Journal of Sciences, Islamic Republic of Iran 17(3): 259-264 (2006) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

On the Convergence Rate of the Law of Large Numbers for Sums of Dependent Random Variables

H.R. Nili Sani,[1,*] H.A. Azarnoosh,[2] and A. Bozorgnia[2]

[1] *Department of Statistics, Faculty of Sciences, Birjand University, Birjand, Islamic Republic of Iran*
[2] *Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University, Mashhad, Islamic Republic of Iran*

Abstract

In this paper, we generalize some results of Chandra and Goswami [4] for pairwise negatively dependent random variables (henceforth r.v.'s). Furthermore, we give Baum and Katz's [1] type results on estimate for the rate of convergence in these laws.

Keywords: Negatively dependent random variables; Complete convergence; Strong law of large numbers

1. Introduction and Preliminaries

Let $\{X_n, n \ge 1\}$ be a sequence of integrable r.v.'s defined on the same probability space.

Chandra and Goswami [4] have proved the following theorem from the arguments of Csorgo *et al.* [5].

Theorem CG1. Let $\{X_n, n \ge 1\}$ be a sequence of non-negative r.v.'s with finite $Var(X_n)$ and $f(n)$ be an increasing sequence such that $f(n) > 0$ for each n and $f(n) \to \infty$ as $n \to \infty$. Put $S_n = \sum_{i=1}^{n} X_i$. Assume that

$$\sup_{n\ge 1} \frac{1}{f(n)} \sum_{i=1}^{n} X_i = A(say) < \infty; \tag{1.1}$$

and there is a double sequence $\{\rho_{ij}\}$ of nonnegative reals such that

$$Var(S_n) \le \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \quad \text{for each } n \ge 1; \tag{1.2}$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} / (f(i \vee j))^2 < \infty, \quad (i \vee j) = \max(i, j). \tag{1.3}$$

Then $(f(n))^{-1}\left[S_n - E(S_n)\right] \to 0$ a.s. as $n \to \infty$.

Nili and Bozorgnia [11] generalized (and corrected) Theorem CG1 for an array of r.v.'s and obtained the following result:

Theorem NB. Let $\{X_{ni}, n \ge 1, i \ge 1\}$ be an array of non-negative r.v.'s with finite $Var(X_{ni})$ and $[\log_\alpha f(n)]$, $\alpha > 1$ be an increasing sequence. Put $S_n = \sum_{i=1}^{l(n)} a_{ni} X_{ni}$, where $l(x)$ stands for a nondecreasing continuous function with inverse l^{-1} such that $l(n)$ is a natural sequence and $l(n) \to \infty$. Assume that there is a double sequence of nonnegative reals $\{\rho_{ij}\}$ such that

*E-mail: nilisani@yahoo.com

$$Var(S_n) \le \sum_{i=1}^{l(n)} \sum_{j=1}^{l(n)} \rho_{ij} \quad \text{for each } n \ge 1; \tag{1.4}$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \rho_{ij} f^{-2}(l^{-1}(i) \vee l^{-1}(j)) < \infty. \tag{1.5}$$

Then $(f(n))^{-1}\left[S_n - E(S_n)\right] \to 0$ completely as $n \to \infty$, in the sense of Hsu and Robbins [6] (see also page 225 of Stout [12]), and hence, a.s.

The question underlying the present work is how one may refine Theorem CG1 to give more information on the law of $\{X_n\}$. We recall the classical answer, the strong law of large numbers Baum and Katz [1] for $p = 2$ (see [2]). In Section 3 we generalize Theorem CG1 and give Baum-Katz's [1] type results on estimate for the rate of convergence in these laws.

Chandra and Goswami [4], also proved Theorem CG2, by Theorem CG1, and extended the results of Landers and Rogge [8] for pairwise independent r.v.'s.

Theorem CG2. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables such that there is a sequence $\{B_n\}$ of Borel subsets of R^1 satisfying the following conditions

(a) $\sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty;$

(b) $\sum_{i=1}^{n} E(X_i I(X_i \in B_i^c)) = o(f(n));$

(c) $\sum_{n=1}^{\infty} (f^{-2}(n) Var(X_n I(X_n \in B_n)) < \infty;$

and

(d) $\sup_{n \ge 1} [\sum_{k=1}^{n} E(|X_k| I(X_k \in B_k))/(f(n)] < \infty;$

here B_n^c is the complement of B_n. Then $(f(n))^{-1}\left[S_n - E(S_n)\right] \to 0$ almost surly as $n \to \infty$.

In Section 3 we also extend Theorem CG2 to negative dependence r.v.'s.

2. Negative Dependence

Definition 1. ([9]). Random variables $X_1, ..., X_n (n \ge 2)$ are said to be pairwise negatively dependent (henceforth pairwise *ND*) if

$$P(X_i > x_i, X_j > x_j) \le P(X_i > x_i) P(X_j > x_j), \tag{2.1}$$

holds for all $x_i, x_j \in \Re, i \ne j$. It can be shown that (2.1) is equivalent to

$$P(X_i \le x_i, X_j \le x_j) \le P(X_i \le x_i) P(X_j \le x_j), \tag{2.2}$$

for all x_i, $x_j \in \Re, i \ne j$.

Events $\{E_n\}$ are said to be pairwise negatively dependent if their indicator functions are pairwise negatively dependent.

Example 1. Let $X + Y = c$, $c \in R$. It is easy to see that X and Y are negatively dependent.

An infinite collection of $\{X_n, n \ge 1\}$ is said to be pairwise *ND* if every finite subcollection is pairwise *ND*. We will need the following results [3,7,10].

Proposition 1. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise *ND* r.v's. Then the following are true:

(i) $Cov(X_i, X_j) < 0, \quad i \ne j$,

(ii) If $\{f_n, n \ge 1\}$ is a sequence of Borel functions all of which are monotone increasing (or all monotone decreasing) then $\{f_n(X_n), n \ge 1\}$ is a sequence of pairwise *ND* r.v's.

(iii) The Borel-Cantelli lemma holds for pairwise *ND* events.

3. Main Results

In the following theorems $\alpha \ge 1/2$ and r is an integer such that $r = 2\alpha - 2$ when $2\alpha - 2$ is integer and $r = [2\alpha - 2] + 1$ ($[x]$ is integer part of x) otherwise. Also in this paper, C stands for a generic constant, not necessarily the same at each appearance. Put $S_n = \sum_{i=1}^{n} X_i$.

Theorem 1. Let $\{X_n, n \ge 1\}$ be a sequence of r.v.'s and $\{f(n), n \ge 1\}$ be a sequence of positive reals such that for some $\beta > 1, [\log_\beta f(n)]$ is an increasing sequence. Assume that there is a double sequence $\{\rho_{ij}\}$ of non-negative reals such that ρ_{ii} is upper bound for $Var(X_i)$ and

$$Var(S_n) \le \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij}. \tag{3.1}$$

If for some $\xi < 2\alpha$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\rho_{ij}(i\vee j)^{r-1/2}}{\beta^{\xi(i\vee j)}}<\infty\ , \tag{3.2}$$

then for every $\varepsilon>0$

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(|S_n-E(S_n)|>\varepsilon f^{\alpha}(n))<\infty. \tag{3.3}$$

Remark. If $\alpha=1$ we can use theorem NB for $X_{ni}=X_i$, $l(n)=n$ and $a_{ni}=1$, it is sufficient to replace (3.2) by (1.5), then (3.3) holds.

Proof. We use sub-sequence method. Replacing X_i by $X_i-E(X_i)$ we may use $E(X_i)=0$. It is easy to show that

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(|S_n|>\varepsilon f^{\alpha}(n))$$

$$\le\sum_{n=1}^{\infty}n^{2(2\alpha-2)}P(|S_{n^2}|>\varepsilon f^{\alpha}(n^2))+$$

$$+\sum_{n^2<k<(n+1)^2}^{\infty}k^{(2\alpha-2)}P(|S_{n^2}|>\varepsilon/2f^{\alpha}(k))+$$

$$+\sum_{n^2<k<(n+1)^2}^{\infty}k^{(2\alpha-2)}P(D_n>\varepsilon/2f^{\alpha}(k))\ ,$$

where $D_n=\max_{n^2<k<(n+1)^2}|S_k-S_{n^2}|$. It is sufficient to show that each of three above series is convergent.

$$\sum_{n=1}^{\infty}n^{2(2\alpha-2)}P(|S_{n^2}|>\varepsilon f^{\alpha}(n^2))$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\rho_{ij}\sum_{n^2\ge i\vee j}\frac{n^{2(2\alpha-2)}}{\beta^{2\alpha(n^2)}}$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\rho_{ij}\int_{\theta-1}^{\infty}\frac{x^{2(2\alpha-2)}}{\beta^{2\alpha(x^2)}}dx\ ,$$

where $\theta=[\sqrt{i\vee j}\,]$. Thus using the change of variable $e^{y}=\beta^{2\alpha x^2}$, $dy=4\alpha x\ln(\beta)dx$, we get RHS

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\rho_{ij}\int_{2\alpha\ln\beta(\theta-1)^2}^{\infty}\frac{y^{(2\alpha-2)}}{e^{y}\sqrt{y}}dy$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\frac{\rho_{ij}}{\sqrt{i\vee j}}\int_{2\alpha\ln\beta(\theta-1)^2}^{\infty}\frac{y^{r}}{e^{y}}dy$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\frac{\rho_{ij}}{\sqrt{i\vee j}}[(\theta-1)^{2r}\beta^{-2\alpha(\theta-1)^2}]$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\frac{\rho_{ij}(i\vee j)^{r-1/2}}{\beta^{2\alpha(\theta-1)^2}}$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\frac{\rho_{ij}(i\vee j)^{r-1/2}}{\beta^{2\alpha(\sqrt{i\vee j}-2)^2}}$$

$$\le C\sum_{i=5}^{\infty}\sum_{j=5}^{\infty}\frac{\rho_{ij}(i\vee j)^{r-1/2}}{\beta^{\xi(i\vee j)}}<\infty.$$

For the second series we have

$$\sum_{n^2<k<(n+1)^2}^{\infty}k^{(2\alpha-2)}P(|S_{n^2}|>\varepsilon f^{\alpha}(k)/2)$$

$$\le C\sum_{n^2<k<(n+1)^2}^{\infty}\frac{n^{2(2\alpha-2)}}{\beta^{2\alpha n^2}}\sum_{i=1}^{n^2}\sum_{j=1}^{n^2}\rho_{ij}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\sum_{n^2\ge i\vee j}\frac{n^{2(2\alpha-2)}}{\beta^{2\alpha n^2}}<\infty.$$

And finally we must show that S_k does not differ enough from nearest S_{n^2} to make any real difference.

$$\sum_{n^2<k<(n+1)^2}^{\infty}k^{2\alpha-2}P(D_n>\varepsilon/2f^{\alpha}(k))$$

$$\le C\sum_{n^2<k<(n+1)^2}^{\infty}\frac{k^{2\alpha-2}}{f^{2\alpha}(k)}E(D_n^2)$$

$$\le C\sum_{n=1}^{\infty}\frac{n^{2(2\alpha-2)+1}}{\beta^{2\alpha n^2}}\sum_{i=n^2+1}^{(n+1)^2-1}\rho_{ii}$$

$$\le C\sum_{i=1}^{\infty}\rho_{ii}\sum_{n:n^2<i<(n+1)^2}\frac{n^{2(2\alpha-2)+1}}{\beta^{2\alpha n^2}},$$

for a fix i the second sum include one statement and we have

$$\le C\sum_{i=1}^{\infty}\rho_{ii}\frac{[\sqrt{i}\,]^{2(2\alpha-2)+1}}{\beta^{2\alpha[\sqrt{i}\,]^2}}.$$

Note that $\frac{[\sqrt{i}\,]^{2(2\alpha-2)+1}}{\beta^{2\alpha[\sqrt{i}\,]^2}}\le C\frac{i^{r-1/2}}{\beta^{\xi i}}$, if i is sufficiently large, thus

$$\sum_{i=1}^{\infty}\rho_{ii}\frac{[\sqrt{i}\,]^{2(2\alpha-2)+1}}{\beta^{2\alpha[\sqrt{i}\,]^2}}\le C\sum_{i=1}^{\infty}\rho_{ii}\frac{i^{r-1/2}}{\beta^{\xi i}}<\infty.$$

In the next theorems we relax the condition that for some $\beta > 1$, $[\log_\beta f(n)]$ is an increasing sequence. The Proofs follow the same lines as the proof of Theorem 1.

Theorem 2. Let $\{X_n, n \geq 1\}$ and $\{\rho_{ij}\}$ be as in Theorem 1 such that

$$Var(\sum_{j=i}^{n} X_i) \leq \sum_{j=i}^{n}\sum_{j=i}^{n} \rho_{ij} \qquad \forall i, n .$$

Let $f(n)$ be an increasing sequence such that $\{n / f(n)\}$ be a bounded sequence. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} /(i \vee j)^{3/2} < \infty , \tag{3.4}$$

then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^{\alpha}(n)) < \infty.$$

Proof. The Chebyshev's inequality, condition (3.4) and a change of order of summation imply that

$$\sum_{n=1}^{\infty} n^{2(2\alpha-2)} P(|S_{n^2}| > \varepsilon f^{\alpha}(n^2))$$

$$\leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)} E(S_{n^2})^2 \leq C \sum_{n=1}^{\infty}\sum_{i=1}^{n^2}\sum_{j=1}^{n^2} \rho_{ij} / n^4$$

$$\leq C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n^2 > (i \vee j)} 1/n^4 \leq C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} /(i \vee j)^{3/2} < \infty .$$

For the second series we have

$$\sum_{n^2 < k < (n+1)^2}^{\infty} k^{2\alpha-2} P(|S_{n^2}| > \varepsilon f^{\alpha}(k)/2)$$

$$\leq C \sum_{n^2 < k < (n+1)^2}^{\infty} \frac{k^{2\alpha-2}}{f^{2\alpha}(k)} E(S_{n^2})^2$$

$$\leq C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n^2 > (i \vee j)} 1/n^4$$

$$\leq C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} /(i \vee j)^{3/2} < \infty.$$

And finally we show that the third series is convergent

$$\sum_{n^2 < k < (n+1)^2}^{\infty} k^{2\alpha-2} P(D_n > \varepsilon / 2 f^{\alpha}(k))$$

$$\leq C \sum_{n=1}^{\infty} \frac{k^{2\alpha-2}}{f^{2\alpha}(k)} E(D_n^2)$$

$$\leq C \sum_{n^2 < k < (n+1)^2}^{\infty} \frac{2n}{n^4} \sum_{i=n^2+1}^{(n+1)^2-1} \rho_{ii} = C \sum_{i=1}^{\infty} \rho_{ii} \sum_{n=(i+1)^{1/2}-1}^{(i-1)^{1/2}} \frac{1}{n^3}$$

$$\leq C \sum_{i=1}^{\infty} \rho_{ii} \frac{1}{(\sqrt{i+1}-1)^3} (\sqrt{i-1} - \sqrt{i+1} + 1)$$

$$\leq C \sum_{i=1}^{\infty} \rho_{ii} \left(\frac{\sqrt{i}}{\sqrt{i+1}-1} \right)^3 (\frac{1}{\sqrt{i}})^3 < \infty.$$

Theorem 3. Let $\{X_n, n \geq 1\}$ be a sequence of r.v.'s and $\{\rho_{ij}\}$ be a double sequence of nonnegative reals such that

$$Var(S_n) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} \quad \text{for each } n \geq 1; \tag{3.5}$$

Assume that $\{f(n)\}$ is an increasing sequence such that $n^{\beta} \leq f(n) \leq (n+1)^{\beta}$ for some $0 < \beta \leq 1$ and for each $n \geq 1$. If

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (f^{-\gamma}(i \vee j)) \rho_{ij} < \infty,$$

where $\gamma = (3 + 4\alpha\beta - 4\alpha)/2\beta$ and $\alpha < \dfrac{3}{4(1-\beta)}$, then for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n - E(S_n)| > \varepsilon f^{\alpha}(n)) < \infty.$$

Proof. Again we are going to use subsequence method. Replacing X_i by $X_i - E(X_i)$, we may assume $E(X_i) = 0$.

$$\sum_{n=1}^{\infty} n^{2(2\alpha-2)} P(|S_{n^2}| > \varepsilon f^{\alpha}(n^2))$$

$$\leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)} E(S_{n^2})^2$$

$$\leq C \sum_{n=1}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)} \sum_{i=1}^{n^2}\sum_{j=1}^{n^2} \rho_{ij}$$

$$= C \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho_{ij} \sum_{n^2 > (i \vee j)}^{\infty} \frac{n^{2(2\alpha-2)}}{f^{2\alpha}(n^2)}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\,(\frac{1}{x^{2\beta\gamma}})_{x+1=(i\vee j)^{1/2}}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\,(\frac{1}{(x+1)^{2\beta\gamma}})_{f((x+1)^2)=f(i\vee j)}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\,(\frac{1}{(((x+1)^2+1)^{\beta})^{\gamma}})_{f((x+1)^2)=f(i\vee j)}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\,(\frac{1}{f^{\gamma}(i\vee j)})<\infty$$

by Chebyshev's inequality and (3.5). For the second sum we have

$$\sum_{n^2<k<(n+1)^2}^{\infty}k^{2\alpha-2}P(\left|S_{n^2}\right|>\varepsilon f^{\alpha}(k)/2)$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\sum_{n^2\ge(i\vee j)}^{\infty}\frac{1}{k^{(2\alpha\beta-2\alpha+2)}}$$

$$\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\rho_{ij}\sum_{n^2\ge(i\vee j)}^{\infty}\frac{1}{n^{2(2\alpha\beta-2\alpha+2)}}<\infty.$$

Thus it remains to verify that the third sum is convergent

$$\sum_{n^2<k<(n+1)^2}^{\infty}k^{2\alpha-2}P(D_n>\varepsilon/2f^{\alpha}(k))$$

$$\le C\sum_{n^2<k<(n+1)^2}^{\infty}\frac{2nk^{2\alpha-2}}{f^{2\alpha}(k)}\sum_{i=n^2+1}^{(n+1)^2-1}E(X_i^2)$$

$$\le C\sum_{i=1}^{\infty}\rho_{ii}\sum_{n^2<i<(n+1)^2}\frac{1}{n^{(4\alpha\beta-4\alpha+3)}}$$

$$\le C\sum_{i=1}^{\infty}\rho_{ii}\sum_{n:f(n^2)<f(i)<f((n+1)^2)}\frac{1}{(n+1)^{2\beta\gamma}}$$

$$\le C\sum_{i=1}^{\infty}\rho_{ii}/f^{\gamma}(i)<\infty$$

Theorem 4. Let α, β, ξ, r and $f(n)$ be as in Theorem 1. Also Let $\{X_n, n\ge 1\}$ be a sequence of pairwise *ND* r.v's such that there is a sequence $\{B_n, n\ge 1\}$ of semi intervals $(-\infty, x_n]$ ($(-\infty, x_n)$, $[x_n, \infty)$ or (x_n, ∞)), $x_n\in R$, satisfying in the following conditions:

(a) $\sum_{n=1}^{\infty}C_nP(X_n\in B_n^c)<\infty$

where

$$C_n=1\vee(\frac{x_n^2}{\beta^{\xi n}}n^{r-0.5});$$

(b) $\sum_{n=1}^{\infty}\frac{n^{r-0.5}}{\beta^{\xi n}}E(X_n^2I(X_n\in B_n))<\infty$;

(c) $\{\left|X_n-x_n\right|I(X_n\in B_n^c)\}$ is uniformly integrable;

here B_n^c is the complement of B_n. Then for every $\varepsilon>0$

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(\left|S_n-E(S_n)\right|>\varepsilon f^{\alpha}(n))<\infty.$$

Proof. Put $Y_n = X_nI(X_n\in B_n) + x_nI(X_n\notin B_n)$, $Z_n = X_n-Y_n$, $S_n=\sum_{i=1}^{n}X_i$, $S_n^*=\sum_{i=1}^{n}Y_i$ and $S_n' = S_n-S_n^*=\sum_{i=1}^{n}Z_i$, $n\ge 1$. It is obvious that $\{Y_n, n\ge 1\}$ and $\{Z_n, n\ge 1\}$ are two sequences of pairwise *ND* r.v.'s. It is sufficient to show that

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(\left|S_n'-E(S_n')\right|>\varepsilon f^{\alpha}(n))<\infty,$$

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(\left|S_n^*-E(S_n^*)\right|>\varepsilon f^{\alpha}(n))<\infty.$$

By Theorem 1, conditions (a) and (b) and Proposition 1 applied to $\{Y_n, n\ge 1\}$ yields

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P(\left|S_n^*-E(S_n^*)\right|>\varepsilon f^{\alpha}(n))$$

$$\le C\sum_{n=1}^{\infty}\frac{n^{2\alpha-2}}{f^{2\alpha}(n)}Var(S_n^*)$$

$$\le C\sum_{n=1}^{\infty}\frac{n^{2\alpha-2}}{f^{2\alpha}(n)}\sum_{i=1}^{n}E(Y_i^2)$$

$$=C\sum_{n=1}^{\infty}\frac{n^{2\alpha-2}}{f^{2\alpha}(n)}[\sum_{i=1}^{n}E(X_i^2I(X_i\in B_i))$$

$$+\sum_{i=1}^{n}x_i^2P(X_i\in B_i^c)]<\infty.$$

Hence, it is sufficient to prove the first sentence. Since

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty ,$$

$\{X_n\}$ and $\{Y_n\}$ are equivalent and

$$\sum_{n=1}^{\infty} n^{2\alpha-2} P(|S_n' - E(S_n')| > \varepsilon f^{\alpha}(n))$$

$$\leq C \sum_{n=1}^{\infty} \frac{n^{2\alpha-2}}{f^{\alpha}(n)} E(|S_n' - E(S_n')|) < \infty ,$$

by (c) and the first Borel Canteli lemma, the desired result follows.

References

1. Baum L.E. and Katz M. Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.*, **120**: 108-123 (1965).
2. Bingham N.H. Moving average. In: G.A. Edgar and L. Sucheston (Eds.), *Almost Everywhere Convergence I*, Academic Press, 129-139 (1989).
3. Bozorgnia A., Patterson R.F., and Taylor R.L. Limit Theorems for dependent random variables. *World Congress Nonlinear Analysts*, **92**: 1639-1650 (1996).
4. Chandra T.K. and Goswami A. Cesaro uniform integrability and the strong law of large numbers. *Sankhya*: *The Indian Journal of Statistics*, Vol. **54**, Series A, Pt. 2, 215-231 (1992).
5. Csorgo S., Tandori K., and Totik V. On the strong law of large numbers for pairwise independent random variables. *Acta Math. Hungarica*, **42**: 319-330 (1983).
6. Hsu P.L. and Robbins H. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Soc.*, **33**: 25-31 (1947).
7. Joag-Dev K. and Proschan F. Negative association of random vaiables with applications. *Ann. Statist*., **11**: 286-295 (1983).
8. Landers D. and Rogge L. Laws of large numbers for pairwise independent uniformly integrable random variables. *Math. Nachr.*, **130**: 189-192 (1986).
9. Lehmann E.L. Some concept of dependence. *Ann. Math. Statist*., **37**: 1137-1153 (1966).
10. Matula P.A. A note on the almost sure convergence of sums of negatively dependent random variables. *Stat. Probab. Letters*, **15**: 209-213 (1992).
11. Nili Sani H.R. and Bozorgnia A. On limit theorems for arrays of rowwise PND random variables. *J. Instit. Math. & Comp. Sci.* (Mathematice series), **16**: 5-12 (2003).
12. Stout W.F. *Almost Sure Convergence*. Academic Press (1974).