ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract

Here we consider some non-autonomous ordinary differential equations of order n and present some results and theorems on the existence of periodic solutions for them, which are sufficient conditions, section 1. Also we include generalizations of these results to vector differential equations and examinations of some practical examples by numerical simulation, section 2. For some special cases that extendibility of the solutions can be verified and under other suitable conditions, we show that the frequency of the periodic solutions can be arbitrary small†.

Introduction

In this paper we consider the following n - th order differential equation:

$$x^{(n)} + f(t, x, \dot{x}, \ldots, x^{(n-1)}) = e(t)$$

our goal is to present sufficient conditions for the existence of periodic solutions of the above system. Our method is based on considering the above system with some boundary conditions and constructing one operator by using Green's function. Then we use Shauder's fixed point theorem to show the existence of at least one solution that satisfies those conditions. By imposing suitable conditions we extend this solution periodically in the future. Then we generalized the obtained results to the space $\Re^m$ ($\Re$ is the set of real numbers) for a system of the form

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$$X^{(n)} + F(t, X, \dot{X}, \ldots, X^{(n-1)}) = E(t)$$

where $X \in \Re^m$ and F, E are vector functions of dimension m. Also we present sufficient conditions for the existence of periodic solutions for some important cases such as

$$x^{(n)} + \sum_{i=0}^{n-1} \varphi_i(t) f_i\left(x^{(i)}\right) = 0$$

and

$$x_+^{(n)} \varphi(t) \prod_{i=0}^{n-1} f_i\left(x^{(i)}\right) = 0$$

and their vector form counterparts. By extendibility of the solutions in the future, it is possible to choose the period of the function $e(l)$ arbitrarily large as well as the period of the periodic solutions.

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In addition, we examine some examples numerically. In this respect, we use the shooting method to obtain proper initial conditions that guarantee periodic oscillation of the systems.

Preliminaries and Notes

In sequel we denote $|u(t)|_T$ as $\max_{t\in[0,T]}|u(t)|$; $|u(t)|_\infty$ as $\max_{t\in\Re^+}|u(t)|$ and $\|u(t)\|_T$ as below

$$\|u(t)\|_T = \max\left\{|u(t)|_T, |\dot{u}(t)|_T, \ldots, |u^{(n-1)}(t)|_T\right\}$$

where n is the dimension of the above differential equation. Also $\delta(t)$ is Dirac delta function; $G(t, s)$ designate Green's function on which

$$G^{(i)}(t,s) = \frac{\partial^i G(t,s)}{\partial t^i}$$

and if $Ux(l)$ is a operator, then

$$U^{(i)}x(t) = \frac{\partial^i Ux(t)}{\partial t^i}$$

Inner product between two vectors X, Y denoted as $X \cdot Y$ in the usual sense. Let us consider the following non-linear equation

$$x^{(n)} + f(t, x, \dot{x}, \ldots, x^{(n-1)}) = e(t) \tag{1}$$

We assume $e(l)$ and f are continuous and smooth enough such that for any set of initial point $(t_0, x^0) \in \Re^{n+1}$ the existence and uniqueness of its solutions are held and the solutions can be extended in the $[0,\tau]\times\Re^n$ (τ is a finite positive real number).

Section 1: Sufficient Conditions

To establish sufficient conditions for the existence of periodic solutions of the system, we require the following theorem.

Theorem 1.1. Suppose $e(l)$ is bounded on $[0,\omega], \omega < \tau$ and $f(t, x, \dot{x}, \ldots, x^{(n-1)})$ is continuous with respect to its arguments, then there exists $x(t)$ solution of (1) in $[0,\omega]$ such that

$$x^{(i)}(0) + x^{(i)}(\omega) = 0, i = 0, \ldots, n-1 \tag{1.1}$$

To prove the above Theorem we require the following Lemma:

Lemma 1.1. Given

$$x^{(n)} = \delta(t-s)$$

then its corresponding Green's function with boundary conditions (1.1) is as follows

$$G(l,s) = \begin{cases} \sum_{i=1}^{n} a_i t^{n-i} & 0 \le t \le s \le \omega \\ \sum_{i=1}^{n} b_i t^{n-i} & 0 \le s \le t \le \omega \end{cases}$$

where, $(n-1)!|b_1| = \frac{1}{2}$, $(n-1)!|a_1| = -\frac{1}{2}$ and for $i > 1$

$$(n-i)!b_i = -\frac{(-1)^i}{2}\frac{s^{i-1}}{(i-1)!} - \sum_{j=1}^{i-1}\frac{(n-i)!}{(i-j)!}b_j\omega^{i-j} \tag{1.2}$$

$$(n-i)!a_i = -\frac{(-1)^i}{2}\frac{s^{i-1}}{(i-1)!} - \sum_{j=1}^{i-1}\frac{(n-i)!}{(i-j)!}b_j\omega^{i-j} \tag{1.3}$$

and we have

$$(n-i)!|a_i| \le \max\{1, \omega^{i-1}\}, (n-i)!|b_i| \le \max\{1, \omega^{i-1}\} \tag{1.4}$$

Proof. By applying the boundary conditions (1.1) to $G(l, s)$ we have $(n-1)!|b_1| + (n-1)!|a_1| = 0$ and for $i > 1$

$$(n-i)!a_i + (n-i)!b_i = -\sum_{j=1}^{i-1}\frac{(n-j)!}{(i-j)!}b_j\omega^{i-j} \tag{1.5}$$

and by continuity conditions it follows $(n-1)!|b_1| - (n-1)!|a_1| = -1$ and for $i > 1$

$$(n-i)!a_i - (n-i)!b_i = -\sum_{j=1}^{i-1}\frac{(n-i)!}{(i-j)!}a_j s^{i-j} + \sum_{j=1}^{i-1}\frac{(n-j)!}{(i-j)!}b_j s^{i-j}$$

but by using mathematical induction on the above relation we obtain

$$(n-i)!a_i - (n-i)!b_i = (-1)^i \frac{s^{i-1}}{(i-1)!} \tag{1.6}$$

Now from (1.5) and (1.6) the recursive relations of (1.2) and (1.3) are obtained. Now we show (1.4). By induction again we have if $\omega \ge 1$ and if $(n-i+1)!|b_{i-1}| \le \omega^{i-2}$ then for $i > 1$

$$(n-i)!|b_i| \le \frac{\omega^{i-1}}{2}\left\{\sum_{j=1}^{i-1}\frac{1}{(i-j)!} + \frac{1}{(i-1)!}\right\} \le \omega^{i-1}$$

and similarly

$$(n-i)!|a_i| \le \frac{\omega^{i-1}}{2}\left\{\sum_{j=1}^{i-1}\frac{1}{(i-j)!}+\frac{1}{(i-1)!}\right\} \le \omega^{i-1}$$

Therefore for any i we have

$$(n-i)!|b_i| \le \omega^{i-1}$$
$$(n-i)!|a_i| \le \omega^{i-1}$$

If $\omega < 1$ then one can show similarly that

$$(n-i)!|a_i| \le 1, (n-i)!|b_i| \le 1$$

Therefore, Lemma 1.1 is proved.

Proof of Theorem 1.1. Let $G(t, s)$ be a Green's function as defined in the above Lemma, and define Banach space B as follows

$$B = \left\{x(t) \in C^n[0,\omega] \Big| \left| x^{(i)}(t) \right|_\omega \le c, i = 0,\dots,n-1\right\}$$
$$\|x(t)\| = \|x(t)\|_\omega$$

We define operator $Ux(l)$ on B as

$$Ux(t) = \int_0^\omega G(t,s)\{e(s) - f(s, x(s), \dot{x}(s), \dots, x^{(n-1)}(s))\}ds$$

and also

$$U^{(i)}x(t) = \int_0^\omega G^{(i)}(t,s)\{e(s) - f(s, x(s), \dot{x}(s), \dots, x^{n-1)}(s))\}ds$$

Now we can obtain $\left|U^{(i)}x(t)\right|_\omega$ as follows

$$\left|U^{(i)}x(l)\right|_T \le (k+M)\int_0^\omega \left|G^{(i)}(l,s)\right|ds \le 3(k+M)\omega^{n-i}, \quad i = 0,\dots,n-1$$

where $\left|e(t)\right|_\omega = k$ and $\left|f(t,x(t),\dot{x}(t),\dots,x^{(n-1)}(t))\right|_\omega = M$. For M small enough, indeed

$$3(k+M)\max\{1,\omega^n\} \le c$$

then

$$\left|U^{(i)}x(t)\right|_\omega \le c, i = 0,\dots,n-1$$

so $Ux(t)$ is a map from B into itself and completely continuous operator.

Then from Schauder's theorem it follows that there exists at least one fixed point of $Ux(l)$, i.e. there exists $x(l)$ on $[0,\omega]$ such that

$$x(l) = \int_0^\omega G(l,s)\{e(s) - f(s, x(s), \dot{x}(s), \dots, x^{(n-1)}(s))\}ds$$

but this $x(t)$ is a solution of system (1) and hence the theorem is proved.

Corollary 1.1. If in addition to assumptions of the above theorem we assume $e(t)$ and $\varphi(t)$ are 2ω-periodic and

$$\begin{cases} e(t+\omega) = -e(t) \\ f(t+\omega, -u_1, \dots, -u_n) = -f(t, u_1, \dots, u_n), \end{cases} \quad (u_1,\dots,u_n) \in \Re^n$$

then there exists at least one 2ω-periodic solution for system (1) such that

$$\int_0^{2\omega} x(t)dt = 0$$

Proof. Because of the existence of $x(l)$ on $[0,\omega]$, we can extend $x(l)$ to the closed interval $[0,2\omega]$ as follows

$$z(t) = \begin{cases} x(t) & 0 \le t \le \omega \\ -x(t+\omega) & -\omega \le t \le 0 \end{cases}$$

By using the above assumptions, it is obvious that

$$z^{(n)} + f(t, z, \dot{z}, \dots, z^{(n-1)}) = e(t)$$

and

$$z^{(i)}(0^+) = z^{(i)}(0^-)$$
$$z^{(i)}(-\omega) = z^{(i)}(\omega)$$

so $z(t)$ is a solution of (1), that can be periodically extended in the future and furthermore we have

$$\int_0^{2\omega} z(t)dt = 0$$

Remark. If in system (1) we have $e(l) = 0$, then results of theorem 1.1 and the above corollary remain valid.

Let us now consider some examples.

Example 1.1. Let us consider the following system:

$$x^{(n)} + \varphi(t)f(x, \dot{x}, \dots, x^{(n-1)}) = 0 \tag{1.7}$$

with conditions of $\varphi(t)$ is 2ω-periodic and

$$\varphi(l+\omega)f(-u_1,\ldots,-u_n)=-\varphi(l)f(u_1,\ldots,u_n),\quad (u_1,\ldots,u_n)\in\Re^n$$

Now assume that $f(x,\dot{x},\ldots,x^{(n-1)})$ is a polynomial of $x,\dot{x},\ldots,x^{(n-1)}$ and its minimum degree is greater than one; then there exists α_j such that

$$\left|f\left(x(t),\dot{x}(t),\ldots,x^{(n-1)}(t)\right)\right|_\omega \le \sum_{i>1}\alpha_j c^j$$

and the solutions of the system can be extended in the future and if $|\varphi(t)|_\omega=\mu/3$, then for any pair of μ,ω there exists c such that

$$\mu\omega^n M \le c$$

where $M=\left|f\left(x(t),\dot{x}(t),\ldots,x^{(n-1)}(t)\right)\right|_\omega$. In fact it suffices to choose $c << 1$ such that above inequality holds.

Example 1.2. Consider the following equation

$$x^{(n)}+\varphi(t)\prod_{i=0}^{n-1}f_i\left(x^{(i)}\right)=0$$

where

$$f_i(u)=f_i(-u), i=0,\ldots,n-2$$
$$f_{n-1}(u)=f_{n-1}(-u)$$

and $\varphi(t)$ is 2ω-periodic real function of t such that

$$\varphi(l+\omega)=\varphi(l)$$

then if $|\varphi_n(l)|_\omega=\mu/3$ is small enough, indeed if

$$\mu\omega^n\prod_{i=0}^{n-1}M_y\le c$$

where $M_i=\left|f_i x^{(i)}(t))\right|_\omega$ then the above system has 2ω-periodic solution, or if at least one of the $f_i(x^{(i)})$ is a polynomial of $x^{(i)}$ with minimum degree greater then one, ad if the solutions of the system can be extended in the future, then for any pair of μ,ω there exists one 2ω-periodic solution for the above system. Extendibility of the solutions of the above system can be established for example by the following Lemma.

Lemma 1.2. For given $T>0$ suppose $e(t)$ is bounded and

$$x^{(n-1)}f(t,x,\dot{x},\ldots,x^{(n-1)})>0$$

on $t\in[0,\mathrm{T}]$, then system (1) has no escape time in the interval.

Furthermore, if such conditions hold for any $T, T>0$ then solutions can be extended in the future.

Proof. Let $x^{(n-1)}=y$ and $V=y^2+1$ be a positive function and $|e(t)|_T=k/2$. We have

$$\dot{V}=2x^{(n-1)}e(t)-2x^{(n-1)}f(t,x,\dot{x},\ldots,x^{(n-1)})$$
$$\le k\left|x^{(n-1)}\right|\le ky^2$$

It is obvious that

$$\left|x^{(n-1)}(t)\right|_T\le e^{kT/2}$$

hence it follows that

$$\left|x^{(n-i)}(t)\right|_T\le\sum_{j=1}^{i}\alpha_j T^{i-j}, i=1,\ldots,n$$

where α_j are constants and $\alpha_1=e^{kT/2}/(i-j)!$.

Now it follows that if the above assumptions hold for any $T\in\Re$ then

$$\left|x^{(n-i)}(t)\right|_T<\infty, i=1,\ldots,n$$

Example 1.3. Also for the following system

$$x^{(n)}+\sum_{i=0}^{n-1}\varphi_i(t)f_i\left(x^{(i)}\right)=0$$

we can obtain similar results for existence of periodic solution. For example if $|\varphi_i(t)|_\omega=\mu_i$ and $\mu'/3=\max\{\mu_0,\ldots,\mu_{n-1}\}$ and $\left|f_i(x^{(i)}(t))\right|_\omega=M_i$ and $m'=\max\{M_0,\ldots,M_{n-1}\}$ then sufficient condition for existence of at least one periodic solution is

$$n\mu'\omega^n m'\le c$$

Similarly if f_i is a polynomial with minimum degree greater than one, then for any value of μ there exists one periodic solution and if the solutions can be extended in the future, then for any values ω there exist 2ω-periodic solutions. Extendibility of the solutions of the above system for example can be established by Lemma 1.3.

Lemma 1.3. The solutions of system (1) have no escape time if there exist K_1 and K_2 such that

$$|f(t, y_1, y_2, \ldots, y_n)| \le K_1 \left(\sum_{i=0}^{n} y_i^2 + K_2 \right)^{1/2}$$

Proof. Similar to Lemma 1.2 and definition of a similar function V and with accordance to the above condition, there exist positive constants k_1 and k_2 such that

$$\dot{V} \le k_1 V + k_2$$

Notes. Given D is the region of origin and assumptions of Lemma 1.2, and Lemma 1.3 are held on complement of D, D^c then the above result remains valid. Also in Lemma 1.3, K_1 and K_2 can be continuous time functions. In fact in this case there exist continuous functions $k_1(t)$ and $k_2(t)$ such that

$$\dot{V} \le k_1(t) V + k_2(t)$$

that implies extendibility of the solutions.

Example 1.4. In [3], presented the sufficient conditions for the existence of periodic solutions of the generalized version of the Reissig's equation of the general form:

$$x^{(n)} + \sum_{i=1}^{n-1} \varphi_i \left(x^{(n-i-1)} \right) x^{(n-1)} + f(x) = p(t) \qquad (R)$$

by imposing different boundary conditions for our conditions and other suitable conditions. With respect to our results obtained above, we can present other conditions that are weakly respect to [3].

Theorem 1.2. Suppose $p(l)$ is 2ω-periodic and

$$\begin{cases} (i) & \varphi_i(u) = \varphi_i(-u) \\ (ii) & f(u) = -f(-u) \\ (iii) & p(t+\omega) = -p(t) \\ (iv) & |\varphi_i(u)| \le \alpha |u|, a > 0 \\ (v) & |f(u)| \le \beta |u|, \beta < \frac{1}{3\max\{1,\omega^n\}} \\ (vi) & |p(t)|_\omega < \frac{1}{n-1}\left(\beta - \frac{1}{3\max\{1,\omega^n\}} \right)^2 \end{cases}$$

then the system (R) has at least one 2ω-periodic solution that

$$\int_0^{2\omega} x(t)dt = 0$$

Proof. By the last three conditions ((iv),(v),(vi)) and obtained results, there exists at least one solution $x(t)$ of the system (R) such that boundary conditions 1.1 are satisfied for this solution; and by the first three conditions ((i),(ii),(iii)), this solution can be extended in the future periodically. In fact for the existence of desired solution for the system (R) in the $[0,\omega]$ we must have

$$3(|p(l)|_\omega + M)\max\{1, \omega^n\} \le c$$

where

$$M = \left| \sum_{i=1}^{n-1} \varphi_i(x^{(n-i-1)}) x^{(n-i)} + f(x) \right| \le (n-1)\alpha c^2 + \beta c$$

then we obtain the following inequality

$$\alpha c^2 + \left(\beta - \frac{1}{3\max\{1,\omega^n\}} \right) \frac{c}{n-1} + \frac{k}{n-1} \le 0$$

where $k = |p(l)|_\omega$ that has two distinct real roots such that at least one of them is positive.

Now we generalize the obtained results to the vector differential equations as below.

Section 2: Generalization

Let us consider the following system:

$$X^{(n)} + F(t, X, \dot{X}, \ldots, x^{(n-1)}) = E(t) \qquad (2.1)$$

where $X(t) \in \Re^m$ is a vector and $F = (f_1, \ldots, f_m)^T$ is a vector function that

$$f_i : \Re \times \Re^{nm} \to \Re$$

and $E(t) = (e_1(t), \ldots e_m(t))^T$ is a vector function of dimension m. Similar to system (1), assume each f_i, e_i are continuous and smooth enough such that existence and uniqueness of solutions for any set of $(t_0, X_0, \ldots, X_0^{(n-1)})$ are held and can be extended in the $[0,\omega] \times \Re^{nm}$. Now we can prove a Theorem similar to Theorem 1.1 that establishes sufficient conditions for existence of solutions that satisfy certain boundary conditions.

Theorem 2.1. Let us assume $E(t)$ is bounded on $[0,\omega]$ such that

$$x_i^{(j)}(0) + x_i^{(j)}(\omega) = 0, i = 1, \ldots, m, j = 0, \ldots, n-1 \qquad (2.2)$$

Proof. Let $G(l, s)$ be as one of Lemma 1 and define Banach space B as

$$B = \{(t) = (X(t), \ldots, X^{(n-1)}(t)) : \|(t)\|_\omega \le c\}$$
$$\|\cdot\|_\omega = \max\left\{\max_i \{|x_i(t)|_\omega\}, \ldots, \max_i \left\{\left|x_i^{(n-1)}(t)\right|_\omega\right\}\right\}$$

Now define operator Γ on B as

$$\Gamma((t)) = \left(U(t), \ldots, U^{(n-1)}(t)\right)$$

where

$$U^{(i)}(t) = \int_0^\omega G^{(i)}(t,s)\left(E(s) - F(s, X(s), \ldots, X^{(n-1)}(s)\right)ds$$

Now we compute $\left|U^{(i)}(t)\right|_\omega$ as

$$\left|U^{(i)}(t)\right|_\omega \le (K_0 + M_0)\int_0^\omega \left|G^{(i)}(t,s)\right| ds$$
$$\le 3(K_0 + M_0)\max\{1, \omega^{n-i}\}$$

where

$$K_0 = \max_i \{|e_i(t)|_\omega\},$$
$$M_0 = \max_i \left\{\left|f_i(t, X(t), \ldots, X^{(n-1)}(t))\right|_\omega\right\}$$

then from Schauder's theorem, Γ has at least one fixed point on B if

$$3(K_0 + M_0)\max\{1, \omega^n\} \le c$$

i.e. there exists $X(t)$ solution of system (1.1) such that

$$X(t) = \int_0^\omega G(t,s)\left(E(s) - F(s, X(s), \ldots, X^{(n-1)}(s)\right)ds$$

Corollary 2.1. If in addition to the above assumptions we assume $E(t)$ and $F(t, X, \ldots, X^{(n-1)})$ are 2ω-periodic with respect to t and

$$\begin{cases} E(t+\omega) = -E(t) \\ F(t+\omega, -U_1, \ldots, -U+n) = -F(t, U-1, \ldots, U_n) \end{cases}$$

then system (2.1) has at least 2ω-periodic solution such that

$$\int_0^{2\omega} x_i(t)dt = 0, i = 1, \ldots, m$$

Proof. Similarly define

$$Z(t) = \begin{cases} X(t) & 0 \le t \le \omega \\ -X(t+\omega) & -\omega \le t \le 0 \end{cases}$$

that satisfies (1.1), i.e.:

$$Z^{(n)} + F(t, Z, \ldots, Z^{(n-1)}) = E(t)$$

and

$$z_i^{(j)}(0^+) = z_i^{(j)}(0^-)$$
$$z_i^{(j)}(-\omega) = z_i^{(j)}(\omega)$$

So $Z(t)$ is a solution of (2.1) that can be extended periodically in the future and

$$\int_0^{2\omega} Z(t)dt = 0$$

Example 2.1. Similar to example (1.2), let us consider the following vector differential equation:

$$x_j^{(n)} + \varphi_j(t)\prod_{i=0}^{n-1} f_{i,j}\left(X^{(i)}\right) = 0, j = 1, \ldots, m$$

where $X \in \Re^m$; $\varphi_j(t)$ are 2ω-periodic real functions and $f_{i,j} : \Re^m \to \Re$. Let us assume

$$\begin{cases} \varphi_j(l+\omega) = \varphi_j(l) \\ f_{i,j}(U) = f_{i,j}(-U), i = 0, \ldots, n-2 \\ f_{n-1,j}(U) = -f_{n-1,j}(-U) \end{cases}$$

and $f_i, j(X)^{(i)})$ be polynomials components $X^{(i)}$ such that at least one of them has a minimum degree greater than one. Then for any pair of μ, ω there exists at least one 2ω-periodic function.

By the extendibility of the solutions that can be verified by the Lemma 2.1 and the result of Example 1.1, for any pair of μ, ω there exists $c > 0$ that

$$\mu_0 M_0 \max\{1, \omega^n\} \le c$$

where $\mu_0 / 3 = \max_i \left\{\left|\varphi_j(t)\right|_\omega\right\}$ and

$$M_0 = \max_j \left\{\prod_{i=0}^{n-1} M_{i,j}\right\}, \quad M_{i,j} = \left|f_{i,j}\left(X^{(i)}\right)\right|_\omega$$

The extendibility of the solutions of the system can be established by the following Lemma.

Lemma 2.1. Let there exists $T > 0$ such that

$$\left\langle X^{(n-1)}, F(t, X, \dot{X}, \ldots X^{(n-1)})\right\rangle > 0$$

then the vector system has no escape time in the interval $[0, T]$.

Furthermore, if such conditions hold for any T then the solutions of the system (2.1) can be extended in the future.

Proof. By definition of $X^{(n-1)} = Y$ and $V(t)$ as

$$V(t) = \frac{1}{2}\langle Y, Y\rangle + m$$

we can compute $\dot{V}$ as

$$\dot{V}(t) = \langle Y, \dot{Y}\rangle \le \langle Y, E\rangle - \langle Y, F\rangle \le k\left(\tfrac{1}{2}\langle Y, Y\rangle + m\right) = kV(t)$$

where $k = \max\{|e_i(t)|_T, i = 1,\ldots,m\}$, that we obtain

$$\left|x_i^{(j)}(t)\right|_T < \infty, j = 0,\ldots,n-1, i = 1,\ldots m$$

Furthermore if such conditions hold for any $T \in \Re^+$ then the solutions can be extended in the future.

Example 2.2. Similar to Example 1.3, let us consider the following vector differential equation

$$x_j^{(n-1)} + \sum_{i=0}^{n-1} \varphi_{i,j}(t) f_{i,j}\left(X^{(i)}\right) = 0, j = 1,\ldots,m$$

where $X \in \Re^m$, $\varphi_{i,j}(t)$ are 2ω-periodic functions and $f_{i,j} : \Re^m \to \Re$. Let us assume

$$\varphi_{i,j}(t+\omega) f_{i,j}(-U) = -\varphi(t) f_{i,j}(U)$$

then the above system has at least one 2ω-periodic solution if

$$\mu_0 M_0 \max\{1, \omega^n\} \le c$$

where

$$\mu_j = \max_i\left\{\left|\varphi_{i,j}(t)\right|_\omega\right\} \quad , \quad \mu_0 / 3 = \max_j\{\mu_j\}$$

$$M_{i,j} = \left|f_{i,j}\left(X^{(i)}\right)\right|_\omega \quad , \quad M_0 = \max_j \left|\sum_{i=0}^{n-1} M_{i,j}\right|$$

Now in addition to the above assumption, let us assume the solutions of the system can be extended in the future, then for any values of ω, there exists sufficiently small μ such that the above sufficient conditions hold. In fact the extendibility of the solutions of the above vector system can be established, for example, by the following Lemma.

Lemma 2.2. The sufficient condition for extendibility of the solutions of system (2.1) is the existence of constants K_{1j} and K_{2j} such that

$$\left|f_j(t, Y_1,\ldots,Y_n\right| \le K_{1j}\left(\sum_{i=1}^{n}\langle Y_i, Y_i\rangle + K_{2j}\right)^{1/2}$$

where $X = Y_1$, $X' = Y_2 \ldots X^{(n-1)} = Y_n$

Proof. By definition of positive definite function V as below

$$V = \sum_{i=1}^{n}\langle Y_i, Y_i\rangle$$

we can compute $\dot{V}$ as following

$$\dot{V} = \sum_{i=1}^{n} 2\left\langle Y_i, \dot{Y}_i\right\rangle = \sum_{i=1}^{n-1} 2\langle Y_i, Y_{i+1}\rangle + 2\left\langle Y_n, \dot{Y}_n\right\rangle$$

$$\le \sum_{i=1}^{n-1}\{\langle Y_i, Y_i\rangle + \langle Y_{i+1}, Y_{i+1}\rangle\} + 2\left\langle Y_n, \dot{Y}_n\right\rangle$$

$$\le 2V + 2\langle Y_n, \|F\|\rangle \le 2V + \langle Y_n, Y_n\rangle + \langle \|F\|, \|F\|\rangle$$

So there exist positive constants k_1 and k_2 such that

$$\dot{V} \le k_1 V + k_2$$

that implies extendibility of the solutions in the future. Notice that in Lemma 2.1 and 2.2, the conditions can be held on the complement of a bounded region of origin. Furthermore in Lemma 2.2, K_{1j} and K_{2j} can be continuous time functions. In this case the K_1 and K_2 are continuous time functions.

Example 2.3. Similar to Example 1.4, we consider vector differential equation of the generalized version of the Reissig's equation:

$$x_j^{(n)} + \sum_{i=1}^{n-1} \phi_{i,j}\left(X^{(n-i-1)}\right) x_j^{(n-i)} + f_j(X) = p_j(t), j = 1,\ldots,m \qquad (VR)$$

where $X \in \Re^m$; $f_i, \phi_{i,j} : \Re^m \to \Re$ are two real valued functions and $p_j(l)$ is a 2ω-periodic time function.

Theorem 2.2. Suppose $P(l)$ is 2ω-periodic vector function and

$$\begin{cases} (i) \quad \phi_{i,j}(U) = \phi_{i,j}(-U) \\ (ii) \quad F(U) = -F(-U) \\ (iii) \quad P(t+\omega) = -P(t) \\ (iv) \quad |\phi_{i,j}(U)| \le \alpha_j \|U\| \\ (v) \quad |f_j(U)| \le \beta_j \|U\|, \beta_0 = \max_j\{\beta_j\}, \beta_0 < \frac{1}{3\max\{1,\omega^n\}} \\ (vi) \quad |p_j(t)|_\omega \le \frac{1}{n-1}\left(\beta_0 - \frac{1}{3\max\{1,\omega^n\}}\right)^2 \end{cases}$$

where $P(l) = (p_1(l),\ldots,p_m(l))$ and $F(U) = (f_1(U),\ldots, f_m(U))$, then the system (VR) has at least one 2ω-periodic solution that

$$\int_0^{2\omega} x_i(t)dt = 0$$

Proof. With respect to the obtained result for the vector equations for the existence of at least one periodic solution we must have

$$3(K_0 + M_0)\max\{1,\omega^n\} \le c$$

where $K_0 = \max_j\{|p_j(t)|_\omega\}$ and

$$M_0 = \max_j\left\{\left|\sum_{i=1}^{n-1}\phi_{i,j}\left(X^{(n-i-1)}\right)x_j^{(n-i)} + f_j(X)\right|_\omega\right\}$$
$$\le \max_j\{(n-1)\alpha_j c^2 + \beta_j c\}$$

Similar to Theorem 1.2, the last three conditions ((*iv*),(*v*),(*vi*)), guarantee the existence of $X(t)$ in $[0,\omega]$ such that boundary conditions (2.2) are satisfied and the first three conditions guarantee the extendibility of this solution periodically in the future.

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