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On Condition ($E'P$)

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Abstract

In this paper we introduce condition ($E'P$) a generalization of Conditions (E'), (E) and (P) of acts over monoids and will give a classification of monoids by comparing this condition of their (Rees factor) acts with other properties. We give also a criterion of cyclic acts to satisfy Condition ($E'P$).

Keywords: Weakly right reversible; Completely right cancellative

1. Introduction and Preliminaries

We deal in this paper with what is generally referred to as homological classification of monoids by flatness properties. Many papers have appeared recently investigating the conditions on a monoid which are necessary and sufficient to make various of the flatness properties coincide, either for all right acts, or for all right acts of certain type. Conditions (P) and (E) of acts over monoids have appeared in many papers and monoids are classified when these properties of acts imply other flatness properties and vice versa. In this paper we introduce a generalization of these conditions and condition (E'), and will give a classification of monoids, when this condition of their (Rees factor) acts implies other flatness properties.

Throughout this paper S will denote a monoid. We refer the reader to ([3]) and ([4]) for basic results, definitions and terminology relating to semigroups and acts over monoids and to ([1] and [6]) for definitions and results on flatness which are used here.

A monoid S is called *right (left) reversible* if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$ $(su = s'v)$. A monoid S is said to be *left collapsible* if for any $p, q \in S$ there exists $r \in S$ such that $rp = rq$. A submonoid $P \subseteq S$ is *weakly left collapsible* if for all $s, s' \in P$, $z \in S$, $sz = s'z$ implies the existence of $u \in P$ such that $us = us'$.

A right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$, there exists $l \in K$ such that $lk = k$ K is called *left annihilating* if for all $t \in S$ and all $x, y \in S \backslash K$, $xt, yt \in K$ implies that $xt = yt$. If for all $s, t \in S \setminus K$ and all homomorphisms $f\colon {}_S(Ss \cup St) \to {}_SS$, $f(s), f(t) \in K$ implies that $f(s) = f(t)$, then K is called *strongly left annihilating*.

A right S-act A satisfies Condition *(P)* if for all $a, a' \in A$, $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. It satisfies *Condition* (P_E) if when ever $a, a' \in A$, $s, s' \in S$, and $as = a's'$, there exist $a'' \in A$ and $u, v, e^2 = e$, $f^2 = f \in S$ such that $ae = a''ue$, $a'f = a''vf$, $es = s, fs' = s'$ and $us = vs'$. It is shown in ([2]) that Condition (P_E) implies weak flatness. A right S-act A satisfies Condition (E) if for all $a' \in A$, $s, s' \in S$, $as = as'$ implies that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = us'$. A right S-act A satisfies Condition (W) if for all $a, a' \in A$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A$, $u \in Ss \cap$

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St such that $as = a't = a''u$. A right S-act A satisfies Condition (E') if for all $a \in A$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = ut$.

We use the following abbreviations,

strong flatness = *SF*.
weak pullback flatness = *WPF*.
equalizer flatness = *EF*
weak kernel flatness = *WKF*.
principal weak kernel flatness = *PWKF*.
translation kernel flatness = *TKF*.
weak homoflatness = (*WP*).
principal weak homoflatness = (*PWP*).
weak flatness = *WF*.
principal weak flatness = *PWF*.
torsion freeness = *TF*.

2. Classification by Condition (*E'P*) of Right Acts

In this section we give a classification of monoids when condition (*E'P*) of their acts implies flatness properties and also a criterion of cyclic acts to satisfy this condition.

Definition 2.1. A right S-act A satisfies Condition (*E'P*) if for all $a \in A$, $s, t, z \in S$, $as = at$ and $sz = tz$ imply that there exist $a' \in A$, $u, v \in S$ such that $a = a'u = a'v$ and $us = vt$.

It is obvious that $(E) \Rightarrow (E') \Rightarrow (E'P)$, but (*E'P*) does not imply (E') in general, otherwise $(P) \Rightarrow (E')$, Since $(P) \Rightarrow (E'P)$. Therefore $(P) \Rightarrow WPF$ and this is not true by ([5, Example 6]). Also (*E'P*) does not im-ply *TF*, otherwise $(E) \Rightarrow TF$. Now if S is a left almost regular monoid which is not regular, for example $(\mathbb{N}, \cdot)$, then by ([4, IV, 6.5]), $TF \Rightarrow PWF$ and so $(E) \Rightarrow PWF$. Then by ([4, IV, 6.7]), S is regular which is a contradiction.

Theorem 2.2. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (E'P) are free.*

(2) *All finitely generated right S-acts satisfying Condition (E'P) are free.*

(3) *All cyclic right S-acts satisfying Condition (E'P) are free.*

(4) *All monocyclic right S-acts satisfying Condition (E'P) are free.*

(5) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Since $(P) \Rightarrow (E'P)$, then by assumption all monocyclic right S-acts satisfying Condition *(P)* are free and so by ([4, IV, 12.8]), $S = \{1\}$.

(5) $\Rightarrow$ (1). If $S = \{1\}$, then it can easily be seen that all right S-acts are free.

Theorem 2.3. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (E'P) are projective generators.*

(2) *All finitely generated right S-acts satisfying Condition (E'P) are projective generators.*

(3) *All cyclic right S-acts satisfying Condition (E'P) are projective generators.*

(4) *All monocyclic right S-acts satisfying Condition (E'P) are projective generators.*

(5) $S = \{1\}$.

Proof. The same argument as the proof of Theorem 2.2, can be used.

Theorem 2.4. *Let S be a monoid and A a regular right S-act. Then for all $a \in A$ and $s, t \in S$, $as = at$ implies that there exists $u \in S$ such that $a = au$ and $us = ut$.*

Proof. Suppose that A is a regular right S-act. Then by ([4, III, 19.2]), for every $a \in A$ there exists $e \in E(S)$ such that $\ker \lambda_a = \ker \lambda_e$ and so $aS \cong eS$. But by ([4, III, 17.8]), eS is projective, and so eS satisfies Condition (E). Thus aS satisfies Condition (E) and so $as = at$ implies that there exist $w_1, w_2 \in S$, such that $a = (aw_1)w_2$ and $w_2 s = w_2 t$. Then $a = a(w_1 w_2)$ and $(w_1 w_2)s = (w_1 w_2)t$. If $w_1 w_2 = u$, then $a = au$ and $us = ut$.

Corollary 2.5. *Let S be a monoid. If A is a regular right S-act, then A satisfies Condition (E).*

Proof. By Theorem 2.4, it is obvious.

Theorem 2.6. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition* (*E'P*) *are WPF.*

(2) *All right S-acts satisfying Condition* (*E'P*) *satisfy Condition (P).*

(3) *All right S-acts satisfying Condition* (*E'P*) *are WKF.*

(4) *All right S-acts satisfying Condition* (*E'P*) *are PWKF.*

(5) *All right S-acts satisfying Condition* (*E'P*) *are TKF.*

(6) *All right S-acts satisfying Condition* (*E'P*) *are (WP).*

(7) *All right S-acts satisfying Condition* (*E'P*) *are (PWP).*

(8) *S is a group.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (6) $\Rightarrow$ (7) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (7) are obvious.

(8) $\Rightarrow$ (1). By ([5, Theorem 3.12]), it is obvious.

(7) $\Rightarrow$ (8). It can be deduced by assumption that all right S-acts satisfying Condition (*E*) are *PWF,* and so by ([4, IV, 6.7]), *S* is regular. Thus by ([4, III, 19.12], S_S is a regular act. But by Corollary 2.5, every regular act satisfies Condition (*E*) and so satisfies Condition (*E'P*). Thus all regular right S-acts are (*PWP*). If *I* is a proper right ideal of *S* such that $A = S \coprod^I S = (1, x)S\dot{\cup}I \dot{\cup} (1, y)S$, then it is obvious that $(1, x)S \dot{\cup} I$ and $(1, y)S\dot{\cup}I$ are subacts of *A* isomorphic to *S.* Since S_S is a regular act, then $(1, x)S \dot{\cup} I$ and $(1, y)S \dot{\cup} I$ are regular and so *A* is regular. Thus *A* is (*PWP*). If $t \in I$, then $t = (1, x)\, t = (1, y)t$ and so there exist $a' \in A$ and $u, v \in S$ such that $(1, x) = a'u$, $(1, y) = a'v$ and $ut = vt$. Thus $(1, x) = a'u$ implies for some $s \in S \setminus I$, that $a' = (s, x)$, similarly, $a' = (s', y)$ for some $s' \in S \setminus I$ which is a contradiction. Thus *S* has no proper right ideal, that is for every $s \in S$, $sS = S$, and so *S* is a group as required.

Corollary 2.7. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition* (*E'P*) *are projective.*

(2) *All right S-acts satisfying Condition* (*E'P*) *are SF.*

(3) *S = {1}.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). By assumption all right S-acts satisfying Condition (*E'P*) are *WP.* Thus by Theorem 2.6, *S* is a group, and so it can easily be seen that all right S-acts satisfy Condition (*E'P*). Thus all right S-acts are strongly flat and, so by ([4, IV, 10.5]), *S = {1}.*

(3) $\Rightarrow$ (1). It is obvious.

Theorem 2.8. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying Condition* (*E'P*) *satisfy Condition* (P_E)*.*

(2) *All right S-acts satisfying Condition* (*E'P*) *are WF.*

(3) *All right S-acts satisfying Condition* (*E'P*) *are PWF.*

(4) *S is a regular monoid.*

Proof. (1) $\Rightarrow$ (2). It is obvious by ([2, Theorem 2.3]).

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (4). Since $(E) \Rightarrow (E'P)$, then by assumption $(E) \Rightarrow PWF$ and so by ([4,IV,6.7]), *S* is a regular monoid.

(4) $\Rightarrow$ (1). Suppose that *A* satisfies Condition (*E'P*) We show that *A* satisfies Condition (*W*). Suppose that $as = a't$ for $a, a' \in A$ and $s, t \in S$. Since *S* is regular, then $a't = a'ts's$, $s' \in V(s)$. Since *A* satisfies Condition (*E'P*) and also $t.s' = ts's.s'$, then there exist $a'' \in A$ and $u, v \in S$, such that $a' = a''u = a''v$, and $ut = vts's$. Then $as = a't = a''ut$ and $ut = vts's \in St \cap Ss$. Thus *A* satisfies Condition (*W*). Since *S* is regular, then by ([4,IV, 6.6]), all right S-acts are *PWF.* Thus *A* is *PWF.* But by ([4, III, 11.4]), $WF \Leftrightarrow (PWF \wedge W)$ and so *A* is *WF.* Since *S* is regular, then *S* is left *PP.* Thus by ([2, Theorem 2.5]), $WF \Leftrightarrow (P_E)$, and so *A* satisfies Condition (P_E).

Theorem 2.9. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are TF.*

(2) *All right S-acts satisfying Condition* (*E'P*) *are TF.*

(3) *Every right cancellative element of S is right invertible.*

Proof. (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). By assumption all right S-acts satisfying Condition *(E)* are *TF.* Thus by ([7, Proposition 4]), all right S-acts are *TF* and so by ([4, IV,6.1]), we are done.

(3) $\Rightarrow$ (1). By ([4, IV, 6.1]), it is obvious.

Condition *(E'P)* of Cyclic Acts

Theorem 2.10. *Let S be a monoid and ρ a right congruence on S. Then the following statements are equiva lent:*

(1) *For all* $x, y, z \in S$, $x\rho y$ *and* $xz = yz$ *imply that there exist* $u, v \in S$ *such that* $1\rho u\rho v$ *and* $ux = vy$.

(2) *For all* $x, t, t', z \in S$, $xt\rho xt'$ *and* $tz = t'z$ *imply that there exist* $u, v \in S$ *such that* $x\rho u\rho v$ *and* $ut = vt'$.

(3) *S/ρ satisfies Condition* (*E'P*).

Proof. (1) $\Rightarrow$ (2). Suppose that for $x, t, t', z \in S$, $xt\rho xt'$ and $tz = t'z$. Then $xt\rho xt'$ and $(xt)z = (xt')z$. Thus by assumption there exist $u', v' \in S$ such that $1\rho u'\rho v'$ and $u'(xt) = v'(xt')$. If $u'x = u$ and $v'x = v$, then $x\rho u\rho v$ and $ut = vt'$.

(2) $\Rightarrow$ (3). Suppose that for $x, t, t' \in S$, we have

$[x]_\rho t = [x]_\rho t'$ and tz = t'z. Then xtρxt', tz = t'z and so by assumption there exist u, v $\in$ S such that xρuρv and ut = vt'. Thus $[x]_\rho$ = $[1]_\rho u$ = $[1]_\rho$v and ut = vt'.

(3) $\Rightarrow$ (1). Suppose that for $x, y, z \in S$, we have $x\rho y$ and $xz = yz$. Then $[1]_\rho x = [1]_\rho y$, $xz = yz$ and so there exist $u', v', w \in S$ such that $[1]_\rho = [w]_\rho u' = [w]_\rho v'$ and $u'x = v'y$. If $wu' = u$ and $wv' = v$, then $1\rho u\rho v$ and $ux = vy$.

Corollary 2.11. *Let S be a monoid and let $z \in S$. Then zS satisfies Condition (E'P) if and only if for all $x, y, t \in S$, $zx = zy$ and $xt = yt$ imply the existence of $u, v \in S$ such that $z = zu = zv$ and $ux = vy$.*

Proof. Since $zS \cong S / ker\lambda_z$, then it suffices in Theorem 2.10, to put $\rho = ker\lambda_z$.

Corollary 2.12. *If S is a right collapsible monoid, then for cyclic acts Conditions (P) and (E'P) Coincide.*

Proof. By Theorem 2.10, and ([4, III, 13.4]), it is obvious.

3. Classification by Condition *(E'P)* of Rees Factor Acts

In this section we give a classification of monoids for which Rees factor acts satisfying Condition (*E'P*) have flatness properties.

Definition 3.1. Let *S* be a monoid. A subsemigroup *P* of *S* is called completely right cancellative if for all $x, y \in P$, $z \in S$, $xz = yz$ implies that $x = y$. A submonoid *P* of *S* is called weakly right reversible if for all $s, s' \in P$, $z \in S, sz = s'z$ implies the existence of $u, v \in P$ such that $us = vs'$.

Corollary 3.2. *Let S be a monoid. Then Θ_S satisfies Condition (E'P) if and only if S is weakly right reversible.*

Proof. Since $\Theta_S \cong S/S_S$, then by Theorem 2.10, Θ_S satisfies Condition (*E'P*) if and only if for all $x, y, z \in S$, $xz = yz$ implies that there exist $u, v \in S$ such that $ux = vy$, and so *S* is weakly right reversible.

Theorem 3.3. *Let S be a monoid and K_S a right ideal of S. Then S/K_S satisfies Condition (E'P) if and only if $K_S = S$ and S is weakly right reversible or K_S is completely right cancellative.*

Proof. Suppose that S/K_S satisfies Condition (*E'P*). If $K_S = S$, then $S/K_S \cong \Theta_S$ and so by Corollary 3.2, *S* is weakly right reversible. Suppose that $K_S \neq S$ and that $k_1 z = k_2 z$ for $k_1, k_2 \in K_S$ and $z \in S$. Since $k_1 \rho_K k_2$, then by Theorem 2.10, there exist $u, v \in S$ such that $1\rho_k u \rho_k v$ and $uk_1 = vk_2$. Then $1 = u = v$ and so $k_1 = k_2$.

Conversely, suppose that $x \rho_k y$ and $xz = yz$ for $x, y, z \in S$. Then there are two cases as follows:

Case 1. $x, y \notin K_S$. Then $x = y$ and so we have (1) of Theorem 2.10.

Case 2. $x, y \in K_S$. If $K_S = S$ and *S* is weakly right reversible, then by Corollary 3.2, S/K_S satisfies Condition (*E'P*). If $K_S \neq S$, then by assumption $x = y$. Thus we have (1) of Theorem 2.10 and so S/K_S satisfies Condition (*E'P*).

Corollary 3.4. *Let S be a monoid and K_S a proper right ideal of S. If S/K_S satisfies Condition (E'P), then K_S is right cancellative.*

Proof. By Theorem 3.3, it is obvious.

Notice that for Rees factor acts Condition (*E'P*) does not imply *TF*, if for example, $S = \{1, \alpha, \alpha^2, ...\}$ is the infinite cyclic monoid generated by α and $K_S = \{\alpha^2, \alpha^3, ...\}$, then K_S is a right ideal of *S* and it can easily be seen that K_S is completely right cancellative. Thus by Theorem 3.3, S/K_S satisfies Condition (*E'P*). On the other hand every element of *S* is right cancellable and $\alpha^2 = \alpha.\alpha \in K_S$, but $\alpha \notin K_S$ and so by ([4, III, 8.10]), S/K_S is not *TF*.

Theorem 3.5. *Let S be a monoid. Then all right Rees factor acts satisfying Condition (E'P) are TF if and only if every proper right ideal K_S of S is not completely right cancellative or for all $s, c \in S$ where c is right cancellable, $sc \in K_S$ implies that $s \in K_S$.*

Proof. By Theorem 3.3, and ([4, III, 8.10]), it is obvious.

Theorem 3.6. *Let S be a monoid. Then all right Rees factor acts satisfying Condition (E'P) are PWF if and* *only if every proper right ideal K_S of S is not completely right cancellative or K_S is left stabilizing.*

Proof. By Theorem 3.3, and ([4, III, 10.11]), it is obvious.

Theorem 3.7. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor S-acts satisfying Condition* (*E'P*) *are flat.*

(2) *All right Rees factor S-acts satisfying Condition* (*E'P*) *are WF.*

(3) *Every proper right ideal K_S of S is not completely right cancellative or K_S is left stabilizing, S is right reversible, and if S is weakly right reversible, then S is right reversible.*

Proof. Since for Rees factors acts flatness and weak flatness are the same, then it suffices to show that (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3). Suppose that K_S is a proper completely right cancellative ideal of *S,* then by Theorem 3.3, S/K_S satisfies Condition (*E'P*). Thus S/K_S is *WF* and so by ([4, III, 12.17]), K_S is left stabilizing and *S* is right reversible. If *S* is weakly right reversible, then by Corollary 3.2, Θ_S satisfies Condition (*E'P*). Thus Θ_S is *WF* and so by ([5, Lemma 2.14]), *S* is right reversible.

(3) $\Rightarrow$ (2). Suppose that K_S is a right ideal of *S* such that S/K_S satisfies Condition (*E'P*). Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/S_k \cong \Theta_S$ satisfies Condition (*E'P*) and so by Corollary 3.2, *S* is weakly right reversible. Thus by assumption *S* is right reversible and hence by ([5, Lemma 2.14]), Θ_S is *WF*.

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative and so by assumption K_S is left stabilizing and *S* is right reversible. Thus by ([4, III, 12.17]), S/K_S is *WF*.

Theorem 3.8. *Let S be a monoid. Then all right Rees factor acts satisfying Condition* (*E'P*) *are (PWP) if and only if every proper right ideal K_S of S is not completely right cancellative or K_S is left annihilating and left stabilizing.*

Proof. By ([5, Lemma 2.8]), and Theorem 3.3, it is obvious.

Theorem 3.9. *Let S be a monoid. Then all right Rees factor acts satisfying Condition* (*E'P*) *are (WP) if and only if S is not weakly right reversible or S is right reversible and if there exists a proper completely right cancellative ideal K_S of S, then S is right reversible, K_S is strongly left annihilating and left stabilizing.*

Proof. Suppose that all right Rees factor acts satisfying Condition (*E'P*) are (*WP*) and that *S* is weakly right reversible. Then by Corollary 3.2, Θ_S satisfies Condition (*E'P*) and by assumption Θ_S is (*WP*). Thus by ([5, Lemma 2.14]), *S* is right reversible. Now let K_S be a proper completely right cancellative ideal of *S*. Then by Theorem 3.3, S/K_S satisfies Condition (*E'P*), and so by assumption S/K_S is (*WP*). Thus by ([5, Lemma 2.13]), K_S is strongly left annihilating and left stabilizing and *S* is right reversible.

Conversely, Let K_S be a right ideal of *S* such that S/K_S satisfies Condition (*E'P*). Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, *S* is weakly right reversible and so by assumption *S* is right reversible. Thus by ([5, Lemma 2.14]), Θ_S is (*WP*).

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative and so by assumption *S* is right reversible and K_S is strongly left annihilating and left stabilizing. Thus by ([5, Lemma 2.13]), S/K_S is (*WP*).

Theorem 3.10. *Let S be a monoid. Then all right Rees factor acts satisfying Condition* (*E'P*) *satisfy Condition (P) if and only if S is not weakly right reversible or S is right reversible and there exists no proper completely right cancellative ideal K_S of S with $|K_S| \geq 2$.*

Proof. Suppose that all right Rees factor acts satisfying Condition (*E'P*) satisfy Condition *(P)* and that *S* is weakly right reversible. Then by Corollary 3.2, $\Theta_S \cong S/S_S$ satisfies Condition (*E'P*). Then by assumption Θ_S satisfies Condition *(P)* and so by ([4, III, 13.7]), *S* is right reversible. Now let K_S be a proper completely right cancellative ideal of *S*. Then by Theorem 3.3, S/K_S satisfies Condition (*E'P*) and so by assumption S/K_S satisfies Condition (*P*). Hence by ([4, III, 13.9]), $|K_S| = 1$.

Conversely, suppose that S/K_S satisfies Condition (*E'P*) for the right ideal K_S of *S*. Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, S is weakly right reversible and so by assumption S is right reversible. Thus by ([4, III, 13.7]), $\Theta_S \cong S/K_S$ satisfies Condition (P).

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative and so by assumption $|K_S| = 1$. Hence $S/K_S \cong S$ is *SF,* and so S/K_S satisfies Condition (P).

Theorem 3.11. *Let S be a monoid. Then all right Rees factor acts satisfying Condition* ($E'P$) *are WPF if and only if S is not weakly right reversible or S is right reversible and weakly left collapsible and there exists no proper completely right cancellative ideal* K_S *of S with* $|K_S| \geq 2$.

Proof. Suppose that all right Rees factor acts satisfying Condition ($E'P$) are *WPF* and that S is weakly right reversible. Then by Corollary 3.2, $\Theta_S \cong S/S_S$ satisfies Condition ($E'P$) and so by assumption Θ_S is *WPF*. Thus by ([5, Corollary 2.20]), S is right reversible and weakly left collapsible. On the other hand by assumption Condition ($E'P$) of Rees factor acts implies Condition (P) and so by Theorem 3.10, there exists no proper completely right cancellative ideal K_S of S with $|K_S| \geq 2$.

Conversely, suppose that S/K_S satisfies Condition ($E'P$) for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, S is weakly right reversible and so by assumption S is right reversible and weakly left collapsible. Thus by ([5, Corollary 2.20]), $\Theta_S \cong S/K_S$ is *WPF.*

Case 2. $K_S \neq S$ .Then by Theorem 3.3, K_S is completely right cancellative and so by assumption $|K_S| = 1$. Hence $S/K_S \cong S$ is *WPF.*

Theorem 3.12. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor acts satisfying Condition* ($E'P$) *are SF.*

(2) *All right Rees factor acts satisfying Condition* ($E'P$) *are EF.*

(3) *All right Rees factor acts satisfying Condition* ($E'P$) *satisfy Condition (E).*

(4) *S is not weakly right reversible or S is left collapsible and there exists no proper completely right cancellative ideal* K_S *of S with* $|K_S| \geq 2$.

Proof. Since for Rees factors acts $SF \Leftrightarrow EF \Leftrightarrow (E)$, then it suffices to show that (3) $\Leftrightarrow$ (4).

(3) $\Rightarrow$ (4). Suppose that S is weakly right reversible. Then by Corollary 3.2, $\Theta_S \cong S/S_S$ satisfies Condition ($E'P$) and so by assumption Θ_S satisfies Condition *(E)*. Thus by ([4, III, 14.3]), S is left collapsible. Also it can be seen by assumption that for Rees factor acts Condition ($E'P$) implies *SF* and so in this case ($E'P$) $\Rightarrow$ (P). Thus by Theorem 3.10, there exists no proper completely right cancellative ideal K_S of S with $|K_S| \geq 2$.

(4) $\Rightarrow$ (3). Suppose that S/K_S satisfies Condition ($E'P$) for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, S is weakly right reversible, and so by assumption S is left collapsible. Thus by ([4, III, 14.3]), $\Theta_S \cong S/S_S$ satisfies Condition (E).

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative and so by assumption $|K_S| = 1$. Hence $S/K_S \cong S$ satisfies Condition *(E).*

Theorem 3.13. *Let S be a monoid. Then all right Rees factor acts satisfying Condition* ($E'P$) *are projective if and only if S is not weakly right reversible or S contains a left zero and there exists no proper completely right cancellative ideal* K_S *of S with* $|K_S| \geq 2$.

Proof. Suppose that all right Rees factor acts satisfying Condition ($E'P$) are projective and that S is weakly right reversible. Then by Corollary 3.2, $\Theta_S \cong S/S_S$ satisfies Condition ($E'P$), and so by assumption Θ_S is projective. Thus by ([4, III, 17.2]), S contains a left zero. Also it can be seen by assumption that for Rees factor acts Condition ($E'P$) implies Condition (P), and so by Theorem 3.10, there exists no proper completely right cancellative ideal K_S of S with $|K_S| \geq 2$.

Conversely, suppose that S/K_S satisfies Condition ($E'P$) for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, S is weakly right reversible and so by assumption S contains a left zero. Thus by ([4, III, 17.2]), $\Theta_S \cong S/S_S$ is projective.

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative and so by assumption

$|K_S| = 1$. Hence $S/K_S \cong S$ is projective.

Theorem 3.14. *For any monoid S the following statements are equivalent:*

(1) *All right Rees factor acts satisfying Condition (E'P) are free.*

(2) *All right Rees factor acts satisfying Condition (E'P) are projective generators.*

(3) *S is not weakly right reversible or* $S = \{1\}$ *and there exists no proper completely right cancellative ideal* K_S *of S with* $|K_S| \geq 2$.

Proof. $(1) \Rightarrow (2)$. It is obvious.

$(2) \Rightarrow (3)$. Suppose that S is weakly right reversible. Then by Corollary 3.2, $\Theta_S \cong S/S_S$ satisfies Condition (*E'P*) and so by assumption Θ_S is projective generator. Thus by ([4, III, 18.7]), $|S| = 1$. Also it can be seen by assumption that for Rees factor acts Condition (*E'P*) implies Condition (*P*) and so by Theorem 3.10, there exists no proper completely right cancellative ideal K_S of S with $|K_S| \geq 2$.

$(3) \Rightarrow (1)$. Suppose that S/K_S satisfies Condition (*E'P*) for the right ideal K_S of *S*. Then there are two cases as follows:

Case 1. $K_S = S$. Then by Corollary 3.2, S is weakly right reversible, and so by assumption $S = \{1\}$. Thus $S/K_S \cong \Theta_S$ is free.

Case 2. $K_S \neq S$. Then by Theorem 3.3, K_S is completely right cancellative, and so by assumption $|K_S| = 1$. Hence $S/K_S \cong S$ is free.

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