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SHIFT OPERATOR FOR PERIODICALLY CORRELATED PROCESSES

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Abstract

The existence of shift for periodically correlated processes and its boundedness are investigated. Spectral criteria for these non-stationary processes to have such shifts are obtained.

Keywords: Periodically correlated; Shift; Stationary; Linearly stationary; Bounded linearly stationary

1. Introduction

Prediction theory of stationary stochastic processes has been extensively developed and is now considered to be complete. The existence of bounded shift for stationary processes has played a major role in this development. The existence and boundedness of shift for non-stationary processes is important [1]. An interesting class of non-stationary stochastic processes is that of periodically correlated (PC) processes. This class of processes has been studied by several authors [2-14]. However, questions concerning their shift have not yet been considered. In this note, we study these questions and obtain spectral criteria for the existence of bounded shift for PC processes.

2. Preliminaries

Let $(\Omega, \beta, \mathrm{P})$ be a probability space and $L_0^2(\Omega, \beta, \mathrm{P})$ denote the space of all complex-valued random variables on Ω with zero mean and finite variance. The inner product and norm here are given by

$$(X, Y) = E(X\overline{Y}) = \int_\Omega X(\omega)\overline{Y(\omega)}\, d\mathrm{P}(\omega)$$

and $\|X\| = \sqrt{(X, X)}$.

Any sequence $\{X_n,\, n \in Z\}$ of random variables in $L_0^2(\Omega, \beta, \mathrm{P})$ will be called a stochastic process and its correlation function $R(m,n)$ is defined by

$$R(m,n) = E(X_m \overline{X}_n)\, .$$

Given a stochastic process X_n, its shift operator V is a linear transformation which sends X_n to X_{n+1}, for each $n \in Z$. In general this operator is not well-defined and in order to make the above definition, it is necessary

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to impose certain restrictions on X_n. Before we proceed further, let's now state the formal definition of shift operator and consider an example where this operator is not well-defined.

Definition 2.1. (a) A stochastic process X_n is said to have a shift if the linear transformation V on $L(X) = SP\{X_n : n \in Z\}$ which sends each X_n to X_{n+1} is well defined. **(b)** A process X_n is said to have a bounded shift if it has a shift which can be extended to

$$H(X) = \overline{SP}\{X_n : n \in Z\}$$

as a bounded operator.

Example. Let Y_n be any nonzero stochastic process and define a new stochastic process X_n by

$$X_n = \begin{cases} Y_k & if \quad n = 2k \\ 0 & if \quad n = 2k+1 \end{cases}$$

The shift operator for X_n is clearly ill-defined because it sends a zero vector, say X_1 to a nonzero vector, say X_2. If we take the original stochastic process Y_n to be a nondeterministic, then we get an example of a stochastic process X_n which has no shift.

Definition 2.2. A stochastic process X_n is called stationary if

$$R(m,n) = R(m+1, n+1)$$

For all $m, n \in Z$.

It is well-known that any stationary stochastic process has a bounded shift and that it is a unitary operator. However, for a non-stationary process as we saw above the shift may not even exist and in order for a stochastic process X_n to have a shift we must impose some restrictions on the process X_n or its correlation function $R(m,n)$. For the following lemma one can see [1].

Lemma 2.3. Let X_n be a stochastic process with correlation function $R(m,n)$ as defined above. Then

(a) In order for X_n to have a shift it is necessary and sufficient that for any finite sequence $\{a_n\}$ of complex numbers

$$\sum a_m \overline{a_n} R(m,n) = 0 \Rightarrow \sum a_m \overline{a_n} R(m+1, n+1) = 0 .$$

(b) In order for X_n to have a bounded shift it is necessary and sufficient to have a positive number M such that for any finite sequence $\{a_n\}$ of complex numbers

$$\sum a_m \overline{a_n} R(m+1, n+1) \le M \sum a_m \overline{a_n} R(m,n) .$$

We close this section with a brief introduction to periodically correlated processes.

Definition 2.4. A stochastic process X_n is called periodically correlated with period p if for all $m, n \in Z$, we have

$$R(m,n) = R(m+p, n+p) .$$

Such a process will be briefly called a PC process. Let X_n be a PC process with period p. Then for each integer τ, the function $R(n, n+\tau)$ is periodic in n with period p. Therefore it has Fourier expansion

$$R(n, n+\tau) = \sum_{k=0}^{p-1} R_k(\tau) \exp(\frac{2\pi ikn}{p}) ,$$

where $R_k(\tau)$ are given by

$$R_k(\tau) = \frac{1}{p} \sum_{n=0}^{p-1} R(n, n+\tau) \exp(-2\pi in\tau) .$$

For convenience, we extend the definition of these $R_k(\tau)$, $k = 0, 1, \ldots, p-1$ to all integers k by $R_k(\tau) = R_{k+p}(\tau)$.

It is shown in [3] that each $R_k(\tau)$ has a spectral representation of the form

$$R_k(\tau) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\tau\theta} dF_k(\theta)$$

where each dF_k is a complex-valued measure on $[0,2\pi)$. One can then see that

$$R(m,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}e^{-i(m\theta-n\lambda)}dF(\theta,\lambda)$$

where the spectral measure dF of X_n is given by

$$F(A,B)=\sum_{k=1-p}^{p-1}F_k(A\cap(B-\frac{2\pi k}{p}))\,.$$

Here $B-a$ stands for the set of all numbers of the form $b-a$ with $b\in B$. This shows that spectral measure dF of any PC process is concentrated on $2p-1$ line segments $\theta-\lambda=2\pi k/p$, $k=1-p,\dots,p-1$, contained in the square $[0,2\pi)\times[0,2\pi)$, and

$$R(m,n)=r(m-n)=\frac{1}{2\pi}\int_0^{2\pi}e^{-i(m-n)\theta}dF(\theta)$$

3. Shift for PC Processes

In this section we study the questions of existence and boundedness of shift for PC processes. We first obtain some basic results and then prove our main result which gives several criteria for existence and boundedness of shift including a spectral criterion.

Lemma 3.1. Let X_n be a PC process with period p.

(a) If X_n has shift V then V is invertible.

(b) If X_n has bounded shift V then V is boundedly invertible.

Proof. (a) Suppose X_n has shift V and assume $\sum a_n X_n=0$ for some finite sequence $\{a_n\}$ of complex numbers. This means that $\sum a_m\overline{a_n}R(m,n)=0$. Applying Lemma 2.3(a) we get $\sum a_m\overline{a_n}R(m+1,n+1)=0$. Applying Lemma 2.3(a) $p-2$ more times to the latter equation, we get

$$\sum a_m\overline{a_n}R(m+p-1,n+p-1)=0\,.$$

Considering that X_n is PC with period p, we get

$$\sum a_m\overline{a_n}R(m-1,n-1)=0\,,$$

which means $\sum a_n X_{n-1}=0$. So we showed that for any sequence of complex numbers a_n

$$\sum a_n X_n=0 \Rightarrow \sum a_n X_{n-1}=0\,.$$

But this is clearly equivalent to the existence of the inverse U of V which sends each X_n to X_{n-1}.

(b) Suppose X_n has a bounded shift V. and let a_n be a finite sequence of complex numbers. Applying Lemma 2.3(b) $p-1$ consecutive times we arrive at

$$\sum a_m\overline{a_n}R(m+p-1,n+p-1)$$
$$\le M^{p-1}\sum a_m\overline{a_n}R(m,n)\,,$$

which in conjunction with the fact that X_n is periodically correlated with period p implies

$$\sum a_m\overline{a_n}R(m-1,n-1)\le M^{p-1}\sum a_m\overline{a_n}R(m,n)\,.$$

Therefore

$$\left\|\sum a_n X_{n-1}\right\|^2\le M^{p-1}\left\|\sum a_n X_n\right\|^2,$$

which implies that backward shift U sending each X_n to X_{n-1} has a bounded extension to $H(X)$. Since clearly $U=V^{-1}$, we conclude that V is boundedly invertible.

The following remarks follows from the proof of Lemma 3.1.

Remarks 3.2. Let X_n be any PC process with period p.

(a) If X_n has a shift (bounded shift) V, then it has shifts (bounded shifts) V_k, of any order $k\in Z$, sending each X_n to X_{n+k}. In fact, it is clear that $V_k=V^k$.

(b) The shift V_p always exists and it is unitary.

(c) If X_n has a bounded shift V, then for any positive integer k,

$$\|V_k\| \le \|V^k\| \quad \text{and} \quad \|V_{-k}\| \le \|V\|^{k(p-1)}.$$

Before we proceed further, we need to introduce some terminologies. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_q$ be q Hilbert spaces. Their direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_q$, equipped with the Euclidean inner product

$$((\mathbf{X}, \mathbf{Y})) = \sum_{i=1}^{q} (X_i, Y_i)$$

becomes a Hilbert space. Here $\mathbf{X} = X_1 \oplus X_2 \oplus \ldots \oplus X_q$ and $\mathbf{Y} = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_q$, with $X_i, Y_i \in \mathcal{H}_i$. Direct sum of q copies of $\mathcal{H}$ will be denoted by $\mathcal{H}^q$.

Lemma 3.3. A stochastic process X_n in $\mathcal{H} = L_0^2(\Omega, \beta, \mathrm{P})$ is periodically correlated with period p if and only if its associated process $\mathbf{Z}_n$ in $\mathcal{H}^q$ defined by

$$\mathbf{Z}_n = X_n \oplus X_{n+1} \oplus \ldots \oplus X_{n+p-1}$$

is stationary.

Proof. "only if" part: If X_n is a PC process with period p, then for any m and n in Z, we can write

$$((\mathbf{Z}_m, \mathbf{Z}_n)) = \sum_{i=0}^{p-1} (X_{m+i}, X_{n+i})$$

$$= (X_m, X_n) + \sum_{i=1}^{p-1} (X_{m+i}, X_{n+i})$$

$$= (X_{m+p}, X_{n+p}) + \sum_{i=1}^{p-1} (X_{m+i}, X_{n+i})$$

$$= \sum_{i=1}^{p} (X_{m+i}, X_{n+i})$$

$$= \sum_{i=0}^{p-1} (X_{m+i+1}, X_{n+i+1})$$

$$= ((\mathbf{Z}_{m+1}, \mathbf{Z}_{n+1})) .$$

This means that $\mathbf{Z}_m$ is stationary. Proof of "if" part is similar.

Lemma 3.4. If X_n is a PC process with period p then the spectral measure of its associated stationary process Z_n introduced in last lemma is $pdF_\circ$, where $dF_\circ$ is the part of the spectral measure dF of X_n supported on the main diagonal of the square mentioned in section 2.

Proof. For the proof, we refer the reader to [7].

Definition 3.5. A stochastic process X_n in the Hilbert space $\mathrm{H} = L_0^2(\Omega, \beta, \mathrm{P})$ is called linearly stationary if there exists a stationary process W_n in another Hilbert space κ and an invertible transformation $T: \kappa \to \mathcal{H}$ such that $X_n = TW_n$, for all $n \in Z$. A linearly stationary process X_n is called bounded linearly stationary if the transformation T can be chosen to be bounded.

It is clear that linearly stationary processes are in general non-stationary. Nevertheless prediction properties of linearly stationary processes can easily be investigated. Because one can transfer a prediction problem concerning a linearly stationary process X_n to one about its stationary counterpart W_n, we find the solution for this stationary process W_n and then transfer the result back to the original process X_n For more detail, one can see [15].

In what follows, we will use the following notations and terminologies.

Let X_n be a PC process with period p in $\mathcal{H} = L_0^2(\Omega, \beta, \mathrm{P})$ and $\mathbf{Z}_n$ in $\mathcal{H}^p$, be its associated stationary process introduced in Lemma 3.3, namely:

$$\mathbf{Z}_n = X_n \oplus X_{n+1} \oplus \ldots \oplus X_{n+p-1}$$

Now let $P: \mathcal{H}^p \to \mathcal{H}^p$ denote the orthogonal projection which maps any vector in $\mathcal{H}^p$ to its first coordinate, i.e.

$$P(X_1 \oplus X_2 \oplus \ldots \oplus X_p) = X_1,$$

for any $X_1, X_2, \ldots, X_p \in \mathcal{H}$. We denote by κ the subspace of $\mathcal{H}^p$ spanned by all Z_n's and Q to stand for the restriction of P to κ.

Theorem 3.6. For any PC process X_n with period p, the following statements are equivalent.

(a) X_n has a bounded shift V.

(b) X_n is bounded linearly stationary.

(c) The operator $Q:\kappa \to H$ defined above is boundedly invertible.

Proof. We prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (c): Take a finite linear combination $\sum a_n X_n$. We can write:

$$\left\|\sum a_n \mathbf{Z}_n\right\|^2 = \left\|\sum a_n X_n\right\|^2 + \left\|\sum a_n X_{n+1}\right\|^2$$

$$+ \ldots + \left\|\sum a_n X_{n+p-1}\right\|^2$$

$$= \left\|\sum a_n X_n\right\|^2 + \left\|V\sum a_n X_n\right\|^2$$

$$+ \ldots + \left\|V^{p-1}\sum a_n X_n\right\|^2$$

$$\leq (1+\|V\|^2 + \ldots + \|V\|^{2(p-1)})\left\|\sum a_n X_n\right\|^2.$$

which shows the inverse of Q exists and is bounded.

(c) $\Rightarrow$ (b): By Lemma 3.3, $X_n = P\mathbf{Z}_n$ where $\mathbf{Z}_n$ is the stationary process associated to X_n. Since each $\mathbf{Z}_n$ is clearly in κ and Q is the restriction of P to κ then we get $X_n = Q\mathbf{Z}_n$ for all $n \in Z$ and this completes the proof.

(b) $\Rightarrow$ (a): Suppose there is a stationary process W_n and boundedly invertible operator $T: H(W) \to H(X)$ such that

$$X_n = T(W_n) \quad \text{for all} \quad n \in Z.$$

Let U be the well-known unitary shift of the stationary process W_n and define $V: H(X) \to H(X)$ by

$$V = TUT^{-1}.$$

One can check that V is the shift of our process X_n. Now since T and U are bounded $V = TUT^{-1}$ is bounded.

Theorem 3.7. Let X_n be a PC process with period p. The following statements are equivalent.

(a) X_n has a shift.

(b) X_n is linearly stationary.

(c) The operator $Q:\kappa \to \mathcal{H}$ defined above is invertible.

Proof. (a) $\Rightarrow$ (c): Suppose X_n has a shift, say V and suppose a finite linear combination of X_n is zero, i.e. $\sum a_n X_n = 0$. Applying V to both sides of this equation $p-1$ times, we get

$$\sum a_n X_{n+1} = 0, \ldots, \sum a_n X_{n+p-1} = 0.$$

Thus

$$\left\|\sum a_n \mathbf{Z}_n\right\|^2 = \sum_{i=0}^{p-1}\left\|\sum a_n X_{n+i}\right\|^2 = 0$$

which means $\sum a_n \mathbf{Z}_n = 0$. Hence Q is invertible.

(c) $\Rightarrow$ (b): By Lemma 3.3, $X_n = P\mathbf{Z}_n$ where $\mathbf{Z}_n$ is the stationary process associated to X_n. Since each $\mathbf{Z}_n$ is clearly in κ and Q is the restriction of P to κ then we get $X_n = Q\mathbf{Z}_n$ for all $n \in Z$ and this completes the proof.

(b) $\Rightarrow$ (a): Let W_n be the stationary process and $T: L(W) \to L(X)$ be the linear transformation with $X_n = TW_n$ and $U: H(W) \to H(W)$ be the standard unitary shift operator of the stationary process W_n, then the linear transformation $V = TUT^{-1}$ clearly serves the desirable shift for X_n.

Next theorem gives our spectral characterization for a PC process to have a shift.

Theorem 3.8. Let X_n be a PC process with period p whose spectral measure $dF(\cdot)$ is concentrated on $2p$

line segments $\theta-\lambda=2\pi k/p$, $k=1-p,\dots,0,\dots,p-1$ of $[0,2\pi)\times(2\pi,0]$ with the measure on the diagonal being dF_0.

(a) X_n has a bounded shift if and only if there exists a positive number K such that

$$\int_0^{2\pi}|\phi(\theta)|^2\,dF_0(\theta)\le K\int_0^{2\pi}\int_0^{2\pi}\phi(\theta)\overline{\phi(\lambda)}dF(\theta,\lambda)$$

for any trigonometric polynomial $\phi(\theta)=\sum a_n e^{-in\theta}$.

(b) X_n has a shift if and only if

$$\int_0^{2\pi}\int_0^{2\pi}\phi(\theta)\overline{\phi(\lambda)}dF(\theta,\lambda)=0$$

$$\Rightarrow\int_0^{2\pi}|\phi(\theta)|^2\,dF_0=0,$$

for any trigonometric polynomial function $\phi(\theta)=\sum a_n e^{-in\theta}$.

Proof. (a) If X_n has a bounded shift then by Theorem 3.6, the operator Q is boundedly invertible. This means there exists some $M>0$ such that

$$\left\|\sum a_n Z_n\right\|\le M\left\|\sum a_n X_n\right\|$$

for every finite sequence a_n of complex numbers. Squaring both sides and rewriting it in terms of the spectral measure we get

$$p\int_0^{2\pi}|\phi(\theta)|^2\,dF_0\le M^2\int_0^{2\pi}\int_0^{2\pi}\phi(\theta)\overline{\phi(\lambda)}dF(\theta,\lambda)$$

where $\phi(\theta)=\sum a_n e^{-in\theta}$. This shows that (1) holds with $K=M^2/p$. Now assume that inequality (1) holds. We can rewrite it as

$$(\ell/p)\left\|\sum a_n \mathbf{Z}_n\right\|^2\le K\left\|\sum a_n X_n\right\|^2,$$

which means the operator Q in part (c) of Theorem 3.6 is boundedly invertible. This in virtue of Theorem 3.6 completes the proof of part (a).

Proof of part (b) is similar to the proof of part (a).

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