$|y_i^{(N+1)}(x)|\le M_i, \quad i=1,2,...,m \tag{3.3}$$

proof. From Taylor expansion of $y_i(x)$ around $x=0$, we have

$$y_i(x)=\sum_{k=0}^{N}\frac{y_i^{(k)}(0)}{k!}x^k+\frac{y_i^{(N+1)}(\xi_i)}{(N+1)!}x^{N+1}, \qquad i=1,2,...,m$$

where $\xi_i\in(0,x)$, hence

$$y_i(x)-y_{iN}(x)=\frac{y_i^{(N+1)}(\xi_i)}{(N+1)!}x^{N+1}, \quad i=1,2,...,m$$

and using (3.3) completes the proof. $\square$

Corollary 3.2 With the conditions of theorem 3.1 we have

$$\lim_{N\to\infty}y_{iN}(x)=y_i(x), \quad i=1,2,...,m \ \square$$

Corollary 3.3 From the above theorem and structure of the differential transform method, it is clear that if the solution $y_i(x), i=1,2,...,m$ of equation (1.1) is a polynomial of degree n, then every approximate solution obtained by differential transform method of degree N with $N\ge n$ will be exact, because in this case we have $M_i=0,\ i=1,2,...,m$. $\square$

Results

In this section, we give some examples to clarify accuracy of the presented method. For solving the problem (1.1) – (1.2) by conditions (1.2) by DT method, we use theorems 2.1, 2.2 and 2.3 to obtain m recurrence relations for $Y_1(n),Y_2(n),\cdots Y_m(n)$ (differential transforms of the unknown functions $y_1(x),y_2(x),\cdots,y_m(x)$ respectively) and solve them to obtain the unknown values $Y_1(n),Y_2(n),\cdots Y_m(n)$. Finally we use the truncated form

$$y_i(x)\approx\sum_{n=0}^{N}Y_i(n)(x-x_0)^n, \quad i=1,2,\ldots,m. \tag{4.1}$$

to get approximations.

All computations were done by programming in Maple software.

3.1. Numerical Examples

Example 1. As the first example consider the following linear system of [15]:

$$\begin{cases} 2xy_1(x)-\int_{-1}^{x}(3ty_1(t)+(2x+1)y_2(t))dt \\ \quad =x^2+x \\ xy_1(x)-2xy_2(x) \\ \quad -\int_{-1}^{x}(2(x+t)y_1(t)-ty_2(t))dt=-2x^3 \end{cases} \tag{4.2}$$

With the exact solutions $y_1(x)=x+1$ and $y_2(x)=-x$.

The method of [15] transforms the system of integral equations to a system of algebraic equations with the help of Taylor series. Then the solution of the algebraic system yields the Taylor coefficients. For more details one can see [15].

Now to solve the system (4.2), firstly, we convert it to the following form:

$$\begin{cases} 2(t-1)u_1(t) \\ \quad -\int_{0}^{t}(3(s-1)u_1(s)+(2t-1)u_2(s))ds=t^2-t \\ (t-1)u_1(t)-2(t-1)u_2(t) \\ \quad -\int_{0}^{t}(2(t+s-2)u_1(s)-(s-1)u_2(s))ds \\ \quad =-2t^3+6t^2-6t+2 \end{cases} \tag{4.3}$$

where $u_i(t)=y_i(t-1),\ \ i=1,2$.

Applying DT method on the both sides of (4.3) yields :

$$\begin{cases} 2U_1(n-1)-2U_1(n) \\ \quad -\frac{3}{n}\sum_{k=0}^{n-1}\delta_{k,1}U_1(n-k-1)+\frac{3}{n}U_1(n-1) \\ \quad -2\sum_{k=0}^{n-1}\delta_{k,1}\frac{U_2(n-k-1)}{n-k} \\ \quad +\frac{1}{n}U_2(n-1)=\delta_{n,2}-\delta_{n,1} \\ U_1(n-1)-U_1(n)-2U_2(n-1) \\ \quad +2U_2(n)-2\sum_{k=0}^{n-1}\delta_{k,1}\frac{U_1(n-k-1)}{n-k} \\ \quad -\frac{2}{n}\sum_{k=0}^{n-1}\delta_{k,1}U_1(n-k-1)+\frac{4}{n}U_1(n-1) \\ \quad +\frac{1}{n}\sum_{k=0}^{n-1}\delta_{k,1}U_2(n-k-1)-\frac{1}{n}U_2(n-1) \end{cases}$$

$$=-2\delta_{n,3}+6\delta_{n,2}-6\delta_{n,1}+2\ \delta_{n,0} \tag{4.4}$$

For $n=1,2,\ldots$.

On the other hand, if we set $t=0$ in (4.3), we obtain:

$$U_1(0)=0,\qquad U_2(0)=1$$

For simplifying the system (4.4) we consider two cases $n=1$ and $n\geq 2$. For $n=1$ we have

$$\begin{cases} -2U_1(1)+1=-1 \\ -U_1(1)+2U_2(1)=-3 \end{cases} \Rightarrow \begin{cases} U_1(1)=1 \\ U_2(1)=-1 \end{cases},$$

And for $n\geq 2$ we obtain

$$\begin{cases} U_1(n) \\ \quad =\frac{1}{2}\left[\left(2+\frac{3}{n}\right)U_1(n-1)+\frac{1}{n}U_2(n-1)\right. \\ \quad \left.-\frac{3}{n}U_1(n-2)-\frac{2}{n-1}U_2(n-2)-\delta_{n,2}\right] \\ U_2(n) \\ \quad =\frac{1}{2}\left[U_1(n)-\left(1+\frac{4}{n}\right)U_1(n-1)\right. \\ \quad +\left(2+\frac{1}{n}\right)U_2(n-1)+\frac{2(2n-1)}{n(n-1)}U_1(n-2) \\ \quad \left.-\frac{1}{n}U_2(n-2)-2\delta_{n,3}+6\delta_{n,2}\right] \end{cases} \tag{4.5}$$

Finally solving the above system yields

$$U_1(n)=0,\quad U_2(n)=0,\quad n=2,3,\ldots$$

or equivalently

$$u_1(t)=t,\quad u_2(t)=1-t$$

and so

$$y_1(t)=t+1,\quad y_2(t)=-t$$

which are the exact solutions of (4.2). Note that this result confirms the corollary 3.3.

Also note that the above solutions are in complete agreement with [15], however as mentioned previously the DT method transforms the system of integral or integro-differential equations to recurrence relations which are solvable more simple than of a system of algebraic equations which done in [15].

Example 2. Consider the nonlinear system of integral

equations of the form

$$\begin{cases} x^2 y_1^{(4)}(x) + y_2^{(4)}(x) \\ \quad - \int_0^x \cos t\, y_1''(t) y_2''(t)\, dt \\ \quad = x^2 \sin x + \cos x + \frac{1}{3}\cos^3 x - \frac{1}{3} \\ y_1^{(4)}(x) - \sin x\, y_2^{(4)}(x) \\ \quad - \int_0^x \sin t\, y_1'''(t) y_2(t)\, dt \\ \quad = \sin x\,(1-\cos x) - \frac{1}{3}\cos^3 x + \frac{1}{3} \end{cases} \tag{4.6}$$

with the conditions

$$\begin{cases} y_1(0) = 0,\ y_1'(0) = 2,\ y_1''(0) = 0,\ y_1'''(0) = -1 \\ y_2(0) = 1,\ y_2'(0) = 0,\ y_2''(0) = -1,\ y_2'''(0) = 0 \end{cases} \tag{4.7}$$

which has the exact solutions $y_1(x) = x + \sin x$ and $y_2(x) = \cos x$.

By applying DT on the both sides of (4.6), we obtain

$$\begin{cases} Y_2(n+4) \\ \quad = \dfrac{1}{(n+1)(n+2)(n+3)(n+4)} \\ \qquad \left[-S1 + S2 + S3 + \dfrac{1}{n!}\cos\dfrac{n\pi}{2} + S4 \right] \\ Y_1(n+4) \\ \quad = \dfrac{1}{(n+1)(n+2)(n+3)(n+4)} \\ \qquad \left[S5 + S6 + \dfrac{1}{n!}\sin\dfrac{n\pi}{2} - S7 - S4 \right] \end{cases} \tag{4.8}$$

for $n = 1, 2, \cdots$.

Where

$$S1 = \sum_{k=0}^{n} \delta_{k,2} (n-k+1)(n-k+2)(n-k+3)(n-k+4) Y_1(n-k+4)$$

$$S2 = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} \frac{1}{k!} \cos\frac{k\pi}{2} (l+1)(l+2)(n-k-l)(n-k-l+1) Y_1(l+2) Y_2(n-k-l+1)$$

$$S3 = \sum_{k=0}^{n} \delta_{k,2} \frac{1}{(n-k)!} \sin\frac{(n-k)\pi}{2}$$

$$S4 = \frac{1}{3} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{1}{k!\, l!\, (n-k-l)!} \cos\frac{k\pi}{2} \cos\frac{l\pi}{2} \cos\frac{(n-k-l)\pi}{2}$$

$$S5 = \sum_{k=0}^{n} \frac{1}{k!} \sin\frac{k\pi}{2} (n-k+1)(n-k+2)(n-k+3)(n-k+4) Y_2(n-k+4)$$

$$S6 = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} \frac{1}{k!} \sin\frac{k\pi}{2} (l+1)(l+2)(l+3) Y_1(l+3) Y_2(n-k-l-1)$$

$$S7 = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \sin\frac{k\pi}{2} \cos\frac{(n-k)\pi}{2}$$

and by conditions (4.7), we have

$$Y_1(0) = 0, \quad Y_1(1) = 2, \quad Y_1(2) = 0, \quad Y_1(3) = -\frac{1}{6}$$

$$Y_2(0) = 1, \quad Y_2(1) = 0, \quad Y_2(2) = -\frac{1}{2}, \quad Y_2(3) = 0$$

also by substituting $x = 0$ in the equations of system (4.6) we obtain

$$Y_1(4) = 0, \qquad Y_2(4) = \frac{1}{24}$$

In this example we solve (4.8) for the cases $N = 12$ and $N = 16$.

For $N = 12$, we obtain the approximate solutions as

$$y_{1,N}(x) = 2x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11}$$

$$y_{2,N}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12},$$

each of these is a truncated Taylor's series of the corresponding exact solution, which is a approximation of $y_i(x)$.

For case $N = 16$, we have

$$y_{1N}(x) = 2x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15}$$

$$y_{2N}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \frac{1}{12!}x^{12} - \frac{1}{14!}x^{14} + \frac{1}{16!}x^{16}$$

Table 1 shows the absolute errors at the some points.

As we mentioned before, since in DT method the solution obtained is by a recurrence relation, we can obtain solution with local arbitrary accuracy. Table 2 shows the absolute errors in some points away from $x = 0$.

Example 3. We consider in this example a system of integro-differential equations as

$$\begin{cases} e^{-x}y_1^{(4)}(x) + (\sin x + 1)y_2^{(4)}(x) \\ \quad - \int_0^x \sin t\, y_1'''(t)y_2''(t)dt = (\sin x + 1)e^{-x} + \cos x \\ (\cos x)y_1^{(4)}(x) - e^x y_2^{(4)}(x) \\ \quad - \int_0^x \cos t\, y_1''(t)y_2'''(t)dt = e^x \cos x + \sin x - 1 \end{cases} \tag{4.9}$$

with supplementary conditions

$$\begin{cases} y_1(0) = y_1'(0) = y_1''(0) = y_1'''(0) = 1 \\ y_2(0) = -y_2'(0) = y_2''(0) = -y_2'''(0) = 1 \end{cases} \tag{4.10}$$

and exact solutions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$.

Using similar methods to those used in of previous examples, we obtain

$$\begin{cases} Y1(n+4) + Y2(n+4) \\ \quad = \frac{1}{(n+1)(n+2)(n+3)(n+4)} \\ \qquad \left[-S1 - S2 + S3 + S4 + \frac{1}{n!}\cos\frac{n\pi}{2} \right] \\ Y_1(n+4) - Y2(n+4) \\ \quad = \frac{1}{(n+1)(n+2)(n+3)(n+4)} \\ \qquad \left[-S5 + S6 + S7 + S8 + \frac{1}{n!}\sin\frac{n\pi}{2} \right] \end{cases} \tag{4.11}$$

for $n = 1, 2, \cdots$.

where

$$S1 = \sum_{k=1}^{n} \frac{(-1)^k}{k!}(n-k+1)(n-k+2)(n-k+3)(n-k+4)Y_1(n-k+4)$$

$$S2 = \sum_{k=1}^{n} \frac{1}{k!}\sin\frac{k\pi}{2}(n-k+1)(n-k+2)(n-k+3)(n-k+4)Y_2(n-k+4)$$

$$S3 = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1} \frac{1}{k!}\sin\frac{k\pi}{2}(l+1)(l+2)(l+3)(n-k-l)(n-k-l+1)Y_1(l+3)Y_2(n-k-l+1)$$

$$S4 = \sum_{k=0}^{n}\left(\frac{1}{k!}\sin\frac{k\pi}{2} + \delta_{k,0}\right)\frac{(-1)^{n-k}}{(n-k)!}$$

$$S5 = \sum_{k=1}^{n} \frac{1}{k!}\cos\frac{k\pi}{2}(n-k+1)(n-k+2)(n-k+3)(n-k+4)Y_1(n-k+4)$$

Table 1. Numerical results of example 2 for N=12, 16

	N = 12		N = 16	
x	Abs.Err.y1	Abs.Err.y2	Abs.Err.y1	Abs.Err.y2
0.2	1.3000e−19	1.0000e−20	0	1.0000e−20
0.4	1.0769e−15	3.0780e−17	1.0000e−20	1.0000e−20
0.6	2.0938e−13	8.9755e−15	5.0000e−19	1.0000e−20
0.8	8.8017e−12	5.0314e−13	6.3100e−17	2.8100e−18
1.0	1.5983e−10	1.1423e−11	2.8032e−15	1.5578e−16
1.2	1.7065e−09	1.4640e−10	6.2114e−14	4.1426e−15
1.4	1.2628e−08	1.2643e−09	8.5236e−13	6.6333e−14
1.6	7.1450e−08	8.1781e−09	8.2362e−12	7.3265e−13
1.8	3.2930e−07	4.2420e−08	6.0878e−11	6.0936e−12
2.0	1.2909e−06	1.8484e−07	3.6423e−10	4.0518e−11

Table 2. Numerical results of example 2 for N=40

x	Abs.Err.y1	Abs.Err.y2
6.00	2.3500e−18	3.3000e −19
7.00	1.2970e−15	2.1643e−16
8.00	3.0697e−13	5.8562e−14
9.00	3.8053e−11	8.1703e−12
10.0	2.8318e−09	6.7586e−10

$$S6=\sum_{k=1}^{n}\frac{1}{k!}(n-k+1)(n-k+2)$$
$$(n-k+3)(n-k+4)Y_2(n-k+4)$$
$$S7=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\frac{1}{k!}\cos\frac{k\pi}{2}(l+1)(l+2)(n-k-l)$$
$$(n-k-l+1)(n-k-l+2)Y_1(l+2)Y_2(n-k-l+2)$$
$$S8=\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\cos\frac{k\pi}{2}$$

and from supplementary conditions we have

$$Y_1(0)=1,\quad Y_1(1)=1,\quad Y_1(2)=\frac{1}{2},\quad Y_1(3)=\frac{1}{6}$$
$$Y_2(0)=1,\quad Y_2(1)=-1,\quad Y_2(2)=\frac{1}{2},\quad Y_2(3)=-\frac{1}{6}$$

also by substituting $x=0$ in the equations of system (4.9) we obtain

$$Y_1(4)=Y_2(4)=\frac{1}{24}.$$

Solving recurrence system (7) for $N=20$, we conclude that the approximate solutions is of the form

$$y_1(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{20!}x^{20}$$
$$y_2(x)=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\cdots+\frac{1}{20!}x^{20}$$

which is the truncated Taylor's series of the exact solutions.

And for $N=30$ we have

$$y_1(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{30!}x^{30}$$
$$y_2(x)=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\cdots+\frac{1}{30!}x^{30}$$

Table 3 shows the absolute errors for this example.

Example 4. We consider the following nonlinear system of [12]

$$\begin{cases} y_1'(x)+\dfrac{2}{x+2}y_2(x) \\ \qquad -\displaystyle\int_0^x t^2y_1^2(t)dt=f_1(x) \\ xy_1(x)+y_2'(x) \\ \qquad -\displaystyle\int_0^x t(2-t)\cos ty_1(t)y_2(t)dt=f_2(x) \end{cases} \tag{4.12}$$

where

$$f_1(x)=1+\sum_{k=1}^{\infty}\left(\frac{x^{2k}}{2^{2k}}+(-1)^k\frac{2^{2k-1}x^{2k+1}+2kx^{2k-1}}{(2k+1)!}\right)$$
$$f_2(x)=\frac{1}{2}+\frac{3}{2}x$$
$$+\sum_{k=1}^{\infty}\left(\frac{(k+2)x^{k+1}}{2^{k+2}}+(-1)^k\frac{2^{2k-1}(2k+1)x^{2k}+x^{2k+1}}{(2k+1)!}\right)$$

with supplementary conditions

$$y_1(0)=1$$
$$y_2(0)=1$$

with the exact solution $y_1(x)=\dfrac{\sin x}{x}$ and $y_2(x)=\dfrac{2}{2-x}$.

By the same way of previous examples for $n=1,2,\cdots,N-1$ we have

$$\begin{cases} Y_1(n+1)=\dfrac{1}{n+1}\left[-S1+S2+F_1(n)\right] \\ Y_2(n+1)=\dfrac{1}{n+1}\left[-Y_1(n-1)+S3+F_2(n)\right] \end{cases} \tag{4.13}$$

where

$$S1=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}Y_2(n-k)$$
$$S2=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\delta_{k,2}Y_1(l)Y_1(n-k-l-1)$$
$$S3=\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\sum_{r=0}^{n-k-l-1}(2\delta_{k,1}-\delta_{k,2})\frac{1}{l!}$$
$$\cos\frac{l\pi}{2}Y_1(r)Y_2(n-k-l-r-1),$$

where F_1 and F_2 denote the differential transforms of f_1 and f_2 respectively. And from supplementary conditions condition

$$Y_1(0)=1,\quad Y_2(0)=1$$

Also by substituting $x=0$ in the second equation of system (4.11) we obtain

$$Y_2(1)=F_2(0)$$

In this example since the second solution is singular

in $x = 2$, we solved the recursive equations (4.12) for high numbers $N = 150$ and $N = 200$. Table 4 shows the absolute errors in points

$$x = (0,1)i, \quad i = 6,7,\ldots,19$$

for $N = 150$, Digits=20 and $N = 200$, Digits = 30 respectively.

Table 3. Numerical results of example 3

	N = 20		N = 30	
x	Abs.Err.y1	Abs.Err.y2	Abs.Err.y1	Abs.Err.y2
1.00	1.0000e−19	2.0000e−20	1.0000e−19	0
1.50	1.0480e−16	9.1390e−17	1.0000e−19	2.0000e−20
2.00	4.5133e−14	3.7615e−14	1.0000e−19	3.0000e−20
2.50	5.0174e−12	3.9944e−12	2.0000e−18	5.0000e−21
3.00	2.3682e−10	1.8006e−10	1.0000e−18	6.4000e−20
3.50	6.1899e−09	4.4937e−09	1.1000e−17	8.1340e−18
4.00	1.0400e−07	7.2762e−08	6.4100e−16	4.9840e−16
4.50	1.2803e−06	8.4671e−07	2.5120e−14	1.8933e−14
5.00	1.2036e−05	7.5931e−06	6.7049e−13	4.8950e−13

Table 4. Numerical results of example 4

	N = 150		N = 200	
x	Abs.Err.y1	Abs.Err.y2	Abs.Err.y1	Abs.Err.y2
0	0	0	0	0
0.2	3.0000e−20	1.0000e−19	1.3000e−29	1.0000e−29
0.4	3.0000e−20	1.0000e−19	3.0000e−30	2.0000e−30
0.6	6.0000e−20	2.0000e−19	2.0000e−30	2.0000e−29
0.8	1.0000e−20	1.0000e−19	7.0000e−30	1.0000e−29
1.0	5.0000e−20	1.0000e−19	7.0000e−30	1.0000e−29
1.2	1.0000e−20	1.3000e−19	2.0000e−30	1.3000e−29
1.4	5.0000e−20	1.0000e−19	5.0000e−30	3.0000e−29
1.6	1.4000e−19	1.1629e−14	1.3000e−29	1.6598e−19
1.7	3.0000e−20	1.4661e−10	1.6000e−29	4.3362e−14
1.8	7.0000e−20	1.2320e−06	2.1000e−29	6.3496e−09
1.9	4.0000e−20	8.6555e−03	1.0000e−30	6.6600e−04

Table 5. Numerical results of example 4 ([12])

x	Abs.Err.y1	Abs.Err.y2
0.0	0	0
0.2	1.8423e−26	1.1111e−16
0.4	1.2070e−21	8.1920e−12
0.6	7.9231e−19	6.1495e−9
0.8	7.8988e−17	7.1583e−7
1.0	2.8033e−15	3.0518e−5

As mentioned above, this example was chosen from [12], where the problem has been solved by the Tau method. In the Tau method [17], we replace the differential and integral parts of the problem by their matrix representation and then convert it to corresponding system of linear algebraic equations. In a similar manner, we convert supplementary conditions to a linear algebraic system of equations. Finally by combining these two linear systems of algebraic equations, we obtain a system of linear algebraic equations and solve it to obtain an approximate solution of the problem. For more details about Tau method see [12], [17] and [18].

For comparing we report the results of [12] in Table 5.

Comparing the results of Tables 4 and 5 shows the high accuracy of the DT method. Also, it is worthy to note that, the results in [12] (Table 5) were reported up to $x = 1$, while the results of Table 4 (DT method) are reported up to $x = 1.9$.

3.2. Conclusion

In this work, the differential transform method has been applied for system of nonlinear Volterra integro-differential equations with variable coefficients. For illustration purpose, some examples have been solved by presented method. As the results of examples show, the method has high accuracy. Also this method has a simple structure, so it can be applied to solve applied problems in applied sciences.

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References

1. Jang, M. J. and Chen, C. K. and liu, Y. C. Two-dimensional differential transform for partial differential equations. *Appl. Math. Comput.* 121 : 261-270 (2001).
2. Zhou, J. K. Differential Transform and its Application for Electric Cicuits. *Huazhong University Press*, Wuhan, China (1986).
3. Ayaz, F. On the two-dimensional differential transform method. *Appl. Math. Comput.* 143 : 361-374 (2003).
4. Ayaz, F. Application of differential transform method to

differential-algebraic equations. *Appl. Math. Comput.* 152: 649-657 (2004).
5. Ayaz, F. Solutions of the system of differential equations by differential transform method. *Appl. Math. Comput.* 147 : 547-567 (2004).
6. Chen, C. K. Solving partial differential equations by two dimensional differential transform. *Appl. Math. Comput.* 106 : 171-179 (1999).
7. Arikoglu, A. and Ozkol, I. Solution of boundary value prablem for integro- differential equations by using differential transform method. *Appl. Math. Comput.* 168 : 1145-1158 (2005).
8. Odibat, Z. M. Differential transform method for solving Volterra integral equations with separable kernels . *Math. Comput. Model.*, 48 (7-8) : 1144-1149 (2008).
9. Tari, A. and Rahimi, M. Y. and Shahmorad, S. and Talati, F. Solving a class of two dimensional linear and nonlinear Volterra integral equations by the differential transform method. *J. of Comput. Appl. Math.* 228 : 70-76 (2009).
10. Jerri, A. J. Introduction to integral equations with applications. *John Wiley and Sons, INC,* (1999).
11. Shakeri, F. and Dehghan, M. Solution of a model describing biological species living together using the variational iteration method, *Mathematical and Computer Modeling.* 48 : 685-699 (2008).
12. Khani, A and Moghadam M. M. and Shahmorad, S. Approximate solution of the system of nonlinear Volterra integro-differential equations, *Computational Methods in Appl. Math.* 8(1) : 77-85 (2008).
13. Biazar, J. and Ghazvini, H. He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind, *Chaos, Solitons and Fractals.* 39(2) : 770-777 (2009).
14. Akyuz-Dascioglu, A and sezer,M. Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations. *J. Franklin Inst.* 342 : 688-701 (2005).
15. Sorkun, H. H. and Yalcinbas, S. Approximate solutions of linear Volterra integral equation systems with variable coefficients. *Applied Mathematical Modelling,* 34 : 3451-3464 (2010).
16. Biazar, J. and Eslami, M. Differential transform method for systems of Volterra integral equation of the second kind. Int. J. of the Physical Sciences, 6(5) : 1207-1212 (2011).
17. Tari, A., Rahimi, M. Y., Shahmorad, S. and Talati, F. Development of the Tau method for the numerical solution of two-dimentional linear Volterra integro-differential equations. J. of Computationa Methods in Applied Mathematics, 9(4) : 421-435 (2009).
18. Hosseini Aliabdi, M. and Shahmorad, S. A matrix formulation of the Tau method for Fredholm and Volterra linear integro-differential differential equations, The Korean jornal of Comput. and Appl. Math. 9(2) : 497-507 (2002).