THE SPINLESS SALPETER EQUATION AND MESON DYNAMICS

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Abstract

Applying the variational method, the spinless reduced Bethe-Salpeter (RBS) equation is solved for the mesonic systems, and the mass spectra are obtained. The method is applied to the Hamiltonian with the Gaussian and hydrogen-type trial wave functions, and different potential models are examined. The results for the different potentials are in challenge in light mesons, while they are consistent in heavy mass region. In spite of this, the consistency for all mesons occurs in Power-law (Rosner, …), Logarithmic (Quigg and Rosner), $K(\Lambda_{\overline{MS}} = 0.4Gev)$ and $G(\Lambda_{\overline{MS}} = 0.5Gev)$ potentials and the theoretical results are in general in agreement with the experimental data. The inconsistency in light mesons has the origin in the nonrelativistic treatment of the potential derivations and the reduced form of the Bethe-Salpeter equation. The consistent and non-consistent interquark potentials show a distinct behaviour at both short and long distances.

Introduction

The most important task in *QCD* and hadron physics is to understand quark confinement. Due to antiscreening effect in *QCD* vacuum, which is the consequence of the color magnetization of the medium caused by the gluons, the interaction of the quarks becomes weaker at the distance $r \to 0$ (asymptotic freedom). On the other hand, the color interaction becomes stronger for large r (infrared slavery). As a result, perturbation techniques in *QCD* are not applicable in bound state systems and low-energy regions. All these may be understood by the non-abelian nature of *QCD* theory.

Keywords: Interquark potential, Quark confinement

Lattice *QCD* calculations show that the interquark potential for a heavy quarkonium in the static limit is well described by a linear confining potential, plus a short-ranged Coulomb potential [1]. The results seem to be consistent with the picture when the linear confinement potential is transformed as a Lorentz scalar, while the Coulomb potential stemmed from one gluon exchange, has feature of the Lorentz vector. Indeed a Lorentz vector potential in Dirac equation can not confine the particles and the confining term of the *QCD* potential must be a scalar which is the consequence of multigluon exchange. Although Lattice *QCD* calculations have had a resounding success in the bound states, but the method has some technical limitations [2,3]. In quantum field theory, a basic description for the bound states is the Bethe-Salpeter

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equation [4]. However, this equation in general is not soluble without important approximations and simplifications. Following the development in various mathematical formalisms for the relativistic treatment of scattering problems for two or many body systems by Dyson, Schwinger, Tomonaga and Feynman [5], Salpeter and Bethe extended the Feynman's formalism to bound state problems involving several particles. On the basis of Feynman's work, by a suitable choice of interpretation of the solutions of Dirac's equation, Bethe and Salpeter obtained

$$\begin{aligned}&(i\nabla_3^a - m_a)(i\nabla_4^b - m_b)\psi(3,4) = \\ &-i\int\int d\tau_1 d\tau_2 G(3,4;1,2)\psi(1,2)\end{aligned} \tag{1}$$

which is a relativistic integral-differential equation for two particles moving from points 1 and 2 to 3 and 4 respectively in four dimensional space under the influence of interaction $G(3,4;1,2)$ [6]. At the same year but before the Bethe and Salpeter's presentation, Gel-Mann and Low [7] had derived a similar integral equation on the basis of quantum field theory. $G(3,4;1,2)$ includes all powers of the coupling constant of the interaction and despite each term of the expansion can, in principle, be calculated, but no closed expression for $G(3,4;1,2)$ does exist. As a result expression (1) is not immediately applicable for bound state problem when the coupling constant is large. However, a further reduction is possible and a tractable approximate reduced form can be obtained. Assuming an instantaneous interaction [8], the *RBS* equation for two heavy relativistic particles is found to be

$$\begin{aligned}&E\varphi(\vec{p}) = [E_a(p) + E_b(p)]\varphi(\vec{p}) + \\ &\int d^3k \Lambda_+^a(\vec{p})\Lambda_+^b(\vec{p})G'(\vec{p},\vec{k})\varphi(\vec{k})\end{aligned} \tag{2}$$

where $\varphi(\vec{p})$ is the relative wave function of two particles in momentum space and the Fourier transform of $G'(\vec{p},\vec{k})$ is instantaneous potential in coordinate space, and the Casimir operator $\Lambda_\pm^i(\vec{p})$ is given by

$$\Lambda_\pm^i(\vec{p}) = \frac{\left[E_i(p) \pm H_i(\vec{p})\right]}{2E_i(p)} \qquad i = a,b \tag{3}$$

and also

$$E_i(p) = \left(m_i^2 + p^2\right)^{\frac{1}{2}} \tag{4}$$

$$H_a(\vec{p}) = \left(m_a\beta^a + \vec{p}.\alpha^a\right),\ H_b(\vec{p}) = \left(m_b\beta^b - \vec{p}.\alpha^b\right) \tag{5}$$

It is the aim of this paper to apply the spinless reduced Bethe-Salpeter equation to quark-antiquark bound states.

Potential Models

The *RBS* equation is a standard eigenvalue equation with solutions algebraically and numerically simpler than of the full equation. This equation may be cast into spin-dependent and spin-independent terms [9]. It is simply given by

$$H = \sqrt{p^2 + m_a^2} + \sqrt{p^2 + m_b^2} + U(r) \tag{6}$$

where $p^2 = -\nabla^2$ and

$$U(r) = V(r) + V_{SD}(r) \tag{7}$$

in which $V_{SD}(r)$ and $V(r)$ are spin-dependent and spin-independent potentials. In Equation (6), r and p are the relative coordinate and its conjugate momentum of two particles respectively, where m_a and m_b represent the constituent quark masses.

Due to the complicated nature of interaction mechanism in *QCD*, different methods have been employed in solving the problem of bound state systems. One of these approaches is the potential models which have had a remarkable success in the spin-independent domain in the case of heavy quarkonium systems. As we mentioned earlier the full version of the relativistic Bethe-Salpeter equation is not applicable and the kernel involved, is in general unknown, while the *RBS* equation (6) has the desired functional from with an interaction potential which may be deduced by nonrelativistic calculations. However, no one has yet completely succeeded in deriving the effective form of the interquark interactions from *QCD* even in the nonrelativistic limit. To fill this gap one may postulate that the gross features of the interaction may be simulated by a confining potential which assumes that the quarks in a bound core move independently and satisfies the asymptotic freedom. This is supported by the lattice *QCD* calculations which reveal the existence of a linear term in r for confinement. On the other hand one gluon exchange interaction gives rise a Coulomb-type contribution. Hence the simplest *QCD* motivated potential may be represented by a linear combination of Coulomb-type interaction and a linear confinement term [10],

$$V(r) = \frac{-\alpha_c}{r} + K_r \tag{8}$$

which is known as Cornell potential and the coefficients given by Eichten and Hagiwara are

$$K = \frac{1\,Gev^2}{(2.34)^2},\ \alpha_c = 0.52 \qquad Eichten,\ldots(Cornell1) \tag{9}$$

$$K = 0.19\,Gev^2,\ \alpha_c = 0.47 \quad Hagiwara,\ldots(Cornell2) \tag{10}$$

Different models have been presented by different authors. In this work we are going to deal with some of them.

Power-law potential of Martin (Power-law 1) [11];

$$V(r) = -8.064\,Gev + (6.898\,Gev)(r \times 1\,Gev)^{0.1}, \quad (1\,Gev^{-1} = 0.197\,fm) \tag{11}$$

Power-law potential of Rosner *et al.* (Power-law 2) [12];

$$V(r) = -0.772\,Gev + 0.801\frac{\left[(r \times 1\,Gev)^{\alpha} - 1\right]}{\alpha}, \quad \alpha = -0.12 \tag{12}$$

Logarithmic potential of Quigg and Rosner [13];

$$V(r) = -0.6635\,Gev + (0.733\,Gev)[\ln(r \times 1\,Gev)] \tag{13}$$

These potentials are phenomenological potentials in correspondence with the empirical evidences. Constituent masses for these potentials are given in Table 1. On the other hand on the basis of *QCD* concepts Buchmüller and Tye found a family of potentials with two free parameters [14]. Exploiting these results, Igi and Ono proposed the following potentials [15];

$$V(r) = V_{AF}(r) + ar \tag{14}$$

$$V(r) = V_{AF}(r) + rd\exp(-gr) + ar \tag{15}$$

Where the asymptotic freedom potential $V_{AF}(r)$ is given by

$$V_{AF}(r) = \frac{-16\pi}{25rf(r)}\left[1 + \frac{2\gamma_E + \frac{53}{75}}{f(r)} - \frac{462\ln(f(r))}{625f(r)}\right] \tag{16}$$

Table 1. Constituent masses and parameters used in different potential models. In all cases m_s=199 Mev and m_d=9.9 Mev

Potential	$\Lambda_{\overline{MS}}\,(Gev)$	**a** (Gev^2)	**g** (Gev)	**d** (Gev^2)	$\mathbf{m_c}\,(Gev)$	$\mathbf{m_b}\,(Gev)$
Cornell 1	-	-	-	-	1.84	5.18
Cornell 2	-	-	-	-	1.32	4.75
Power-law 1	-	-	-	-	1.80	5.174
Power-law 2	-	-	-		1.56	4.96
Logarithmic	-	-	-	-	1.50	4.906
Potential I: b=23.3	0.31	0.1585	-	-	1.494	4.874
Potential J: b=20	0.1	0.1733	0.3076	0.4344	1.134	4.563
	0.2	0.1587	0.3436	0.2550	1.322	4.731
	0.3	0.1443	0.3280	0.0495	1.471	4.868
	0.4	0.1387	2.903	0.582	1.515	4.910
	0.5	0.1391	2.955	1.476	1.514	4.911
Potential K: b=5	0.1	0.1762	0.2753	0.4720	1.120	4.551
	0.2	0.1734	0.3479	0.5362	1.267	4.684
	0.3	0.1615	0.4482	0.6020	1.416	4.815
	0.4	0.1389	0.6219	0.5632	1.604	4.986
	0.5	0.1137	1.0029	0.7368	1.748	4.118
Potential G: b=2	0.1	0.1755	0.2849	0.421	1.125	4.553
	0.2	0.1668	0.2948	0.352	1.264	4.679
	0.3	0.0956	0.0993	0.1729	1.450	4.849
	0.4	0.1375	0.2430	0.1311	1.535	4.924
	0.5	0.1425	0.4275	0.1811	1.583	4.972

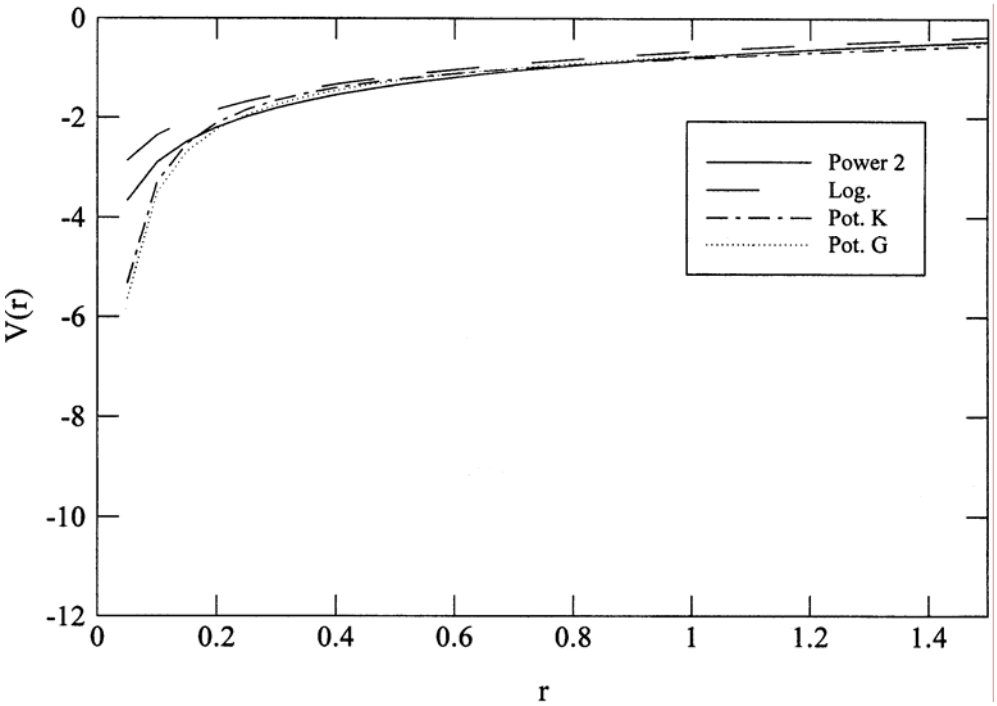

Figure 1. The consistent confining potentials. The radial distance is in the unit of 1/*Gev*, and the potential energy is in the unit of *Gev*.

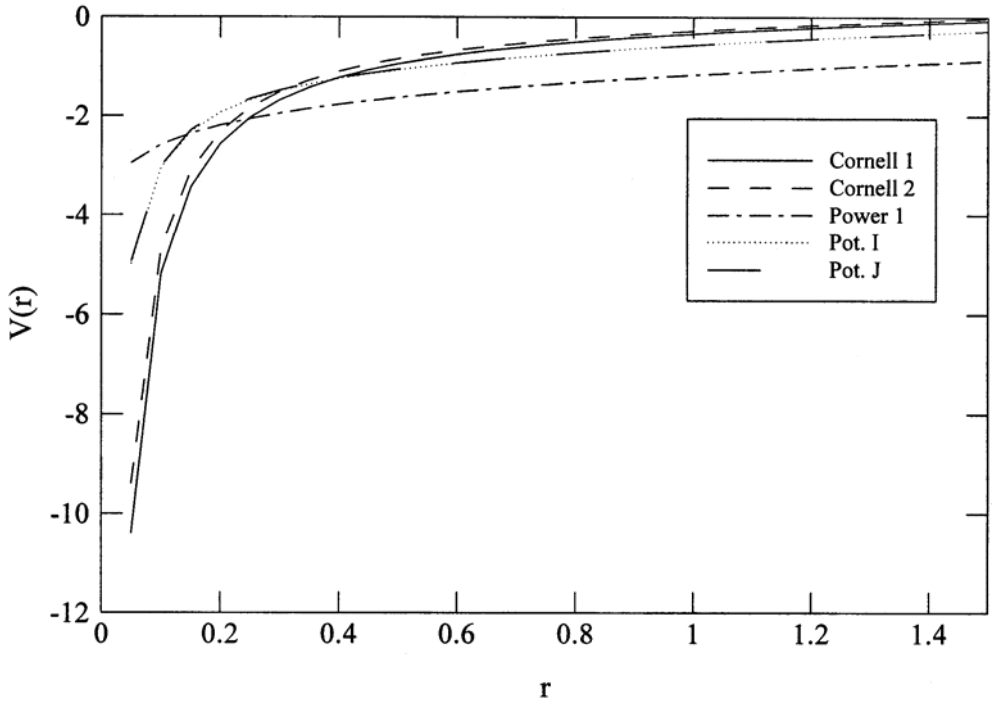

Figure 2. The non-consistent potentials. The radial distance is in the unit of 1/*Gev*, and the potential energy is in the unit of *Gev*.

Here $\gamma_E = 0.5772$ is the Euler's constant and a, b and $\Lambda_{\overline{MS}}$ are the free parameters and

$$f(r) = \ln\left[\frac{1}{\left(r\Lambda_{\overline{MS}}\right)^2} + b\right] \tag{17}$$

The potential (14) is called *the potential I* and the potential (15) with $b = 20$ is called *the potential J*, while the one with $b = 5$ is called *the potential K*. The potential (15) with $f(r)$ replaced by

$$f(r) = \ln\left[\frac{1}{(r\Lambda_{\overline{MS}})} + b\right]^2 \tag{18}$$

is known as *the potential G*.

Different parameters calculated by Igi and Ono are represented in Table 1. The extent of asymptotic freedom for the relevant potentials displayed in Figure 1 and Figure 2 may be compared.

Calculation Scheme

One of the powerful techniques in solving nonrelativistic bound state problems, is the variational method. The extensive application of the method rests on the existence of a definite lower bound to the energy eigenvalue spectrum.

The *RBS* equation does not possess the negative energy solutions, hence the variational method can be applied. The solutions are variationally stable. We apply the method to the Hamiltonian (6) with the Gaussian and hydrogen-type wave function [16] for a variety of the mesonic systems

$$\psi_G(r) = \left[\left(\frac{\mu}{\sqrt{\pi}}\right)^{\frac{3}{2}}\right]\exp\left(\frac{-\mu^2 r^2}{2}\right) \tag{19}$$

$$\psi_H(r) = \frac{2}{\sqrt{4\pi\nu^3}}\exp\left(\frac{-r}{\nu}\right). \tag{20}$$

Here μ and ν are the variational parameters and the subscripts G and H denote the Gaussian and hydrogen-type wave functions.

The Gaussian wave function emphasizes on confinement of the quarks and has a stronger fall off at large r, while the other trial wave function is more effective in short range due to Coulomb attraction. Consequently we try each trial function separately. The stationary value of the expectation value of the *RBS* Hamiltonian under the variation of the variational parameters determines the ground state energy spectrum. This can be achieved by minimizing

$$\langle H\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = E(\lambda) \tag{21}$$

with respect to the variational parameter λ, that is

$$\frac{\partial E(\lambda)}{\partial\lambda} = 0 \qquad at \quad \lambda = \lambda_{\min} \tag{22}$$

$E(\lambda)$ represents the mass of the system.

The Fourier transform of the trial wave functions (19) and (20) in momentum space which are needed in the matrix elements evaluations, are respectively given by

$$\varphi_G(p) = \frac{1}{\left(\sqrt{\pi}\mu\right)^{\frac{3}{2}}}\exp\left(\frac{-p^2}{2\mu^2}\right) \tag{23}$$

$$\varphi_H(p) = \frac{2\sqrt{2}\nu^{\frac{3}{2}}}{\pi\left(\nu^2 p^2 + 1\right)^2} \tag{24}$$

Results and Discussion

The results of variational calculations with two different trial wave functions are tabulated in Table 2 and Table 3 for a wide range of the mesonic systems. The mass spectra are obtained employing the spinless Bethe-Salpeter equation and they are compared with the experimental spin-averaged energies [17]. The best fit for the potentials J, K, and G occur at $\Lambda_{\overline{MS}} = 0.3\,Gev$, $\Lambda_{\overline{MS}} = 0.4\,Gev$, and $\Lambda_{\overline{MS}} = 0.5\,Gev$ respectively.

The results obtained by the Gaussian wave function are in general slightly higher than those obtained by the hydrogen-type wave function. The mass spectra in light and heavy-light sectors are in general inconsistent with the experimental findings. However, the consistency for all of the cases in Power-law (Rosner,…), $K(\Lambda_{\overline{MS}} = 0.4\,Gev)$, Logarithmic (Quigg and Rosner) and $G(\Lambda_{\overline{MS}} = 0.5\,Gev)$ potentials is remarkable. The agreement between variational and experimental results in these cases is encouraging and it can be deduced that the light quarkonium systems are much more sensitive to the form of the interaction and the nature of confinement. However, lower discrepancies between our computational results and the experimental data in light sectors in the case of hydrogen-type wave function emphasizes on the importance of the trial function.

A comparison between the consistent potentials displayed in Figure 1 and those shown in Figure 2 reveals an important characteristics of the confining potentials. These two groups of potentials behave completely different. We observe generally in the consistent potentials the tendency that the potentials which have higher (lower) values in the short range, have also higher (lower) values in the long range, while the situation is reversed in the second group of the

Table 2. The meson spectra predicted by a variety of the potentials for hydrogen-type trial wave function. The results are in *Gev* and *v* is the variational parameter

Model		**Meson**									
		$\mathbf{d\bar{d}}$	$\mathbf{s\bar{d}}$	$\mathbf{s\bar{s}}$	$\mathbf{c\bar{d}}$	$\mathbf{c\bar{s}}$	$\mathbf{c\bar{c}}$	$\mathbf{b\bar{d}}$	$\mathbf{b\bar{s}}$	$\mathbf{b\bar{c}}$	$\mathbf{b\bar{b}}$
Cornell 1	m	1.136	1.193	1.247	2.582	2.626	3.898	5.845	5.884	7.077	10.152
	v	2.073	1.995	1.925	1.605	1.549	1.033	1.310	1.260	0.804	0.552
Cornell 2	m	1.183	1.240	1.294	2.143	2.189	3.024	5.472	5.512	6.283	9.424
	v	2.075	2.001	1.933	1.625	1.570	1.229	1.360	1.309	0.951	0.629
Power-law 1	m	0.360	0.419	0.476	1.779	1.823	3.074	5.078	5.118	6.307	9.480
	v	2.215	2.106	2.012	1.561	1.504	1.105	1.353	1.305	0.915	0.714
Power-law 2	m	0.748	0.810	0.869	1.951	1.997	3.031	5.264	5.303	6.264	9.410
	v	2.415	2.250	2.118	1.602	1.532	1.115	1.317	1.264	0.888	0.658
Logarithmic	m	0.854	0.914	0.972	1.997	2.043	3.031	5.314	5.354	6.278	9.446
	v	2.314	2.183	2.070	1.660	1.593	1.167	1.360	1.308	0.942	0.705
Potential I: $\Lambda_{\overline{MS}} = 0.31$	m	0.950	1.006	1.061	2.071	2.117	3.100	5.361	5.401	6.321	9.460
	v	2.099	2.018	1.944	1.592	1.535	1.174	1.354	1.305	0.944	0.962
Potential J: $\Lambda_{\overline{MS}} = 0.3$	m	0.934	0.992	1.049	2.043	2.090	3.057	5.352	5.392	6.297	9.453
	v	2.181	2.088	2.006	1.630	1.571	1.195	1.378	1.325	0.955	0.702
Potential K: $\Lambda_{\overline{MS}} = 0.4$	m	0.703	0.761	0.818	1.936	1.982	3.065	5.236	5.277	6.298	9.459
	v	2.167	2.079	1.996	1.605	1.546	1.161	1.376	1.324	0.947	0.711
Potential G: $\Lambda_{\overline{MS}} = 0.5$	m	0.740	0.798	0.854	1.952	1.998	3.061	5.258	5.298	6.295	9.449
	v	2.163	2.075	1.992	1.603	1.544	1.152	1.368	1.316	0.927	0.678
Exp	m	0.667	0.794	1.004	1.973	2.075	3.067	5.313	5.410	–	9.448

Table 3. The meson spectra predicted by a variety of the potentials for Gaussian trial wave function. The results are in *Gev* and μ is the variational parameter

Model		**Meson**									
		$\mathbf{d\bar{d}}$	$\mathbf{s\bar{d}}$	$\mathbf{s\bar{s}}$	$\mathbf{c\bar{d}}$	$\mathbf{c\bar{s}}$	$\mathbf{c\bar{c}}$	$\mathbf{b\bar{d}}$	$\mathbf{b\bar{s}}$	$\mathbf{b\bar{c}}$	$\mathbf{b\bar{b}}$
Cornell 1	m	1.174	1.227	1.279	2.620	2.661	3.934	5.896	5.933	7.130	10.228
	μ	0.351	0.364	0.377	0.478	0.495	0.709	0.544	0.565	0.900	1.313
Cornell 2	m	1.217	1.270	1.322	2.174	2.217	3.048	5.516	5.553	6.321	9.480
	μ	0.352	0.365	0.377	0.451	0.466	0.601	0.530	0.550	0.767	1.166
Power-law 1	m	0.410	0.465	0.517	1.815	1.855	3.083	5.119	5.154	6.315	9.478
	μ	0.341	0.359	0.375	0.493	0.510	0.707	0.567	0.587	0.859	1.117
Power-law 2	m	0.806	0.864	0.920	2.000	2.042	3.056	5.318	5.353	6.289	9.425
	μ	0.308	0.330	0.351	0.475	0.495	0.692	0.577	0.600	0.874	1.204
Logarithmic	m	0.907	0.964	1.018	2.038	2.080	3.049	5.360	5.396	6.295	9.451
	μ	0.325	0.344	0.362	0.472	0.490	0.663	0.565	0.586	0.829	1.129
Potential I: $\Lambda_{\overline{MS}} = 0.3$	m	0.988	1.041	1.093	2.102	2.144	3.111	5.399	5.435	6.334	9.463
	μ	0.352	0.366	0.380	0.473	0.490	0.654	0.556	0.577	0.821	1.142
Potential J: $\Lambda_{\overline{MS}} = 0.3$	m	0.976	1.031	1.084	2.079	2.122	3.075	5.395	5.432	6.317	9.468
	μ	0.338	0.353	0.367	0.460	0.478	0.641	0.544	0.566	0.805	1.114
Potential K: $\Lambda_{\overline{MS}} = 0.4$	m	0.746	0.801	0.853	1.971	2.013	3.077	5.277	5.313	6.312	9.470
	μ	0.340	0.355	0.370	0.473	0.491	0.667	0.554	0.573	0.815	1.093
Potential G: $\Lambda_{\overline{MS}} = 0.5$	m	0.782	0.837	0.889	1.989	2.031	3.079	5.302	5.339	6.319	9.471
	μ	0.340	0.355	0.370	0.468	0.485	0.662	0.547	0.567	0.825	1.138
Exp	m	0.667	0.794	1.004	1.973	2.075	3.067	5.313	5.410	–	9.448

potentials, that is the potentials with higher values in the short range have lower values in the long range and vice versa. In comparison with the experimental data the Cornell potentials and the power-law potential of Martin with two different behaviour in the short range asymptotic region and long range confining region produce higher and lower mass spectra respectively.

Our calculation in real systems exhibit the validity of the *RBS* equation, also known as the nopair equation, when the criteria outlined by M. G. Olsson, S. Veseli and K. Williams are satisfied [18]. They have discussed on the conditions under which the difference in the results obtained by the *RBS* and the full Salpeter equations is nonsignificant. The critical factors are found to be the constituent mass, the physical state and the nature of the interaction. Hence, the inconsistency in light sectors for most of the potentials has root in the nonrelativistic treatment of the potential derivations as well as the other factors.

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