TESTING FOR "RANDOMNESS" IN SPATIAL POINT PATTERNS, USING TEST STATISTICS BASED ON ONE-DIMENSIONAL INTER-EVENT DISTANCES

M. Q. Vahidi-Asl*

Department of Statistics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Islamic Republic of Iran

Abstract

To test for "randomness" in spatial point patterns, we propose two test statistics that are obtained by "reducing" two-dimensional point patterns to the one-dimensional one. Also the exact and asymptotic distribution of these statistics are drawn.

1. Introduction

Data in the form of a set of points, irregularly distributed within a region of space, is usually called a *spatial point pattern*. Examples, in different biological contexts, include locations of trees in a forest, of nests in a breeding colony of birds, or of cell nuclei in a microscopic section of tissue. The locations are called *events* to distinguish them from arbitrary points of the region in question.

Figures 1, 2, and 3 show three spatial point patterns in a square region, all taken from Diggle [4]. The first due to Numata [5], shows 65 Japanese black pine samplings in a square of side 5.7 m, the second, extracted by [6] from [7], shows 62 redwood seedlings in a square of side approximately 23 m, and finally the third, due to Crick and Lawrence [3], shows the centers of 42 biological calls distributed more or less regularly over the unit square.

Figure 1 shows no obvious structure and might be regarded as a "completely random" pattern, to be defined formally below. In Figure 2, on the other hand, the strong clustering is apparent, which is termed "aggregated" by Diggle [4] to avoid the mechanistic connotations of the perhaps more obvious term clustered. Patterns such as the ones in Figure 3 are called "regular" for obvious reasons.

The classification of patterns as regular, random or aggregated may seem an over-simplification, but it is useful at an early stage of analysis. At a later stage, this simplistic approach can be abandoned in favour of a more detailed, and essentially multidimensional description of pattern which can be obtained either by identifying different "scales of patterns" or by formulating an explicit model of the underlying process. Diggle [4] develops methods for the analysis of spatial patterns based on stochastic models, which assumes that the events are generated by some underlying random mechanism.

The hypothesis of *complete spatial randomness* (henceforth *CSR*) for a spatial point pattern asserts that (i) the number of events in any planar region A with area $|A|$ follows a Poisson distribution with mean $\lambda|A|$,

Keywords: Spatial patterns; Complete spatial randomness; Poisson processes

* *E-mail: m-vahidi@cc.sbu.ac.ir*

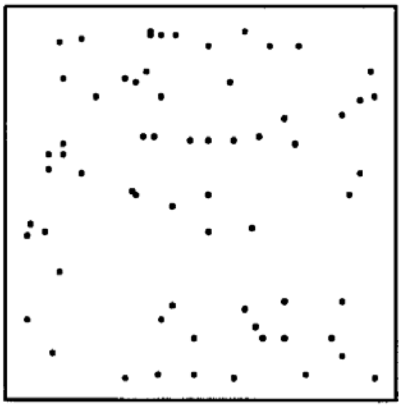

Figure 1

Figure 2

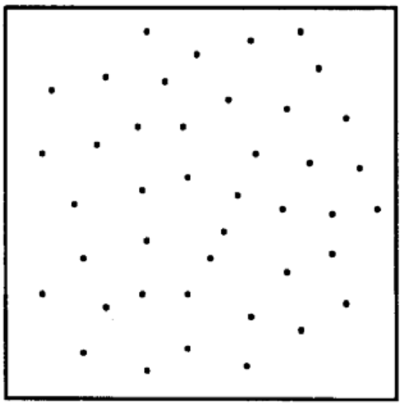

Figure 3

(ii) given n events x_i in a region A, the x_i is an independent random sample from the uniform distribution on A.

Most analyses begin with a test of *CSR*, and there are several good reasons for this. Firstly, a pattern for which *CSR* is not rejected hardly merits any further formal statistical analysis. Secondly, tests are used as a means of exploring a set data, rather than because rejection of *CSR* is of intrinsic interest. Greig-Smith, in the discussion of Bartlett [2], has emphasized that ecologists often know *CSR* to be untenable but nevertheless use tests of *CSR* as an aid to the formulation of ecologically interesting hypotheses concerning pattern and genesis. Thirdly, *CSR* acts as a dividing hypothesis to distinguish between patterns which are broadly classifiable as “regular” or “aggregated”.

There are numerous methods for testing a point pattern against *CSR* on top of which is the use of Monte Carlo tests [1].

Quite generally, let u_1 be the observed value of a statistic U and let u_i, $i=2,\ldots,s$, be the corresponding values generated by independent random sampling from the distribution of U under a simple hypothesis $\mathcal{H}$. Let $u_{(j)}$ denote the jth largest amongst u_i, $i=1,\ldots,s$. Then, under $\mathcal{H}$,

$$P\{u_1 = u_{(j)}\} = s^{-1}, \quad j = 1,\ldots,s,$$

and rejection of $\mathcal{H}$ on the basis that u_1 ranks kth larger or higher gives an exact, one-sided test of size k/s. Usually, s is taken as 100 in most examples. For complete details and other related topics, we refer the reader to Ref. [4]. A complete and updated list of statistical tests for testing *CSR*, with less emphasis on Monte Carlo tests, is given in chapter 8 of the more recent book by Cressie [8].

Our concern here is to introduce test statistics whose exact and asymptotic distributions are known and test *CSR* against data without using any simulation. Therefore it may be included among the many simple existing tests, nevertheless it proves to be as effective as Monte Carlo tests, as emphasized in [4]. In Section 2 we introduce these statistics and some theoretical results regarding their distribution, and in Section 3 we use this method on the data given in Figures 1-3 [4].

2. Theoretical Results

Since the statistics to be introduced are based on points distributed along a line with exponential distribution for the distance between two consecutive points, therefore we consider the one-dimensional case first.

Let the points $X_1,\ldots, X_{n+1}$ be distributed randomly along some stretch of a line so that the random variables

$$T_i = X_{i+1} - X_i,\ i=1,2,\ldots,n$$

are iid. The random process $\{T_n\}_{n\geq 1}$ with $T_n=X_n-X_{n-1}$, $n=1,2,\ldots$, forms an iid sequence of random variables with mean λ^{-1} if and only if $\{X_n\}_{n\geq 1}$ is a labeling of a homogenous Poisson process of intensity λ [9]. Here, the points $\{X_i\}_{1\leq i\leq n+1}$ are a restriction of a Poisson point process to some stretch of a line. Let c be a constant. We place line segments of length c on every point X_i along the supporting line of the X_i’s so that X_i is the midpoint of this line segment. These lines either overlap or there is a “gap” between two consecutive line segments. If we denote the gap between X_i and X_{i+1} by Y_i, then we have

$$Y_i = (T_i - c)^+,\ i=1,2,\ldots,n,$$

in which $x^+=\max(x,0)$. This holds because the right half of the line of length c centering at X_i extends to the right of this point and the left half of the line of length c centering at X_{i+1} extends to the left of this point, and there is no "gap" between X_i and X_{i+1} if and only if $T_i \le c$, and therefore Y_i=0.

Let $U = \sum_{i=1}^{n} Y_i$. We discuss the distribution of U first.

Theorem 1. If the $T_1, T_2, \ldots, T_n$ are iid random variables with a common exponential distribution with mean λ^{-1} then

$$F_U(u) = \sum_{j=1}^{n} \binom{n}{j} h_j(jc+u)\left(1-e^{-\lambda c}\right)^{n-j} + \left(1-e^{-\lambda c}\right)^n \delta_0(u)$$

where

$$h_j(z) = P\left(\sum_{i=1}^{j} T_i \le z, T_1 \ge c, \ldots, T_i \ge c\right) = \int_c^z h_{j-1}(z-t)\lambda e^{-\lambda t} dt, \quad j \ge 2,$$

$$h_1(z) = e^{-\lambda t} - e^{-\lambda z}, \quad z \ge c,$$

and

$$\delta_t(x) = \begin{cases} 0 & \text{if} \quad x < t \\ 1 & \text{if} \quad x \ge t \end{cases}.$$

Proof. Let N be the number of T_i, s such that $T_i \ge c$, then,

$$\begin{aligned} F_U(u) &= P(U \le u) \\ &= P\left(\sum_{i=1}^{n} (T_i - c)^+ \le u\right) \\ &= \sum_{j=0}^{n} P\left(\sum_{i=1}^{n} (T_i - c)^+ \le u, \quad N = j\right) \\ &= \sum_{j=1}^{n} P\left(\sum_{i=1}^{n} (T_i - c)^+ \le u, \quad N = j\right) \\ &\quad + P(T_1 < c, \ldots, T_n < c)\delta_0(u) \end{aligned}$$

Now by using the iid property of T_i, s we have

$$\begin{aligned} F_U(u) &= \sum_{j=1}^{n} \binom{n}{j} P\left\{\sum_{i=1}^{n} (T_i - c)^+ \le u, T_1 \ge c, \ldots T_j \ge c,\right. \\ &\qquad \left. T_{j+1} < c, \ldots, T_n < c\right\} + \left(1-e^{-\lambda c}\right)^n \delta_0(u) \\ &= \sum_{j=1}^{n} \binom{n}{j} P\left\{\sum_{i=1}^{n} (T_i - c) \le u, T_1 \ge c, \ldots T_j \ge c, \ldots,\right. \\ &\qquad \left. \ldots, T_{j+1} < c, \ldots, T_n < c\right\} + \left(1-e^{-\lambda c}\right)^n \delta_0(u) \\ &= \sum_{j=1}^{n} \binom{n}{j} P\left\{\sum_{i=1}^{j} T_i \le jc + u, T_1 \ge c, \ldots T_j \ge c\right\} \times \\ &\qquad \times P\left\{T_{j+1} < c, \ldots, T_n < c\right\} + \left(1-e^{-\lambda c}\right)^n \delta_0(u) \\ &= \sum_{j=1}^{n} \binom{n}{j} P\left\{\sum_{i=1}^{j} T_i \le jc + u, T_1 \ge c, \ldots T_j \ge c\right\} \times \\ &\qquad \times \left(1-e^{-\lambda c}\right)^{n-j} + \left(1-e^{-\lambda c}\right)^n \delta_0(u) \\ &= \sum_{j=1}^{n} \binom{n}{j} h_j(jc+u)\left(1-e^{-\lambda c}\right)^{n-j} + \left(1-e^{-\lambda c}\right)^n \delta_0(u) \end{aligned}$$

in which $h_j(z) = P(\sum_{i=1}^{j} T_i \le z, T_1 \ge c, \ldots, T_j \ge c)$ as mentioned in the statement of the theorem. But

$$\begin{aligned} h_j(z) &= \int_0^\infty P\left(\sum_{i=1}^{j} T_i \le z, T_1 \ge c, \ldots, T_j \ge c \mid T_j = t\right) f_T(t) dt \\ &= \int_c^z P\left(\sum_{i=1}^{j-1} T_i \le z - t, T_1 \ge c, \ldots, T_{j-1} \ge c\right) \lambda e^{-\lambda t} dt \\ &= \int_c^z h_{j-1}(z-t)\lambda e^{-\lambda t} dt, \quad j \ge 2. \end{aligned}$$

The statement for $h_1(z)$ is obvious. □

The application of this result for computing probabilities needed in the last section requires tedious recursive integrations. An upper bound for the probability $P(U \le u)$ may be obtained as in the following corollary.

Corollary 1. With T_i*'s* as in Theorem 1, we have

$$P(U \le u) = F_U(u) \le \sum_{j=0}^{n} \binom{n}{j} \min\left\{a_j(u), b_j\right\},$$

where,

$$a_j(u) = e^{-\lambda c j} P\left\{\sum_{i=1}^{j} T_i \le jc + u\right\}$$

and

$$b_j = e^{-jc}\left(1-e^{-\lambda c}\right)^{n-j}.$$

Proof. Let $u > 0$. Then

$$F_U(u) = P(U \le u)$$

$$= \sum_{j=0}^{n} \binom{n}{j} P\{\sum_{i=1}^{n} (T_i - c)^+ \le u, T_1 \ge c, \ldots T_j \ge c,$$

$$T_{j+1} < c, \ldots, T_n < c\}.$$

$$= \sum_{j=0}^{n} \binom{n}{j} P\{\sum_{i=1}^{n} T_i \le jc + u, T_1 \ge c, \ldots T_j \ge c,$$

$$T_{j+1} < c, \ldots, T_n < c\}$$

But,

$$c_j(u) = P\left(\sum_{i=1}^{j} T_i \le jc + u, T_1 \ge c, \ldots T_j \ge c, T_{j+1} < c, \ldots, T_n < c \right)$$

$$\le P\left(\sum_{i=1}^{j} T_i \le jc + u, T_{j+1} < c, \ldots, T_n < c \right)$$

$$= P\left(\sum_{i=1}^{j} T_i \le jc + u \right) \left(1 - e^{-\lambda c}\right)^{n-j}$$

and

$$c_j(u) \le P(T_1 \ge c, \ldots, T_j \ge c, T_{j+1} < c, \ldots, T_n < c$$

$$= e^{-jc}(1 - e^{-\lambda c})^{n-j}.$$

Therefore $c_j(u) \le \min\{a_j(u), b_j\}$ and the result follows.

Note that $a_j(u)$ may be computed using χ^2 tables. □

The above result gives approximate values of the probabilities needed later on in the next section, but as is seen, it still requires lengthy calculations, which we are trying to avoid. It is fortunate that the asymptotic distribution for U is very easy to come by and at the same time, as will be shown, very helpful.

Theorem 2. Let $T_1, \ldots, T_n$ be iid random variables with a common exponential distribution with mean λ^{-1}. Then the random variable $U = \sum_{i=1}^{n} (T_i - c)^+$ has an asymptotic normal distribution with mean $n\lambda^{-1}e^{-\lambda c}$ and variance $n\lambda^{-2}e^{-\lambda c}(2 - e^{-\lambda c})$.

Proof. Considering the iid sequence $\{T_n\}$ we note that the sequence $\{(T_n\text{-}c)^+\}$ is iid also, therefore we can apply the central limit theorem, iid case, to the latter sequence and conclude that

$$U' = \frac{\sum_{i=1}^{n} (T_i - c)^+ - n\mu}{\sqrt{n}\sigma}$$

has an asymptotic standard normal distribution, once we show that

$$\mu = E(T_1 - c)^+ = \lambda^{-1}e^{-\lambda c}$$

and

$$\sigma^2 = \mathrm{var}(T_1 - c)^+$$

$$= \lambda^{-2}e^{-\lambda c}(2 - e^{-\lambda c}) < \infty.$$

But

$$E(T_1 - c)^+ = \int_c^{\infty} (t - c)\lambda e^{-\lambda t} dt$$

$$= \int_0^{\infty} u e^{-\lambda(u+c)} du$$

$$= \lambda^{-1} e^{-\lambda c},$$

and

$$E[(T_1 - c)^+]^2 = \int_c^{\infty} (t - c)\lambda e^{-\lambda t} dt$$

$$= \lambda^{-2} e^{-\lambda c} \int_0^{\infty} y^2 e^{-y} dy$$

$$= 2\lambda^{-2} e^{-\lambda c},$$

hence the desired result. □

For arbitrary n, we may use a statistic W whose definition and distribution, both exact and asymptotic, are very easy to find.

Theorem 3. Let $W = \sum_{i=1}^{n} 1_{\{T_i > c\}}$ where T_i, s are as in Theorem 2 and 1_A is the indicator function of the set A. Then W has a binomial distribution with a success probability of $p = e^{-\lambda c}$. □

The above results are readily applicable to "line data" or points randomly distributed along a line. The idea of exploiting the statistic U above is very simple: if the n+1 points are "almost regularly" distributed along their supporting line, and c is chosen close to the average distance between all of the points, then U will be "small". For points randomly distributed under *CSR* assumption, the value of U should be "moderate" and for "aggregated data" or points showing somehow clustering, the value of U should be "large". We now try to adapt it to planar data. In the next section, we will use the above statistics to test the data referred to in the Introduction.

Consider a rectangle A with sides a and b. The hypothesis of *CSR* is valid in this rectangular region.

We divide the sides with length a and b to l and m equal parts, respectively and draw lines parallel to the sides so that the original rectangle is subdivided to m

rectangles with sides a and $\frac{b}{m}$ and rectangles with sides b and $\frac{a}{l}$. Now consider the "horizontal" rectangles first. If we show these rectangles by $A_1, A_2, \ldots, A_m$, we project all the points inside A_j; $j=1,2,\ldots,m$ on the "base" of the rectangle, i.e., the side with length a. If we denote two consecutive projections in the rectangle A_j by X_{ij} and $X_{i+1,j}$ and the distance between the points X_{ij} and $X_{i+1,j}$ by T_{ij}, then under *CSR*

$$P(T_{ij} > t) = e^{-\lambda \frac{b}{m} t}$$

and therefore T_{ij} has an exponential distribution with mean $m(\lambda b)^{-1}$, and hence we are back to the one-dimensional case again. Doing the same on all the strips $A_1, \ldots, A_m$, we may consider the statistic

$$U = \sum_{i,j} (T_{ij} - c)^+$$

which is the sum of n-m independent identically distributed random variables with a common exponential distribution with mean $m(\lambda b)^{-1}$. Using the same procedure for vertical strips, we obtain a statistic

$$V = \sum_{i,j} (T'_{ij} - c')^+$$

which is the sum of n-1 independent identically distributed random variables with a common exponential distribution with mean $l(\lambda a)^{-1}$. Let $c = \frac{ma}{n-m}$ and $c' = \frac{lb}{n-l}$. As for the "linear" data, "small values" of U and V simultaneously correspond to the "regular" data and "large values" of U and V correspond to "aggregated" data. Therefore, we will reject the *CSR* hypothesis whenever

$$(U \le u_1 \text{ and } V \le v_1) \text{ or } (U \ge u_2 \text{ and } V \ge v_2).$$

3. Application to Certain Data

We now apply the statistics presented in Theorems 2 and 3 to the data given in Figures 1-3 of the Introduction, but before doing so we note that these three sets of data are distributed in a square region, so we may take $m=l$ and $c=c'$ in which case U and V have the same distribution and therefore we may reject the *CSR* hypothesis whenever

$$U \le \min(u_1, v_1) \text{ or } U \ge \max(u_1, v_1).$$

We consider the three cases separately.

3.1. Location of Japanese Black Pine Trees

Consider the data presented in Figure 1 of the Introduction. We take m=n=5. We have doubled the size of squares in Figures 1-3 so that we have a square with 10 cm sides in this case. Therefore, according to Theorem 2,

$$\mu_U = n(2\lambda)^{-1} e^{-2\lambda c}, \ \sigma_U = (2\lambda)^{-1} \sqrt{n e^{-2\lambda c}(2 - e^{-2\lambda c})}.$$

To obtain numerical values of μ_U and σ_U we need an estimate for λ. Under *CSR*, $\hat{\lambda} = \frac{65}{|A|} = 0.65$ is the maximum likelihood estimate of λ [4].

Also in order to take into account the "gaps" between the two events which are the "last" and "first" events in two consecutive rectangular strips, we take n=65. This is tantamount to "piecing together" all the strips and obtaining one strip of 5×10 cm long and 2 cm wide. This is reasonable under the *CSR* hypothesis as long as we ignore the edge effects. We will do the same in Sections 3.2 and 3.3 without further mentioning. Therefore,

$$\mu_U \approx 18.45, \quad \sigma U \approx 4.81$$

$$u_1 = 11.9, \quad v_1 = 16.8.$$

We have

$$p = P(U < 11.9) = 0.0869.$$

Note that we have a two-sided test here, that the attained significant level will be 2×(0.0869)=0.1738, and hence the *CSR* hypothesis is accepted.

It should be noted that doubling the one-sided P-values in asymmetric cases is somewhat controversial but is advocated by some authors, including R. A. Fisher [10]. We will adhere to this fact without further mentioning.

We now apply Theorem 3 for this set of data. The statistic $W = \sum_{i=1}^{n} 1_{\{T_i > c\}}$ has a binomial distribution with success probability $p = e^{-\lambda c}$. Here $p = e^{-1} \approx 0.37$. Since n is large enough, we may use the normal approximation to the binomial. This time "small values" of W correspond to "aggregated" data and "large values" of W correspond to "regular" ones. The observed value of W here is w_1=20 for the horizontal strips and w_2=23 for the vertical strips. Hence,

$$P(W < \min(w_1, w_2) = P(W < 20) \approx P(Z < -1.02) = 0.1539$$

Therefore the attained significant level is 2(0.1530)= 0.3078 and the *CSR* hypothesis is accepted again.

3.2. Locations of 62 Redwood Seedlings

Proceeding as before, we have,

$n=62, \quad \hat{\lambda}=\frac{62}{100}, \quad c=\frac{50}{62}\approx 0.80, \quad \mu_V=18.45,$

$\sigma_U=4.92, \quad u_1=28.3, \quad u_2=26, \quad \max(u_1,u_2)=28.3$

Therefore,

$$P(U>28.3)\approx P(Z>\frac{28.3-18.45}{4.92})=P(Z>2.00)=0.0228$$

Hence the attained significant level is 0.0456 and the *CSR* hypothesis is rejected.

Now applying Theorem 3 and noting that w_1=14 for the horizontal strips and w_1=11 for the vertical strips, we have μ_W=22.88, σ_W=3.80, and therefore

$$P(W\le 11)\approx P(Z\le -3.12)<0.001,$$

and the *CSR* hypothesis is emphatically rejected.

3.3. Locations of 42 Cell Centers

For this set of data, we have u_1=9.3 for the horizontal strips and u_2=7.5 for the vertical ones, μ_U=18.45, σ_U=5.99. Hence

$$P(U<7.5)\approx P(Z<-1.83)=0.0336.$$

The attained significance level is 2(0.0336)=0.0676, and we may be inclined to reject the *CSR* hypothesis.

Regarding the other statistic presented in Theorem 3, we have μ_W=15.59 and σ_W=3.128. Since w_1=13 for the horizontal strips and w_2=18 for the vertical ones, therefore

$$P(W<13)\approx P(Z<-0.812)=0.209,$$

and this leads to the acceptance of the *CSR* hypothesis, contrary to what we except as a result of applying the majority of the tests presented by [4].

To investigate the effects of increasing the number of the rectangular strips partitioning the region *A*, we take $m=l=10$. Once again μ_W=15.54, σ_W=3.128, w_1=10 and w_2=14. Hence

$$P(W<10)\approx P(Z<-1.77)=0.384,$$

and this time the *CSR* hypothesis is rejected.

It is interesting to use Theorem 2 again for the new partitioning. This time, $\mu_U\approx$36.76, $\sigma_U\approx$11.95, u_1=26.2, u_2=18.1, and

$$P(U<18.1)\approx P(Z<-1.56)\approx 0.0594,$$

so the result hints to the acceptance of the *CSR* hypothesis. Even partitioning the region to $m=l=15$ strips does not lead to the rejection of *CSR* hypothesis for this set of data. This may be an indication of the weakness of these tests against "regular" alternatives. But referring to the statistical tests discussed in [4], we note that some of these tests justify the acceptance of this set of data as being completely spatial random. The majority of the tests supporting the "regularity" hypothesis are specifically based on "small distances". Examining the empirical distribution function plot, a complete absence of small inter-event distances is observed [4]. Translating this idea in terms of the statistic, denoted by *W* in Theorem 2 of Section 3.2, we may consider a variant of this statistic, defined by

$$W'=\sum_{i,j}^{n} 1_{\{T_{ij}\le c'\}}$$

where c' is any positive constant. Though there is an element of arbitrariness in choosing the value of c', we may note that any reasonable "extreme value" would suffice for this purpose. For our examples, we choose $c'=\frac{c}{2}$ where $c=\frac{ma}{n}$ when $m=l$. W' denotes the number of inter-event distances shorter than c', and has a binomial distribution with parameters *n* and $p=1-e^{-\lambda' c'}$ where $(\lambda')^{-1}$ is the mean of T_{ij}. For the pattern given as the location of 42 cell centers in Figure 3, we have $c\approx 0.60$, $w_1'=7$, $w_2'=6$, therefore the attained significance level is approximately

$$2P(Z<\frac{5-42.39}{\sqrt{42(0.39)(0.61)}})=2P(Z<-3.60)$$

and the *CSR* hypothesis is emphatically rejected.

For the location of Japanese black pine trees given in Figure 2, we have $c'\approx 0.34$, $w_1'=24$, $w_2'=19$, so the attained significance level is approximately

$$2P(Z<-1.601)$$

which leads to the acceptance of the *CSR* hypothesis.

Finally for the locations of 62 redwood seedlings presented in Figure 2, we have $c'\approx 0.40$, $w_1'=38$, $w_2'=42$, and hence the attained significance level is approximately,

$$2P(Z>4.64)$$

which leads to the emphatic rejection of the *CSR* hypothesis.

Remark. We should note that even though we have used rectangular regions for reducing our test statistics, regions of any shape are amenable to this method as long as this region can be partitioned to rectangular strips of equal heights, but not necessarily equal lengths. In fact, no matter what the shape the region, we may contain it in a rectangular or square region and generate additional events, the number of which is proportional to the area of the added area. This is justified under the *CSR* hypothesis.

Another advantage of this method is that, we may allow the existence of some "excluded" areas in the study region, such as lakes in a breeding colony of birds or deserts in a country in studying population centers.

Acknowledgements

The helpful comments of the referees are gratefully acknowledged.

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