J. Sci. I. R. Iran **Vol. 12, No. 2, Spring 2001**

FUZZY IDEALS OF NEAR-RINGS WITH INTERVAL VALUED MEMBERSHIP FUNCTIONS

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Abstract

In this paper, for a complete lattice $\mathcal{L}$, we introduce interval-valued $\mathcal{L}$-fuzzy ideal (prime ideal) of a near-ring which is an extended notion of fuzzy ideal (prime ideal) of a near-ring. Some characterization and properties are discussed.

1. Introduction

Zadeh in [19] introduced the concept of a fuzzy subset of a non-empty set X as a function from X to [0,1]. Goguen in [10] generalized the fuzzy subset of X, to $\mathcal{L}$-fuzzy subset, as a function from X to a lattice $\mathcal{L}$.

Since Rosenfeld [18] in 1971 introduced the concept of fuzzy subgroups following Zadeh, fuzzy algebra theory has been developed by many researchers. Liu [12] defined the fuzzy ideals of a ring and discussed the operations on fuzzy ideals. Mukherjee and Sen [16], Malik and Mordeson [16], Mashinchi and Zahedi [14], Zahedi [21], shown the meaning of the fuzzy prime ideals and its nature. The notion of fuzzy ideals and its properties were applied to various areas: distributive lattice [2], BCK-algebra [17], hyperrings [6,8], near-rings [1,11], hypernear-rings [7].

In 1975, Zadeh [20] introduced the concept of interval-valued fuzzy subsets (in short written by i-v fuzzy sets), where the values of the membership functions are intervals of numbers instead of the numbers. In [4], Biswas defined interval-valued fuzzy subgroups of the same nature of Rosenfeld's fuzzy subgroups.

In this paper, for a complete lattice $\mathcal{L}$, we define Interval-valued $\mathcal{L}$-fuzzy ideals (prime ideals) of a near-ring, and we obtain an exact analogue of fuzzy ideals. In particular, we show there exists a one-to-one correspondence between the set of all f-invariant i-v $\mathcal{L}$-fuzzy prime ideals of R and the set of all i-v $\mathcal{L}$-fuzzy prime ideals of R', where R and R' are near-rings and f is a homomorphism from R onto R'.

Keywords: Fuzzy set; Near-ring; Fuzzy ideal; Level set

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2. Basic Definitions

From now on this paper $\mathcal{L}$ is a complete lattice [3], i.e. there is a partial order $\leq$ on $\mathcal{L}$ such that, for any $S \subseteq \mathcal{L}$, infimum of S and supremum of S exist and these will be denoted by $\bigwedge_{s\in S}\{s\}$ and $\bigvee_{s\in S}\{s\}$, respectively. In particular for any elements $a,b\in\mathcal{L}$, in $f\{a,b\}$ and $\sup\{a,b\}$ will be denoted by $a\wedge b$ and $a\vee b$, respectively. Also, $\mathcal{L}$ is a ditributive lattice with a least element 0 and a greatest element 1. If $a,b\in\mathcal{L}$; we write $a\geq b$ if $b\leq a$, and $a>b$ if $a\geq b$ and $a\neq b$.

Definition 2.1. Given two elements $a,b\in\mathcal{L}$ with $a\leq b$, we define the following closed interval set:

$$[a,b]=\{c\in\mathcal{L}\,|\,a\leq c\leq b\}.$$

Suppose $\mathcal{D}(\mathcal{L})$ denotes the family of all closed intervals of $\mathcal{L}$.

Definition 2.2. Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ and $I_i = [a_i, b_i]$ be elements of $\mathcal{D}(\mathcal{L})$ then we define

$$I_1 \wedge I_2 = [a_1 \wedge a_2, b_1 \wedge b_2],$$

$$I_1 \vee I_2 = [a_1 \vee a_2, b_1 \vee b_2],$$

$$\bigwedge_i \{I_i\} = [\bigwedge_i \{a_i\}, \bigwedge_i \{b_i\}],$$

$$\bigvee_i \{I_i\} = [\bigvee_i \{a_i\}, \bigvee_i \{b_i\}].$$

We call $I_2 \le I_1$ if and only if $a_2 \le a_1$ and $b_2 \le b_1$.

Definition 2.3. Let X be a non-empty set. An $\mathcal{L}$-fuzzy subset F defined on X is given by

$$F = \{(x, \mu_F(x) \mid x \in X\}, \text{ where } \mu_F : X \to \mathcal{L}.$$

Definition 2.4. Let X be a non-empty set. An interval-valued $\mathcal{L}$-fuzzy subset F defined on X is given by

$$F = \{(x, [\mu_F^L(x), \mu_F^U(x)]) \mid x \in X\},$$

where μ_F^L and μ_F^U are two $\mathcal{L}$-fuzzy subsets of X such that $\mu_F^L(x) \le \mu_F^U(x)$ for all $x \in X$.

Suppose $\hat{\mu}_F(x) = [\mu_F^L(x), \mu_F^U(x)]$. If $\mu_F^L(x) = \mu_F^U(x) = c$ where $0 \le c \le 1$, then we have $\hat{\mu}_F(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $\mathcal{D}(\mathcal{L})$. Thus $\hat{\mu}_F(x) \in \mathcal{D}(\mathcal{L})$ for all $x \in X$. Therefore the i-v fuzzy subset F is given by

$$F = \{(x, \hat{\mu}_F(x)) \mid x \in X\}, \text{ where } \hat{\mu}_F : X \to \mathcal{D}(\mathcal{L}).$$

Definition 2.5. Let f be a mapping from a set X into a set Y. Let A be an i-v $\mathcal{L}$-fuzzy subset of X. then the image of A, i.e., $f[A]$ is the i-v fuzzy subset of Y with the membership function defined by

$$\hat{\mu}_{f[A]}(y) = \begin{cases} \bigvee_{z \in f^{-1}(y)} \{\hat{\mu}_A(z)\} & \text{if } f^{-1}(y) \neq \emptyset \\ [0,0] & \text{otherwise} \end{cases} \quad \text{for all } y \in Y$$

Let B be an i-v $\mathcal{L}$-fuzzy subset of Y. Then the inverse image of B, i.e., $f^{-1}[B]$ is the i-v $\mathcal{L}$-fuzzy subset of X with the membership function given by

$$\hat{\mu}_{f^{-1}[B]} = \hat{\mu}_B(f(x)) \quad \text{for all } x \in X.$$

Definition 2.6. Let X and Y be any two non-empty sets and $f : X \to Y$ be any function. An i-v $\mathcal{L}$-fuzzy subset of F of X is called f-invariant if

$$f(x) = f(y) \Rightarrow \hat{\mu}_F(x) = \hat{\mu}_F(y), \quad \text{where } x, y \in X.$$

Definition 2.7. A non-empty set R with two binary operations $+$ and $\cdot$ is called a near-ring [5,15] if

1) $(R,+)$ is a group,

2) $(R,\cdot)$ is a semigroup,

3) $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

To be more precise, they are left near-rings because the left distributive law is satisfied. We will use the word near-ring to mean left near-ring. We denote xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for all $x \in R$ [15, Lemma 1.10]. A near-ring R is called a zero symmetric if $0x = 0$ for all $x \in R$.

Definition 2.8. Let $(R,+,\cdot)$ be a near-ring. An ideal of R is a subset I of R such that

1) $(I,+)$ is a normal subgroup of $(R,+)$,

2) $RI \subseteq I$,

3) $(r+i)s - rs \in I$ for all $i \in I$ and $r, s \in R$.

Note that if I satisfies (1) and (2) then it is called a left ideal of R. If I satisfies (1) and (3) then it is called a right ideal of R. Let P be an ideal of R. We call P a prime ideal if for any ideal $I, J \subseteq R$, $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

i-v $\mathcal{L}$-Fuzzy Ideals in a Near-Ring

In this section first we define interval-valued $\mathcal{L}$-fuzzy subnear-rings and ideals and then we explain some results in this connection.

Definition 3.1. Let $(R,+,\cdot)$ be a near-ring. An i-v $\mathcal{L}$-fuzzy subset F of R is called an i-v $\mathcal{L}$-fuzzy subnear-ring, if the following hold:

1) $\hat{\mu}_F(x) \wedge \hat{\mu}_F(y) \le \hat{\mu}_F(x-y)$ for all $x, y \in R$,

2) $\hat{\mu}_F(x) \wedge \hat{\mu}_F(y) \le \hat{\mu}_F(x \cdot y)$ for all $x, y \in R$.

Furthermore F is called an i-v $\mathcal{L}$-fuzzy ideal of R, if F is an i-v $\mathcal{L}$-fuzzy subnear-ring of R and

3) $\hat{\mu}_F(x) = \hat{\mu}_F(y + x - y)$ for all $x, y \in R$,

4) $\hat{\mu}_F(x) \le \hat{\mu}_F(xy)$ for all $x, y \in R$,

5) $\hat{\mu}_F(i) \le \hat{\mu}_F((x+i)y - xy)$ for all $x, y, i \in R$.

Note that F is an i-v $\mathcal{L}$-fuzzy left ideal of R if it satisfies (1), (3) and (4), and F is an i-v $\mathcal{L}$-fuzzy right ideal of R if it satisfies (1), (2), (3) and (5).

Now, we give an example of an i-v $\mathcal{L}$-fuzzy ideal of a near-ring.

Example 3.2. Let $R = \{0, a, b, c\}$ be a set with two binary operations as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	a	0
c	c	b	0	a

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	a	a

Then $(R, +, \cdot)$ is a near-ring. Define an i-v $\mathcal{L}$-fuzzy subset F by membership function $\hat{\mu}_F : R \to \mathcal{D}(\mathcal{L})$ by $\hat{\mu}_F(b) = \hat{\mu}_F(c) < \hat{\mu}_F(a) < \hat{\mu}_F(0)$. Then F is an i-v $\mathcal{L}$-fuzzy ideal of R.

Lemma 3.3. For an i-v $\mathcal{L}$-fuzzy ideal F of a near-ring R, we have

$$\hat{\mu}_F(x) = \hat{\mu}_F(-x) \le \hat{\mu}_F(0) \text{ for all } x \in R.$$

Proposition 3.4. Let F be an i-v $\mathcal{L}$-fuzzy ideal of R. If $\hat{\mu}_F(x-y) = \hat{\mu}_F(0)$ then $\hat{\mu}_F(x) = \hat{\mu}_F(y)$.

Proof. Assume that $\hat{\mu}_F(x-y) = \hat{\mu}_F(0)$. Then

$$\begin{aligned} \hat{\mu}_F(x) &= \hat{\mu}_F(x - y + y) \\ &\ge \hat{\mu}_F(x-y) \wedge \hat{\mu}_F(y) \\ &= \hat{\mu}_F(0) \wedge \hat{\mu}_F(y) \\ &= \hat{\mu}_F(y). \end{aligned}$$

Similarly, using $\hat{\mu}_F(y-x) = \hat{\mu}_F(x-y) = \hat{\mu}_F(0)$, we get $\hat{\mu}_F(y) \ge \hat{\mu}_F(x)$.

Corollary 3.5. $[\mu_F^L, \mu_F^U]$ is an i-v $\mathcal{L}$-fuzzy ideal of a near-ring R if and only if μ_F^L, μ_F^U are $\mathcal{L}$-fuzzy ideals of R. Now, we define

$$F_t^L = \{x \in X \mid \mu_F^L(x) \ge t\} \quad \text{and}$$
$$F_s^U = \{x \in X \mid \mu_F^U(x) \ge s\}.$$

Then $\hat{\mu}_F$ is an i-v $\mathcal{L}$-fuzzy ideal of R if and only if for every t, s where $0 \le t \le s \le 1, F_t^L, F_s^U \neq \emptyset$ are ideals of R.

Definition 3.6. Let F_1 and F_2 be two i-v $\mathcal{L}$-fuzzy subsets of a near-ring R. Then $F_1 \cap F_2$ and $F_1 o F_2$ are defined as follows:

$$\hat{\mu}_{F_1 \cap F_2} = \hat{\mu}_{F_1}(x) \wedge \hat{\mu}_{F_2}(x),$$

$$\hat{\mu}_{F_1 o F_2}(x) = \begin{cases} \bigvee\limits_{x=yz} \{\hat{\mu}_{F_1}(y) \wedge \hat{\mu}_{F_2}(z)\} \\ [0,0] \quad \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Lemma 3.7. Let R be a near-ring, we have

1) If F_1, F_2 are two i-v $\mathcal{L}$-fuzzy ideals of R (right or left) then $F_1 \cap F_2$ is an i-v $\mathcal{L}$-fuzzy ideal of R (right or left), respectively;
2) If R is a zero-symmetric and if F_1 is an i-v $\mathcal{L}$-fuzzy right ideal and F_2 is an i-v $\mathcal{L}$-fuzzy left ideal, then $F_1 o F_2 \subseteq F_1 \cap F_2$.

Proof. (1) It is an immediate consequence of Corollary 3.5 and Definition 3.6.

(2) We assume R is a zero symmetric near-ring. If $\hat{\mu}_{F_1 o F_2}(x) = 0$, there is nothing to prove. Otherwise

$$\hat{\mu}_{F_1 o F_2}(x) = \bigvee_{x=yz} \{\hat{\mu}_{F_1}(y) \wedge \hat{\mu}_{F_2}(z)\}.$$

Since F_1 is an i-v $\mathcal{L}$-fuzzy left ideal, we have

$$\hat{\mu}_{F_1}(z) \le \hat{\mu}_{F_1}(yz) = \hat{\mu}_{F_1}(x),$$

and since F_1 is an i-v $\mathcal{L}$-fuzzy right ideal, we have

$$\hat{\mu}_{F_1}(x) = \hat{\mu}_{F_1}(yz) = \hat{\mu}_{F_1}((0+y)z - 0z)) \ge \hat{\mu}_{F_1}(y).$$

Therefore

$$\hat{\mu}_{F_1 o F_2}(x) \le \hat{\mu}_{F_1}(x) \wedge \hat{\mu}_{F_2}(x) = \hat{\mu}_{F_1 \cap F_2}(x).$$

Definition 3.8. Let X be a non-empty set and F be an i-v $\mathcal{L}$-fuzzy subset of X. Then we define

$$F_{[t,s]} = \{x \in X \mid \hat{\mu}_F(x) \ge [t,s]\}.$$

The set $F_{[t,s]}$ is called the "level set" of F.

It is easy to see that $F_{[t,s]} = F_t^L \cap F_s^U$.

Now, we obtain the relation between an i-v $\mathcal{L}$-fuzzy ideal and level ideals. This relation is expressed in terms of a necessary and sufficient condition.

Theorem 3.9. Let R be a near-ring and F be an i-v $\mathcal{L}$-fuzzy subset of R. Then F is an i-v $\mathcal{L}$-fuzzy ideal of R if and only if for every t, s where $0 \le t \le s \le 1, F_{[t,s]} \ne \emptyset$ is an ideal of R.

Proof. The proof is similar to the proof of Theorem 3.4 of [7], by considering the suitable modification with using Definitions 2.4 and 3.1.

Definition 3.10. An i-v $\mathcal{L}$-fuzzy ideal P of a near-ring R is said to be prime if P is not constant function and for any i-v $\mathcal{L}$-fuzzy ideals F_1, F_2 in $R, F_1 o F_2 \subseteq P$ implies $F_1 \subseteq P$ or $F_2 \subseteq P$.

Proposition 3.11. Let P be an i-v $\mathcal{L}$-fuzzy prime ideal of a near-ring R. Define

$$\pi = \{x \in R \mid \hat{\mu}_P(x) = \hat{\mu}_P(0)\},$$

then π is a prime ideal in R.

Proof. The proof is similar to the proof of Theorem 3.7 in [1].

Proposition 3.12. Let R be a near-ring and F_1, F_2 are i-v $\mathcal{L}$-fuzzy prime ideals of R, then $F_1 \cap F_2$ is an i-v $\mathcal{L}$-fuzzy prime if and only if $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.

Proof. The proof is straightforward, in view of the fact that $F_1 o F_2 \subseteq F_1 \cap F_2$.

We have the following corollary which plays an important role in the determination of i-v $\mathcal{L}$-fuzzy prime ideals.

Corollary 3.13. Let R be a near-ring. Then every ideal of R is a level ideal of an i-v $\mathcal{L}$-fuzzy ideal of R.

Proof. Let I be any ideal of a near-ring R and let $[\alpha_1, \alpha_2] \le [\beta_1, \beta_2] \ne [0,0]$ be elements in $\mathcal{D}(\mathcal{L})$. Then the fuzzy subset F is defined as follows:

$$\hat{\mu}_F(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in I \\ [\alpha_1, \alpha_2] & \text{otherwise.} \end{cases}$$

We have $I = F_{[\beta_1, \beta_2]}$ and by Theorem 3.9, it is enough to prove that F is an i-v $\mathcal{L}$-fuzzy ideal.

An element $[\alpha_1, \alpha_2] \ne [1,1]$ in $\mathcal{D}(\mathcal{L})$ is called "prime" if for any $[a_1, a_2], [b_1, b_2] \in \mathcal{D}(\mathcal{L})$, $[a_1, a_2] \wedge [b_1, b_2] \le [\alpha_1, \alpha_2]$ implies either $[a_1, a_2] \le [\alpha_1, \alpha_2]$ or $[b_1, b_2] \le [\alpha_1, \alpha_2]$.

Theorem 3.14. Let I be a prime ideal of a near-ring R and let $[\alpha_1, \alpha_2]$ a prime element in $\mathcal{D}(\mathcal{L})$. Let P be the fuzzy subset of R defined by

$$\hat{\mu}_P(x) = \begin{cases} [1,1] & \text{if } x \in I \\ [\alpha_1, \alpha_2] & \text{otherwise.} \end{cases}$$

Then P is an i-v $\mathcal{L}$-fuzzy prime ideal.

Proof. By Corollary 3.13, P is clearly a non-constant i-v $\mathcal{L}$-fuzzy ideal. Let F_1 and F_2 be any i-v $\mathcal{L}$-fuzzy ideals and let $F_1 \not\subseteq P, F_2 \not\subseteq P$. Then there exist x, y in R, such that $\hat{\mu}_{F_1}(x) \not\le \hat{\mu}_P(x)$ and $\hat{\mu}_{F_2}(x) \not\le \hat{\mu}_P(x)$. This implies that $\hat{\mu}_P(x) = \hat{\mu}_P(y) = [\alpha_1, \alpha_2]$ and hence $x \notin R$ and $y \notin R$. Since I is prime, there exists $r \in R$ such that $xry \notin I$. Now, we have $\hat{\mu}F_1(x) \not\le [\alpha_1, \alpha_2]$ and $\hat{\mu}F_2(ry) \not\le [\alpha_1, \alpha_2]$ (otherwise $\hat{\mu}F_2(y) \le [\alpha_1, \alpha_2]$ and since $[\alpha_1, \alpha_2]$ is prime, $\hat{\mu}_{F_1}(x) \wedge \hat{\mu}_{F_2}(ry) \not\le [\alpha_1, \alpha_2]$ and hence $(F_1 o F_2)(xry) \not\le [\alpha_1, \alpha_2] = \hat{\mu}_P(xry)$ so that $F_1 o F_2 \not\subseteq P$. Hence P is an i-v $\mathcal{L}$-fuzzy prime.

Lemma 3.15. Let f be a mapping from a non-empty set X into a non-empty set Y, and let A, B are i-v $\mathcal{L}$-fuzzy subsets of X, Y, respectively, such that

$$\hat{\mu}_A = [\mu_A^L, \mu_A^U] : X \to \mathcal{D}(\mathcal{L}) \text{ and}$$

$$\hat{\mu}_B = [\mu_B^L, \mu_B^U] : Y \to \mathcal{D}(\mathcal{L}).$$

Then

$$\hat{\mu}_{f[A]} = [f(\mu_A^L), f(\mu_A^U)] \text{ and}$$

$$\hat{\mu}_{f^{-1}[B]} = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)].$$

Using Lemma 3.15, the following propositions are obvious.

Proposition 3.16. Let f be a homomorphism from a near ring R onto a near-ring R', and A be any f-invariant i-v $\mathcal{L}$-fuzzy prime ideal of R. Then $f[A]$ is an i-v $\mathcal{L}$-fuzzy prime ideal of R'.

Proposition 3.17. Let f be a homomorphism from a near ring R onto a near-ring R', and B be any f-invariant i-v $\mathcal{L}$-fuzzy prime ideal of R'. Then $f^{-1}[B]$ is an i-v $\mathcal{L}$-fuzzy prime ideal of R.

Theorem 3.18. Let f be a homomorphism from a near ring R onto a near-ring R', then the mapping $A \to f[A]$ defines a one-to-one correspondence between the set of all f-invariant i-v $\mathcal{L}$-fuzzy prime ideals of R and the set of all i-v $\mathcal{L}$-fuzzy prime ideals of R'.

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