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The Minimum Universal Cost Flow in an Infeasible Flow Network

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Abstract

In this paper the concept of the *Minimum Universal Cost Flow* (MUCF) for an infeasible flow network is introduced. A new mathematical model in which the objective function includes the total costs of changing arc capacities and sending flow is built and analyzed. A polynomial time algorithm is presented to find the MUCF.

Keywords: Minimum cost network flow problem; Infeasible flow network

1. Introduction

Let G = (N,A, **u**,**c**) be a network with node set N, arc set A, capacity vector **u** and cost vector **c**. Each components u_{ij} and c_{ij} of **u** and **c** represent the arc capacity and flow cost of $(i,j)\in A$, respectively. A minimum cost flow problem (MCF) on G is defined as:

$$\min. \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1-1a}$$

$$s.t: \quad \sum_{j\in N} x_{ij} - \sum_{k\in N} x_{ki} = b_i \qquad \forall i\in N \tag{1-1b}$$

$$0\le x_{ij}\le u_{ij} \qquad \forall (i,j)\in A \tag{1-1c}$$

A flow **x** that satisfies Equations (1-1b) and (1-1c) is called a feasible flow and the network is feasible if such a flow exists. Feasibility of a flow network depends on node numbers, b_i s and arc capacities u_{ij} s. Flow network infeasibility was first considered by Hoffman [1]. He proved that a flow network is infeasible if and only if there exists a cut $(S,\bar{S})$ such that

$$\sum_{i\in S} b_i > \sum_{(i,j)\in(S,\bar{S})} u_{ij} \tag{1-2}$$

In such a case, S is called isolation by Greenberg [2] and a witness by Aggarwal, Hao and Orlin [3]. Aggarwal *et al.* [3] showed that the problem of finding a minimum witness in an infeasible flow network is Np-hard. But rather efficient heuristic procedures have been introduced for practical instances, Greenberg [2,4,5].

Having a flow network been diagnosed as infeasible, the next task is to convert it to a feasible one by the least cost. For this purpose McCormick [6] introduced the following model:

$$\min. \sum_{(i,j)\in A} c'_{ij}\alpha_{ij} \tag{1-3a}$$

$$s.t: \quad \sum_{j\in N} x_{ij} - \sum_{k\in N} x_{ki} = b_i \qquad \forall i\in N \tag{1-3b}$$

$$0\le x_{ij}\le u_{ij}+\alpha_{ij} \qquad \forall (i,j)\in A \tag{1-3c}$$

where c'_{ij} is the cost of changing the capacity of arc

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$(i, j) \in A$ by one units. When the above problem is solved the optimal amount of capacity change and a feasible flow in the resulting network would be obtained. Such a feasible flow was called a Least Infeasible Flow (LIF) [6].

As it is seen, the objective function of McCormick's model consists of the modification cost only. The optimal flow cost has to be computed by solving the minimum cost flow problem using the resulted arc capacities.

In this paper we first construct a comprehensive model that includes both modification and flow costs. Then a polynomial time algorithm for computing the optimal capacities and flow is presented. This algorithm has two applications, *i.e.* modifying the arc capacities and obtaining the optimal flow in the resulting feasible flow network at the same time, and minimizing the sum of modification and flow costs instead of the modification cost only. In Sections 2 and 3 we construct the model and obtain the optimality conditions. In Section 4 a polynomial time algorithm that finds the optimal arc capacities and flow is introduced and is verified.

2. Problem Formulation and Analysis

Suppose that due to the current arc capacities the flow network G = (N,A) and the minimum cost flow problem defined by (1-1) are infeasible. Thus there exists a witness S such that (1-2) is true. In order to change the network to a feasible one, arc capacities have to be modified so that there does not exists any witness satisfying (1-2). In the procedure of computing LIF, arc capacities are changed so as the total changing cost is minimized. The computed LIF is merely a feasible flow in the resulted network and hence its sending cost may be non-optimal, when both costs are taken into account. The following example proves this claim.

Example 2.1. Consider the flow network shown in Figure (2.1), where c_{ij} denotes the flow cost, c'_{ij} denotes the capacity changing cost and u_{ij} denotes the current capacity of arc $(i, j) \in A$. Let α_{ij} be the amount of the increment of the capacity of arc $(i, j) \in A$. The LIF that minimizes $\sum_{(i,j)\in A} c'_{ij}\alpha_{ij}$ is $x^*_{12} = 1$, $x^*_{23} = 4$, $x^*_{13} = 1$, $\alpha^*_{13} = 1$, $\alpha^*_{12} = 2$, $\alpha^*_{23} = 0$ and $x^*_{13} = 1$, and the minimum cost of the converting the flow network to a feasible one is 48. The total costs of converting and flow cost, $\sum_{(i,j)\in A} c_{ij}x^*_{ij} + \sum(c_{ij} + c'_{ij})\alpha^*_{ij}$ is 88. If we let $x^*_{12} = 1, \alpha^*_{12} = 3, x^*_{13} = 1$ and $\alpha^*_{13} = 0$, then $\sum_{(i,j)\in A} c_{ij}x^*_{ij} + \sum(c_{ij} + c'_{ij})\alpha^*_{ij}$ becomes 87. □

The above example shows that a LIF is not the best solution of an infeasible flow network.

In order to change the infeasible network G to a feasible one, according to (2-1) the current arc capacities have to be increased, as much as all witnesses are vanished. Since the amount of increments must incur the least possible cost, any feasible solution of the resulting flow network has a flow amount equal to the new arc capacities. In other words if **x** is a flow satisfying the current capacity constraints, the amount of change on x_{ij}, in order to satisfy the conservation constraints, is equal to the increasing amount of u_{ij}. Now let α_{ij} and c'_{ij} respectively denote the number of capacity units to be added to u_{ij} and the cost of each unit, the minimum universal cost flow model is defined as:

$$\min. \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{(i,j)\in A} (c_{ij} + c'_{ij})\alpha_{ij} \qquad (2\text{-}1a)$$

$$s.t: \sum_{j\in N}(x_{ij} + \alpha_{ij}) - \sum_{k\in N}(x_{ki} + \alpha_{ki}) = b_i \quad \forall i \in N \qquad (2\text{-}1b)$$

$$0 \le x_{ij} \le u_{ij} \qquad \forall (i, j) \in A \qquad (2\text{-}1c)$$

$$\alpha_{ij} \ge 0 \qquad \forall (i, j) \in A \qquad (2\text{-}1d)$$

It is obvious that the problem (2-1) is not a minimum cost flow problem and can't be solved by the related algorithms. Since $c'_{ij} \ge 0$, problem (2-1) may be solved by a minimum convex cost flow algorithm. But we introduce a different algorithm, to solve the problem, without duplication of arcs. First we obtain the optimality conditions.

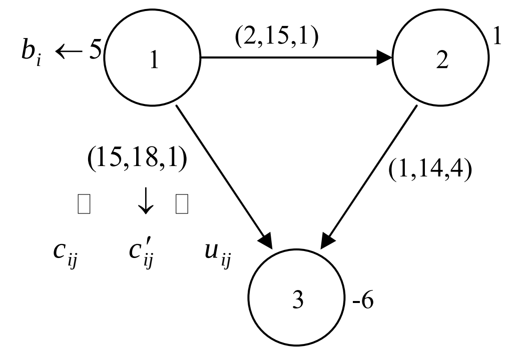

Figure 2.1

2.1. Optimality Conditions

Let π_i and λ_{ij} denote the dual variables corresponding to (2-1b) and (2-1c), respectively. The dual problem of (2-1) is defined as:

$$\max. \sum_{i\in N} b_i\pi_i - \sum_{(i,j)\in A}\lambda_{ij}u_{ij} \qquad (2\text{-}2a)$$

$$s.t: \quad \pi_i - \pi_j - \lambda_{ij} \le c_{ij} \qquad \forall (i,j)\in A \qquad (2\text{-}2b)$$

$$\pi_i - \pi_j \le c_{ij} + c'_{ij} \qquad \forall (i,j)\in A \qquad (2\text{-}2c)$$

$$\pi_i \ \textit{unrestricted} \qquad (2\text{-}2d)$$

$$\lambda_{ij} \ge 0 \qquad (2\text{-}2e)$$

Theorem 2.1. Suppose $c_{ij}^{\pi} = c_{ij} - \pi_i + \pi_j$ and $\bar{c}_{ij}^{\pi} = c_{ij}^{\pi} + c'_{ij}$ denote the reduced costs of arc $(i,j)\in A$ with c_{ij} and $c_{ij} + c'_{ij}$ costs respectively. $(\boldsymbol{x}^*, \boldsymbol{\alpha}^*)$ and $\boldsymbol{\pi}^*$ are respectively optimal if and only if the following conditions hold:

$$1)\ c_{ij}^{\pi^*} < 0 \Rightarrow x_{ij}^* = u_{ij}\ , \begin{cases} \bar{c}_{ij}^{\pi^*} \ne 0 \Rightarrow \alpha_{ij}^* = 0 \\ \bar{c}_{ij}^{\pi^*} = 0 \Rightarrow \alpha_{ij}^* \ge 0 \end{cases} \qquad (2\text{-}3a)$$

$$2)\ c_{ij}^{\pi^*} > 0 \Rightarrow x_{ij}^* = 0\ , \alpha_{ij}^* = 0 \qquad (2\text{-}3b)$$

$$3)\ c_{ij}^{\pi^*} = 0 \Rightarrow 0 \le x_{ij}^* \le u_{ij}\ , \alpha_{ij}^* = 0 \qquad (2\text{-}3c)$$

Proof. Considering the problems (2-1) and (2-2) and the complementary slackens conditions we have:

$$x_{ij}^*(c_{ij} - \pi_i^* + \pi_j^* + \lambda_{ij}^*) = 0 \quad \Rightarrow x_{ij}^*(c_{ij}^{\pi^*} + \lambda_{ij}^*) = 0$$

$$\alpha_{ij}^*(c_{ij} + c'_{ij} - \pi_i^* + \pi_j^*) = 0 \quad \Rightarrow \alpha_{ij}^*\bar{c}_{ij}^{\pi^*} = 0$$

$$\lambda_{ij}^*(u_{ij} - x_{ij}^*) = 0$$

Since (2-2a) is to be maximized and the coefficient of λ_{ij} in (2-2a) is $-u_{ij}$, in an optimal solution, λ_{ij} should be at the least possible value. Thus if $x_{ij}^* \ne 0$ then $\lambda_{ij}^* = -c_{ij}^{\pi^*}$. If $c_{ij}^{\pi^*} \ne 0$ then $\lambda_{ij}^* \ne 0$ and hence we have $x_{ij}^* = u_{ij}$. Noting $\alpha_{ij}^*\bar{c}_{ij}^{\pi^*} = 0$, (2-3a) is obtained. Now if $c_{ij}^{\pi^*} > 0$, Since $\lambda_{ij}^* \ge 0$, then $c_{ij}^{\pi^*} + \lambda_{ij}^* > 0$ and the condition $x_{ij}^*(c_{ij}^{\pi^*} + \lambda_{ij}^*) = 0$ implies $x_{ij}^* = 0$, and $\bar{c}_{ij}^{\pi^*} > 0$ implies $\alpha_{ij}^* = 0$. If $c_{ij}^{\pi^*} = 0$, then $x_{ij}^*\lambda_{ij}^* = 0$, $\bar{c}_{ij}^{\pi^*} > 0$. Hence $0 \le x_{ij}^* \le u_{ij}$ and $\alpha_{ij}^* = 0$. □

3. Computing the Optimal Solution

In the model (2-1) the flow on an arc may exceed its current capacity. The extra flow is denoted by α_{ij}. If an arc has a positive residual capacity it is no need to increase its capacity. In other words whenever arc (i,j) is saturated, then α_{ij} embarks to be positive. This fact can be easily extracted from the complementary slackens conditions (2-3). Sending α_{ij} units of extra flow on arc (i,j), causes a cost equal to $(c_{ij} + c'_{ij})\alpha_{ij}$, since the capacity increment cost is $c'_{ij}\alpha_{ij}$ and sending flow cost is $c_{ij}\alpha_{ij}$. Therefore whenever the flow on an arc reaches its capacity, the arc cost becomes $c_{ij} + c'_{ij}$. Now let $B = \sum_{b_i > 0} b_i$ and $\boldsymbol{x}^{\boldsymbol{\alpha}} = \boldsymbol{x} + \boldsymbol{\alpha}$, for a pseudo flow **x** and $\boldsymbol{\alpha} \ge 0$. It is very easy to see that, within any feasible flow network it is no need for arc capacities to be greater than *B*. The model (2-1) can now be reformulated as:

$$\min. \sum_{(i,j)\in A} c_{ij}x_{ij}^{\alpha} + \sum_{(i,j)\in A} c'_{ij}\alpha_{ij} \qquad (3\text{-}1a)$$

$$s.t: \quad \sum_{j\in N} x_{ij}^{\alpha} - \sum_{k\in N} x_{ki}^{\alpha} = b_i \qquad \forall i \in N \qquad (3\text{-}1b)$$

$$0 \le x_{ij}^{\alpha} \le B \qquad \forall (i,j)\in A \qquad (3\text{-}1c)$$

$$\alpha_{ij} \ge 0 \qquad \forall (i,j)\in A \qquad (3\text{-}1d)$$

4. The Minimum Universal Cost Flow (MUCF) Algorithm

As many minimum cost flow algorithms, MUCF algorithm uses residual network for adjusting the arc capacities. The residual network that is used in our algorithm differs in arc capacities and costs from the residual networks defined so far. First we describe this network which we call it *expanded residual network*. As it will be seen later the algorithm at each iteration generates a pseudo flow $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in which $\alpha_{ij} = 0$ while $x_{ij} < u_{ij}$. Now define:

$$A_1 = \{(i,j): 0 < x_{ij}^{\alpha} < u_{ij}\} = \{(i,j): 0 < x_{ij} < u_{ij}\}$$

and

$$A_2 = \{(i,j): x_{ij}^{\alpha} \ge u_{ij}\} = \{(i,j): x_{ij} = u_{ij}\}$$

The expanded residual network, $\bar{G}(\boldsymbol{x}^{\boldsymbol{\alpha}})$ is constructed as follows:

$\bar{G}(\boldsymbol{x}^{\alpha})$ Contains all nodes of G. For each arc $(i,j) \in A$ two arcs (i,j) and (j,i) belong to $\bar{G}(\boldsymbol{x}^{\alpha})$. The former one has residual capacity $r_{ij} = u_{ij} - x_{ij}^{\alpha}$ and cost c_{ij}. The residual capacity and cost of the latter one are $r_{ji} = x_{ij}^{\alpha}$ and $c_{ji} = -c_{ij}$, respectively. For each arc $(i,j) \in A_2$ there are two arcs (i,j) and (j,i) in $\bar{G}(x^{\alpha})$. The residual capacity and cost of (i,j) are $B - x_{ij}^{\alpha}$ and $c_{ij} + c'_{ij}$, respectively. For arc (j,i), if $x_{ij}^{\alpha} = u_{ij}$ then $r_{ji} = x_{ij}^{\alpha}$ and $c_{ji} = -c_{ij}$, otherwise $r_{ji} = x_{ij}^{\alpha} - u_{ij}$ and $c_{ji} = -(c_{ij} + c'_{ij})$.

Now let **x** be a pseudo flow and $\boldsymbol{\alpha} \geq 0$, for $i \in A$ let:

$$e(i) = b_i + \sum_{j \in N} x_{ji}^{\alpha} - \sum_{k \in N} x_{ik}^{\alpha}$$

Define $E = \{i : e(i) > 0\}$ as the set of excess nodes and $D = \{i : e(i) < 0\}$ as the set of deficit nodes. The algorithm begins with a pseudo flow **x** = 0, extra capacity vector $\boldsymbol{\alpha} = 0$ and node potentials $\boldsymbol{\pi} = 0$. At each step it selects a node $s \in E$ and a node $t \in D$ and finds a shortest path P from s to t in $\bar{G}(x^{\alpha})$, using c_{ij}^{π} and $\bar{c}_{ij}^{\pi}$ as the length of arcs with $x_{ij} \leq u_{ij}$ and $\alpha_{ij} > 0$, respectively. Then

$$\delta = \min\{e(s), -e(t), \min\{r_{ij} : (i,j) \in P\}\}$$

units of flow is augmented along P in $\bar{G}(\boldsymbol{x}^{\alpha})$. The equivalent operations in G are as follows:

Suppose $(i,j) \in P$ in $\bar{G}(\boldsymbol{x})$. Two cases are possible:

Case I: $(i,j) \in G$, if $(i,j) \in A_1$, set $x_{ij} = x_{ij} + \delta$, otherwise set $\alpha_{ij} = \alpha_{ij} + \delta$.

Case II: $(j,i) \in G$, if $0 < x_{ji} < u_{ji}$, set $x_{ji} = x_{ji} - \delta$, otherwise if $\alpha_{ji} = 0$, set $x_{ji} = x_{ji} - \delta$, and if $\alpha_{ji} > 0$, set $\alpha_{ji} = \alpha_{ji} - \delta$.

After updating **x** and $\boldsymbol{\alpha}$ the algorithm updates π to $\boldsymbol{\pi} - \boldsymbol{d}$, where **d**, denotes the shortest path distances from s to all other nodes. The above steps are repeated until E and/or D are empty.

4.1. Algorithm Verification

Lemma 4.1. A feasible flow $\boldsymbol{x}^{\alpha}$ is an optimal solution of problem (3-1) if and only if the expanded residual network, $\bar{G}(\boldsymbol{x}^{\alpha})$, contains no negative cost directed cycle.

Proof. In $\bar{G}(\boldsymbol{x}^{\alpha})$ define $A_3 = \{(j,i) : (i,j) \in A_1\}$ and $A_4 = \{(j,i) : (i,j) \in A_2\}$. For a given $\boldsymbol{x}^{\alpha}$, suppose that $\bar{G}(\boldsymbol{x}^{\alpha})$ contains a negative cost directed cycle as w. The cost of w equals to

$$C(w) = \sum_{(i,j) \in A_1 \cap w} c_{ij} + \sum_{(i,j) \in A_2 \cap w} c_{ij} + c'_{ij} + \sum_{(i,j) \in A_3 \cap w} -c_{ij} + \sum_{(i,j) \in A_4 \cap w} -(c_{ij} + c'_{ij})$$

which is negative. Let $\delta = \min\{r_{ij} : (i,j) \in w\}$, augmenting δ units of flow along w reduces the objective function of problem(3-1) by $\delta C(w)$ units. Thus $\boldsymbol{x}^{\alpha}$ could not be optimal. Conversely suppose that for a given feasible flow $\boldsymbol{x}^{\alpha}$ of (3-1), $\bar{G}(\boldsymbol{x}^{\alpha})$ contains no negative cost directed cycle. Let $\boldsymbol{x}^{\alpha^*}$ denotes the optimal solution of problem (3-1). Then according to [7], $\boldsymbol{x}^{\alpha^*} - \boldsymbol{x}^{\alpha}$ is a feasible solution of problem (3-1) and can be decomposed in to at most m directed cycles in $\bar{G}(\boldsymbol{x}^{\alpha})$. The sum of the cost of flows on these cycles according to (3-1a) is

$$\sum_{(i,j) \in A} c_{ij}(x_{ij}^{\alpha^*} - x_{ij}^{\alpha}) + \sum_{(i,j) \in A} c'_{ij}(\alpha_{ij}^* - \alpha_{ij}).$$

Since the cost of all cycles in $\bar{G}(\boldsymbol{x}^{\alpha})$ are nonnegative we have

$$\sum_{(i,j) \in A} c_{ij}(x_{ij}^{\alpha^*} - x_{ij}^{\alpha}) + \sum_{(i,j) \in A} c'_{ij}(\alpha_{ij}^* - \alpha_{ij}) \geq 0$$

or

$$\sum_{(i,j) \in A} c_{ij} x_{ij}^{\alpha^*} + \sum_{(i,j) \in A} c'_{ij} \alpha_{ij}^* \geq \sum_{(i,j) \in A} c_{ij} x_{ij}^{\alpha} + \sum_{(i,j) \in A} c'_{ij} \alpha_{ij}.$$

According to the optimality of $\boldsymbol{x}^{\alpha^*}$ we get

$$\sum_{(i,j) \in A} c_{ij} x_{ij}^{\alpha^*} + \sum_{(i,j) \in A} c'_{ij} \alpha_{ij}^* \leq \sum_{(i,j) \in A} c_{ij} x_{ij}^{\alpha} + \sum_{(i,j) \in A} c'_{ij} \alpha_{ij}.$$

Hence $\boldsymbol{x}^{\alpha}$ is also an optimal solution of problem (3-1). □

Remark 4.1. A feasible flow for problem (3-1) is optimal if and only if, some set of node potentials, $\boldsymbol{\pi}$, satisfy the following conditions:

$$c_{ij}^{\pi} \geq 0, \quad \forall (i,j) \in A_1 \cup A_3$$

$$\textit{and} \qquad (4\text{-}1)$$

$$\bar{c}_{ij}^{\pi} \geq 0, \quad \forall (i,j) \in A_2 \cup A_4 .$$

$$\textit{in } \bar{G}(\boldsymbol{x}^{\boldsymbol{\alpha}}).$$

Proof. Given a feasible flow $\boldsymbol{x}^{\boldsymbol{\alpha}}$ for problem (3-1), suppose that there exists some node potentials $\boldsymbol{\pi}$, satisfying condition (4-1). Therefore for every directed cycle w,

$$\sum_{(i,j)\in(A_1\cup A_3)\cap w} c_{ij}^{\pi} + \sum_{(i,j)\in(A_2\cup A_4)\cap w} \bar{c}_{ij}^{\pi} \geq 0 .$$

Thus we have

$$\sum_{(i,j)\in(A_1\cup A_3)\cap w} c_{ij} + \pi_i - \pi_j$$
$$+ \sum_{(i,j)\in(A_2\cup A_4)\cap w} c_{ij} + c'_{ij} + \pi_i - \pi_j \geq 0.$$

Since w is a directed cycle we get:

$$\sum_{(i,j)\in(A_1\cup A_3)\cap w} c_{ij} + \sum_{(i,j)\in(A_2\cup A_4)\cap w} c_{ij} + c'_{ij} \geq 0 .$$

i.e. the cost of cycle w is nonnegative. Thus $\bar{G}(\boldsymbol{x}^{\boldsymbol{\alpha}})$ contains no negative cost directed cycle. Conversely, if contains no negative cost directed cycle, then the shortest path distances from node 1 to all other nodes, d, are well defined and satisfy the conditions

$$d(j) \leq d(i) + c_{ij} \quad , \quad \forall (i,j) \in A_1 \cup A_3$$

and

$$d(j) \leq d(i) + c_{ij} + c'_{ij} \quad , \quad \forall (i,j) \in A_2 \cup A_4 .$$

If we let $\boldsymbol{\pi} = -\boldsymbol{d}$, then

$$c_{ij} - (-d(i) + (-d(j)) = c_{ij}^{\pi} \geq 0 , \quad \forall (i,j) \in A_1 \cup A_3$$

and

$$c_{ij} + c'_{ij} - (-d(i) + (-d(j)) = \bar{c}_{ij}^{\pi} \geq 0 ,$$
$$\forall (i,j) \in A_2 \cup A_4 . \ \square$$

The following two lemmas are counterparts of Lemmas 9-11 and 9-12 in [7] and are similarly proved.

Lemma 4.2. Suppose that $\boldsymbol{x}^{\boldsymbol{\alpha}}$ satisfies (4-1) with respect to some node potentials $\boldsymbol{\pi}$. Let vector **d** denotes the shortest path distances from node s to all other nodes in $\bar{G}(\boldsymbol{x}^{\boldsymbol{\alpha}})$ with c_{ij}^{π} and $\bar{c}_{ij}^{\pi}$ as the lengths of arcs $(i,j) \in A_1 \cup A_3$ and $(i,j) \in A_2 \cup A_4$, respectively. Then the following properties are valid.

a) The pseudo flow $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *also satisfies the optimality conditions (4-1) with respect to* $\boldsymbol{\pi}' = \boldsymbol{\pi} - \boldsymbol{d}$.

b) $c_{ij}^{\pi'} \geq 0, \forall (i,j) \in (A_1 \cup A_3) \cap P$ *and* $\bar{c}_{ij}^{\pi'} \geq 0$, $\forall (i,j) \in (A_2 \cup A_4) \cap P$, *where P denotes the shortest path from s to all other nodes.* □

Lemma 4.3. Suppose that a pseudo flow $\boldsymbol{x}^{\boldsymbol{\alpha}}$ satisfies condition (4-1) and we obtain $\boldsymbol{x}^{\boldsymbol{\alpha}'}$ from $\boldsymbol{x}^{\boldsymbol{\alpha}}$ by sending flow along a shortest path from node s to some other node k in $\bar{G}(\boldsymbol{x}^{\boldsymbol{\alpha}})$, then $\boldsymbol{x}^{\boldsymbol{\alpha}'}$ also satisfies condition (4-1). □

Theorem 4.1. Algorithm MUCF solves problem (2-1) in polynomial time.

Proof. The algorithm begins with a pseudo flow $(\boldsymbol{x},\boldsymbol{\alpha})$ satisfying (2-3) or equivalently a pseudo flow $\boldsymbol{x}^{\boldsymbol{\alpha}}$ satisfying (4-1) with respect to some node potentials $\boldsymbol{\pi}$. At each step, the algorithm attempts to reduce the infeasibility of the solution and meanwhile attempts to preserve the optimality conditions. Lemmas (4-2) and (4-3) show that the algorithm at each step preserves the optimality conditions and reduces the infeasibility of the solution by sending flow from an excess node to a deficit node. The algorithm terminates, when an optimal and feasible flow is found. Let $U = \max\{u_{ij} : (i,j) \in A\} + B$, and $C = \max\{c_{ij} : (i,j) \in A\}$ and $S(m,n,C)$ denotes the time required to solve a shortest path problem with m arcs, n nodes and nonnegative cost whose values are no more than C. In each iteration, the algorithm finds a shortest path from an excess node to a deficit node which takes $o(S(m,n,nC))$ time, and the number of iterations is bounded by $o(nU)$. Thus the running time of the algorithm is $o(nU\ S(m,n,nC))$. By scaling the capacity similar to [7], the polynomial version of the algorithm, $o(mlog\ U\ S(m,n,nC))$, is obtained. □

5. Conclusion

This paper showed that the total costs of converting an infeasible flow network to a feasible one and solving the obtained problem may not be minimized by a LIF. A flow that minimizes the sum of these costs, MUCF, called Minimum Universal Cost Flow, was introduced and computed by a proposed algorithm of polynomial time order.

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