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On The Mean Convergence of Biharmonic Functions

A. Abkar*

Department of Mathematics, Imam Khomeini International University, P.O. Box 288,
Qazvin 34194, Islamic Republic of Iran

Abstract

Let T denote the unit circle in the complex plane. Given a function $f \in L^p(T)$, one uses t usual (harmonic) Poisson kernel $\mathrm{P}(\zeta, z)$ for the unit disk to define the Poisson integral of f, namely $h = P[f]$. Here we consider the biharmonic Poisson kernel $F(\zeta, z)$ for the unit disk to define the notion of F-integral of a given function $f \in L^p(T)$; this associated biharmonic function will be denoted by $u = F[f]$. We then consider the dilations $u_r(z) = u(rz)$ for $z \in T$ and $0 \le r < 1$. The main result of this paper indicates that the dilations u_r are convergent to f in the mean, or in the norm of $L^p(T)$.

Keywords: Biharmonic function; Biharmonic Poisson kernel; Mean convergence (Convergence in the Mean)

1. Introduction

We denote by D the unit disk and by T the unit circle in the complex plane. The Laplace operator in the complex plane is defined by

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z\, \partial \overline{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad z = x + iy.$$

A function u defined on D is said to be biharmonic provided that $\Delta^2 u = 0$. It is known, (see [3]) although not so well, that the biharmonic Green function for the unit disk has the form

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log\left|\frac{z - \zeta}{1 - z\overline{\zeta}}\right|^2 + \left(1 - |z|^2\right)\left(1 - |\zeta|^2\right)$$

$$(z, \zeta) \in D \times D.$$

We define the *biharmonic Poisson kernel* for the unit disk to be the function

$$F(\zeta, z) = -\frac{1}{2}\partial_{n(z)}\Delta_z \Gamma(z, \zeta)$$
$$= \frac{1}{2}\left\{\frac{\left(1 - |z^2|\right)^2}{\left|1 - \overline{z}\zeta\right|^2} + \frac{\left(1 - |z^2|\right)^3}{\left|1 - \overline{z}\zeta\right|^4}\right\}$$

$$(\zeta, z) \in T \times D.$$

Being asymmetric, the kernel function $F(\zeta, z)$ is biharmonic in its second argument.

For a function $f \in L^1(T)$, we define the F*-integral* of f by

$$u(z) = F[f](z) = \int_T F(\zeta, z) f(\zeta)\, d\sigma(\zeta)$$

$$z \in D, \; f \in L^1(T)$$

**E-mail: abkar@ikiu.ac.ir*

where $d\sigma = (2\pi)^{-1} d\theta$ is the normalized arc length measure on the unit circle. It follows that $F[f]$ is biharmonic in the unit disk, moreover it extends $f \in L^1(T)$ to a biharmonic function throughout the unit disk D; a property which explains the phrase "*biharmonic Poisson kernel*" we have associated to $F(\zeta, z)$.

Similarly, given a Borel measure ν on the unit circle T, we define its F-integral by the formula

$$F[\nu](z) = \int_T F(\zeta, z) d\nu(\zeta) \qquad z \in D.$$

In this paper we aim to study the biharmonic functions defined by the biharmonic Poisson kernel $F(\zeta, z)$, or the F-integrals of functions. To do this, we begin with a careful examination of the kernel function $F(\zeta, z)$ itself. It turns out that $F(\zeta, z)$ is an approximate identity. We then study the convergence in the mean of F-integrals of functions in $L^p(T)$. Given $f \in L^p(T)$, we consider $u = F[f]$, and define its dilations in the following manner:

$$u_r(z) = u(rz), \qquad 0 \le r < 1, \quad z \in \overline{D}.$$

Suppose f be a function in $L^p(T)$ and $h = P[f]$ be its Poisson integral which extends f to a harmonic function in the unit disk. It is well-known that the dilations h_r are convergent to h in the mean, or in the norm of L^p (see for instance [4]). The main result of this paper says that the functions u_r are convergent to f in the mean as well, therefore from this perspective, the biharmonic Poisson kernel resembles the usual (harmonic) Poisson kernel.

The biharmonic poisson kernel played a significant role in a Riesz-type representation formula found by Abkar and Hedenmalm [2]. Moreover, in a recent paper [1], the current author studied the boundary behavior of the potentials associated to the biharmonic Poisson kernel, *i.e.*, $u = F[f]$. Here we have been able to establish a Fatou theorem on the existence almost everywhere of nontangential limits on the boundary of the unit disk. For more information on the origin of the biharmonic Poisson kernel and its relevance to the potential theory of the complex plane we refer the interested reader to [2].

2. Convergence in the Mean

In this section we study the F-integrals of functions $f \in L^p(T)$ for $1 \le p \le +\infty$. Putting $u = F[f]$, we shall see that the dilations u_r are convergent to f in the norm of $L^p(T)$ when $1 \le p < +\infty$, and they converge weak-star to f when $p = +\infty$. This is a dual statement to the convergence in the mean of Poisson integrals of $L^p(T)$ functions to the same function.

In the following proposition we collect some intrinsic properties of the biharmonic Poisson kernel. In particular, it follows from parts (a), (b), and (d) of the proposition that the biharmonic Poisson kernel F is an approximate identity. We mention that the proof of the proposition follows readily from the definition of the biharmonic Poisson kernel.

Proposition 2.1. Let $F(\zeta, z)$ denote the biharmonic Poisson kernel for the unit disk. Then we have

(a) $F(\zeta, z) > 0$ for $(\zeta, z) \in T \times D$,

(b) $\int_T F(\zeta, z) d\sigma(\zeta) = 1$ for $z \in D$,

(c) $F(\zeta, rz) = F(z, r\zeta)$ for $(\zeta, z) \in T \times T$ and $0 \le r < 1$,

(d) $F(\zeta, z) \to 0$ uniformly as $|z| \to 1$ and $z^* \in T \setminus I_\zeta$, where $z^* = z/|z|$ for $z \ne 0$, and I_ζ is an arc centered at ζ.

For a subset X of the complex plane, we shall use the notation $C(X)$ for the space of bounded continuous functions on X. Let f be a bounded function defined on X; we write

$$\|f\|_X = \sup\{|f(z)| : \ z \in X\}.$$

Let $f \in L^1(T)$ and $u = F[f]$ be the F-integral of f. We can then extend f to $\overline{D}$ by putting $\tilde{f}(rz) = u(rz)$ for $0 \le r < 1$ and $z \in T$. We shall see that the F-integral of a continuous function f on the boundary of the unit disk behaves very well in the closure of D.

Proposition 2.2. Let $f \in L^1(T)$ and $z \in T$. Define $\tilde{f}$ on $\overline{D}$ by

$$\tilde{f}(rz) = \begin{cases} f(z), & r = 1, \\ u(rz), & 0 \le r < 1. \end{cases}$$

Then $\tilde{f}$ is a biharmonic function inside the unit disk.

Proof. The proposition follows from the biharmonicity of the Poisson biharmonic kernel in the z variable.

Proposition 2.3. Let $z \in T$ *and* $f(z)=\sum_{n=-k}^{k} c_n z^n$ be a trigonometric polynomial on T. Then we have

$$\tilde{f}(rz)=\sum_{n=-k}^{k} c_n r^{|n|} z^n \left(1+\frac{n}{2}(1-r^2)\right),$$

$$0 \le r \le 1, \quad z \in T.$$

Proof. The case $r=1$ follows from the definition. Assume now that $0 \le r < 1$ and that $z \in T$. By definition

$$\tilde{f}(rz)=\int_T F(\zeta, rz) f(\zeta) d\sigma(\zeta)$$
$$=\sum_{n=-k}^{k} c_n \int_T F(\zeta, rz)\zeta^n d\sigma(\zeta).$$

We note that

$$\int_T F(\zeta, rz)\zeta^n \, d\sigma(\zeta)=\frac{1}{2}\int_T \frac{(1-|rz|^2)^2}{|\zeta-rz|^2}\zeta^n \, d\sigma(\zeta)$$
$$+\frac{1}{2}\int_T \frac{(1-|rz|^2)^3}{|\zeta-rz|^4}\zeta^n \, d\sigma(\zeta).$$

We first assume that n is nonnegative. As for the first integral above, we see from the definition of the usual Poisson kernel $P(w,\zeta)$, for $w \in D$ and $\zeta \in T$, that

$$\int_T \frac{(1-|rz|^2)^2}{|\zeta-rz|^2}\zeta^n \, d\sigma(\zeta) = (1-r^2)\int_T \frac{1-|rz|^2}{|\zeta-rz|^2}\zeta^n \, d\sigma(\zeta)$$
$$=(1-r^2)\int_T P(rz,\zeta)\zeta^n d\sigma(\zeta)$$
$$=(1-r^2)(rz)^n.$$

On the other hand,

$$\int_T \frac{\zeta^n}{\left|1-rz\bar{\zeta}\right|^4} d\sigma(\zeta)$$
$$=\sum_{p,q=0}^{\infty}(p+1)(q+1)r^{p+q}\int_T (z\bar{\zeta})^p(\bar{z}\zeta)^q \; \zeta^n \, d\sigma(\zeta)$$
$$=\sum_{p=0}^{\infty}(p+n+1)(p+1)r^{2p+n}z^{p+n} \; (\bar{z})^p$$
$$=(rz)^n\left\{\sum_{p=0}^{\infty}(p+1)^2 r^{2p}+n\sum_{p=0}^{\infty}(p+1)r^{2p}\right\}$$
$$=(rz)^n\left\{\frac{1+r^2}{(1-r^2)^3}+\frac{n}{(1-r^2)^2}\right\},$$

and consequently,

$$\int_T \frac{(1-r^2)^3}{|\zeta-rz|^4}\zeta^n \, d\sigma(\zeta)=(rz)^n\left\{1+r^2+n(1-r^2)\right\}.$$

Hence

$$\int_T F(\zeta, rz)\zeta^n \, d\sigma(\zeta)$$
$$=\frac{1}{2}(rz)^n\left\{(1-r^2)+(1+r^2)+n(1-r^2)\right\}$$
$$=(rz)^n\left(1+\frac{n}{2}(1-r^2)\right).$$

Now assume that $n=-p$ is a negative integer. Similar argument, using the fact that for $\zeta \in T$ we have $\zeta^n=\bar{\zeta}^p$, we see that

$$\int_T F(\zeta, rz)\zeta^n \, d\sigma(\zeta)=r^{|n|}z^n\left(1+\frac{n}{2}(1-r^2)\right),$$

from which it follows that

$$\tilde{f}(rz)=\int_T F(\zeta, rz) f(\zeta)\, d\sigma(\zeta)$$
$$=\sum_{n=-k}^{k} c_n \int_T F(\zeta, rz)\zeta^n \, d\sigma(\zeta)$$
$$=\sum_{n=-k}^{k} c_n r^{|n|} z^n\left(1+\frac{n}{2}(1-r^2)\right).$$

This completes the proof.

Proposition 2.4. Let $f \in C(T)$ and $u=F[f]$. Then $\tilde{f}$ is uniformly continuous on $\bar{D}$. In particular, the functions u_r converge uniformly to f on T, as $r \to 1$.

Proof. It follows from Proposition 2.1(b) that for $0 \le r < 1$ we have

$$|u(rz)|=\left|\int_T F(\zeta, rz) f(\zeta)\, d\sigma(\zeta)\right|$$
$$\le \sup_{\zeta\in T}\left|f(\zeta)\right|=\|f\|_T.$$

Hence

$$\|\tilde{f}\|_{\bar{D}}=\|f\|_T, \qquad f \in C(T). \tag{2-1}$$

Let $p(z)=\sum_{n=-k}^{k} c_n z^n$ be a trigonometric polynomial on T . By Proposition 2.3

$$\tilde{p}(rz) = \sum_{n=-k}^{k} c_n r^{|n|} z^n \left(1+\frac{n}{2}(1-r^2)\right),$$

$$0 \le r < 1, \quad z \in T.$$

In particular, $\tilde{p}$ is continuous on $\overline{D}$. Since the trigonometric polynomials are dense in $C(T)$, we can find a sequence of polynomials $\{p_n\}$ on T such that $\|p_n - f\|_T \to 0$ as $n \to \infty$. It now follows from (2-1) that

$$\|\tilde{p}_n - \tilde{f}\|_{\overline{D}} = \|p_n - f\|_T \to 0, \qquad n \to \infty.$$

Hence the sequence $\tilde{p}_n$ converges uniformly to $\tilde{f}$ on $\overline{D}$. As we already observed, each $\tilde{p}_n$ is continuous on $\overline{D}$, so that $\tilde{f}$ is uniformly continuous on $\overline{D}$. In particular, if $0 \le r < 1$, then continuity of $\tilde{f}$ on $\overline{D}$ yields

$$\lim_{r\to 1} \tilde{f}(rz) = f(z), \qquad z \in T,$$

or equivalently,

$$\|u_r - f\|_T \to 0, \qquad r \to 1.$$

The proof is complete.

It is now time to state the main result of this paper.

Theorem 2.5. Let $f \in L^p(T)$ for $1 \le p < \infty$. Put $u = F[f]$, and $u_r(z) = u(rz)$ for

$z \in T$, and $0 \le r < 1$. Then

(a) $u_r \in L^p(T)$ and $\|u_r\|_{L^p(T)} \le \|f\|_{L^p(T\}}$,

(b) for $1 \le p < +\infty$, the functions u_r converge to f in the mean, that is

$$\|u_r - f\|_{L^p(T)} \to 0, \qquad r \to 1,$$

(c) for $p = +\infty$, the functions u_r converge weak-star to f as $r \to 1$.

Proof. In proving (a), the case $p = +\infty$ follows readily from the definition and Proposition 2.1(b). Assume now that $1 \le p < +\infty$ and note that

$$u_r(z) = F[f](rz) = \int_T F(\zeta, rz) f(\zeta) d\sigma(\zeta), z \in D,$$

so that

$$|u_r(z)| \le \int_T |f(\zeta)| F(\zeta, rz)\, d\sigma(\zeta).$$

Putting $F(\zeta, rz) d\sigma(\zeta) = d\lambda(\zeta)$, we have $\int_T d\lambda(\zeta) = 1$ in accordance with Proposition 2.1(b). On the other hand, the function $x \mapsto x^p$ is convex for $1 \le p < +\infty$, and $x > 0$, hence we can apply Jensen's formula (see [5], p. 62) to obtain

$$\begin{aligned} |u_r(z)|^p &\le \left(\int_T |f(\zeta)| d\lambda(\zeta)\right)^p \\ &\le \int_T |f(\zeta)|^p\, d\lambda(\zeta). \end{aligned} \tag{2-2}$$

We now integrate both sides of (2-2) to get

$$\begin{aligned} \|u_r\|_{L^p(T)}^{\ p} &= \int_T |u(rz)|^p d\sigma(\zeta) \\ &\le \iint_{T\times T} |f(\zeta)|^p d\lambda(\zeta) d\sigma(z) \\ &\le \iint_{T\times T} |f(\zeta)|^p\ F(\zeta, rz) d\sigma(\zeta) d\sigma(z). \end{aligned} \tag{2-3}$$

It follows from Fubini's theorem and Proposition 2.1(b, c) that

$$\begin{aligned} &\iint_{T\times T} |f(\zeta)|^p\ F(\zeta, rz) d\sigma(\zeta) d\sigma(z) \\ &= \iint_{T\times T} |f(\zeta)|^p\ F(z, r\zeta) d\sigma(z) d\sigma(\zeta) \\ &= \int_T F(z, r\zeta) d\sigma(z) \int_T |f(\zeta)|^p\, d\sigma(\zeta) \\ &= \int_T |f(\zeta)|^p\, d\sigma(\zeta) \\ &= \|f\|_{L^p(T)}^{\ p}. \end{aligned}$$

This together with (2-3) implies that

$$\|u_r\|_{L^p(T)} \le \|f\|_{L^p(T)}$$

completing the proof of part (a).

As for part (b) we fix $\varepsilon > 0$. Since the trigonometric polynomials are dense in $L^p(T)$, we choose atrigonometric polynomial $g(z) \in L^p(T)$ such that

$$\|f - g\|_{L^p\{T\}} < \varepsilon.$$

Putting $v = F[g]$, we have

$$\|u_r - f\|_{L^p\{T\}} \le \|u_r - v_r\|_{L^p(T)}$$
$$+\|v_r - g\|_{L^p(T)} + \|g - f\|_{L^p(T)}.$$

Since $u - v = F[f - g]$, it follows from part (a) that

$$\|u_r - v_r\|_{L^p(T)} = \|(u-v)_r\|_{L^p(T)} \le \|f - g\|_{L^p(T)} < \varepsilon,$$

hence

$$\|u - f_r\|_{L^p(T)} \le 2\varepsilon + \|v_r - g\|_{L^p(T)}. \qquad (2\text{-}4)$$

According to Proposition 2.4,

$$\|v_r - g\|_{L^p(T)} \le \|v_r - g\|_T \to 0, \qquad r \to 1,$$

which means that for r sufficiently close to 1 we have

$$\|v_r - g\|_{L^p(T)} \le \varepsilon.$$

This together with (2-4) implies that

$$\|u_r - f\|_{L^p(T)} < 3\varepsilon.$$

We now manage to prove part (c). Let $g \in L^1(T)$ and $v = F[g]$. It follows from Proposition 2.1(c) and Fubini's theorem that

$$\begin{aligned}\int_T u_r(z)g(z)\,d\sigma(z) &= \int_T\left(\int_T F(\zeta, rz) f(\zeta)\,d\sigma(\zeta)\right) g(z)\,d\sigma(z) \\ &= \int_T\left(\int_T F(z, r\zeta) g(z)\,d\sigma(z)\right) f(\zeta)\,d\sigma(\zeta) \qquad (2\text{-}5)\\ &= \int_T v_r(z) f(z)\,d\sigma(z).\end{aligned}$$

Therefore

$$\int_T u_r(z)g(z)\,d\sigma(z) - \int_T f(z)g(z)\,d\sigma(z) = \int_T (v_r(z) - g(z)) f(z)\,d\sigma(z),$$

in accordance with (2-5). It follows that

$$\left|\int_T u_r(z)g(z)\,d\sigma(z) - \int_T f(z)g(z)\,d\sigma(z)\right| \le \|f\|_{L^\infty(T)} \|v_r - g\|_{L^1(T)}.$$

But, according to part (b), the right hand side of the above inequality tends to zero as r approaches 1, so that

$$\lim_{r\to 1}\int_T u_r(z)g(z)\,d\sigma(z) = \int_T f(z)g(z)\,d\sigma(z),$$
$$g \in L^1(T),$$

meaning that the functions u_r are weak-star convergent to f as $r \to 1$.

3. The *F*-Integrals of Measures

Let ν be a Borel measure on the unit circle. The F-integral of ν is defined in the following manner:

$$\mu(z) = F[\nu](z) = \int_T F(\zeta, z)\,d\nu(\zeta), \qquad z \in D.$$

The following theorem states that the mapping $\nu \mapsto \mu = F[\nu]$ is injective, moreover, for a finite Borel measure ν on T, the functions μ_r converge weak-star to ν as $r \to 1$.

Theorem 3.1. Let ν be a finite Borel measure on the unit circle T. Then

(a) the functions μ_r (or the measures $\mu_r d\sigma$) converge weak-star to ν as $r \to 1$,

(b) the mapping $\nu \mapsto \mu = F[\nu]$ is injective,

(c) if ν is positive, then

$$\|\mu_r\|_{L^1(T)} = \nu(T), \qquad 0 < r < 1.$$

Proof. Let g be a continuous function on T, and $v = F[g]$. As in (2-5), we use Fubini's theorem together with Proposition 2.1(c) to obtain

$$\int_T \mu_r(z)g(z)\,d\sigma(z) = \int_T v_r(z)\,d\nu(z).$$

Since g is uniformly continuous on the unit circle, it follows from Proposition 2.4 that $v_r \to g$ uniformly on T as $r \to 1$. Hence for every $g \in C(T)$,

$$\lim_{r\to 1}\int_T \mu_r(z)g(z)\,d\sigma(z) =$$
$$\lim_{r\to 1}\int_T v_r(z)\,d\nu(z) = \int_T g(z)\,d\nu(z),$$

from which the result follows.

To prove (b), we let $\mu = 0, f \in C(T)$ and $u = F[f]$. As before, we see that

$$\int_T u_r(z)\,d\nu(z) = \int_T \mu_r(z) f(z)\,d\sigma(z).$$

By Proposition 2.4, the functions u_r are uniformly convergent to f on the unit circle, as $r \to 1$. Hence for every $f \in C(T)$ we have

$$\int_T f(z)\,d\nu(z) = \lim_{r\to 1} \int_T u_r(z)\,d\nu(z) = 0,$$

which means that the measure ν vanishes identically.

As for part (c), we see by a direct calculation that

$$\begin{aligned}\|\mu_r\|_{L^1(T)} &= \int_T \mu_r(z)\,d\sigma(z)\\ &= \int_T \left(\int_T F(\zeta, rz)\,d\nu(\zeta)\,d\sigma(z) \right)\\ &= \int_T \left(\int_T F(\zeta, rz)\,d\sigma(z) \right) d\nu(\zeta)\\ &= \frac{1}{2}\int_T \left((1-r^2) + (1+r^2) \right) d\nu(\zeta) = \nu(T).\end{aligned}$$

The proof is complete.

Concluding remark. Let $B(D)$ denote the class of all biharmonic functions on the unit disk, and let $M(T)$ denote the class of all finite Borel measures on the unit circle. We have already observed that both the F-integral of an $L^1(T)$ function and the F-integral of a finite Borel measure on T are elements of $B(D)$. Moreover, the mapping $\nu \mapsto F[\nu]$ from $M(T)$ to $B(D)$ is injective. It is desirable to solve the following problem.

Open problem. Let u be an element of $B(D)$. Find condition(s) under which there exists a finite Borel measure $\nu \in M(T)$ such that $u = F[\nu]$.

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